The role of a representative reinsurer in optimal reinsurance

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Abstract

In this paper, we consider a one-period optimal reinsurance design model with n reinsurers and an insurer. For very general preferences of the insurer and that all reinsurers use a distortion premium principle, we establish the existence of a representative reinsurer and this in turn facilitates solving the optimal reinsurance problem with multiple reinsurers. The insurer determines its optimal risk that it wants to reinsure via this pricing formula. The risk to be reinsured is then shared by the reinsurers via tranching. The optimal ceded loss functions among multiple reinsurers are derived explicitly under the additional assumptions that the insurer's preferences are given by an inverse-S shaped distortion risk measure and that the reinsurer's premium principles are some functions of the Conditional Value-at-Risk. We also demonstrate that under some prescribed conditions, it is never optimal for the insurer to cede its risk to more than two reinsurers.

Key Words: optimal reinsurance design, distortion risk measure, distortion premium principle, multiple reinsurers, representative reinsurer.

1 Introduction

The optimal risk sharing between an insurer and a reinsurer is one of the most challenging problems that has been heavily studied in the academic literature and actuarial practice. This problem is first formally analyzed by Borch (1960) who demonstrates that, under the assumption the reinsurance premium is calculated by the expected value principle, the stop-loss reinsurance treaty is the optimal strategy that minimizes the variance of the retained loss of the insurer. By maximizing the expected utility of the terminal wealth of a risk-averse insurer, Arrow (1963) similarly shows that the stop-loss reinsurance treaty is optimal. These pioneering results have subsequently been refined to incorporating more sophisticated optimality criteria and/or more realistic premium principles. See, for example, Kaluszka (2005) and Chi and Tan (2011) for a small sample of these generalizations. These results indicate that more exotic strategies such as that based on the limited stop-loss or truncated stop-loss could be optimal, as opposed to the stop-loss reinsurance.

While most of the existing literature on optimal reinsurance have predominantly confined to analyzing the optimal risk sharing between two parties, i.e., an insurer and a reinsurer, recently some progress has been made on addressing the optimal reinsurance in the presence of multiple reinsurers. See, for example, Asimit et al. (2013), Chi and Meng (2014), and Cong and Tan (2016). Such formulation is more reasonable since in a well established reinsurance market, an insurer could always use more than one reinsurer to reinsure its risk. In fact it may be desirable for the insurer to do so in view of the differences in reinsurers' risk attitude and the competitiveness of the reinsurance market. Some reinsurers may have higher risk tolerance and maybe more aggressive in pricing certain layers of risk. As a result, insurer that exploits such discrepancy among reinsurers might be able to achieve better risk sharing profile.

Motivated by these results, this paper studies the problem of optimal reinsurance in the presence of multiple reinsurers. The significant contributions of our proposed study can be described as follows. First, we allow for very general preferences of the insurer. In contrast, Cong and Tan (2016) assume that the insurer's objective to minimize its value at risk (VaR), while both Asimit et al. (2013) and Chi and Meng (2014) assume conditional value at risk (CVaR), in addition to VaR. Second, we allow for more than two reinsurers while the optimal reinsurance models of Asimit et al. (2013) and Chi and Meng (2014) explicitly assume two reinsurers. Third, both Asimit et al. (2013) and Chi and Meng (2014) impose the condition that one of the reinsurers adopts the expected value premium principle. Our proposed model, on the other hand, does not have such constraint. In fact, our premium principle is quite general in that we assume the reinsurers use distortion premium principles. Many authors, including Wang (1995, 2000), Wang et al. (1997), De Waegenaere et al. (2003), Chen and Kulperger (2006), Cheung (2010), Cui et al. (2013), and Assa (2015), use distortion functions to price risk. Special cases include the pricing principles induced by Wang transform, VaR, and expected value principle. Fourth, we also analyze the uniqueness of proposed solution. Finally, we demonstrate that under some additional assumptions, it is never optimal for an insurer to cede its loss to more than two reinsurers.

If there is only one reinsurer and the insurer maximizes dual utility (Yaari, 1987), the optimal reinsurance contract is given by tranching of the total insurance risk as shown by Cui et al. (2013), and Assa (2015). We extend this result to the case of multiple reinsurers and show that the optimal ceded loss functions could be in the form of multiple tranches, with each tranche being allocated to each reinsurer.

The layout of the remaining paper is as follows. The model setup is described in Section 2. In Section 3, we show our main results that characterize the representative reinsurer. Section 4 describes the optimal reinsurance contracts if the insurer uses dual utility. Section 5 provides an example where reinsurance prices are determined via the well-known CVaR. Finally, Section 6 concludes the paper.

2 Model Setup

The purpose of this section is to describe various important concepts including the distortion premium principles and our proposed formulation of the optimal reinsurance model in the presence of multiple reinsurers. These are described in Subsections 2.1 and 2.2, respectively.

2.1 Distortion premium principles

In this subsection, we introduce the pricing formula of the reinsurers. Let $(\Omega, F, \mathbb{P})$ be a probability space and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the class of bounded random variables on it. For brevity, we use the notation L^∞ to denote $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ when there is no confusion. We interpret random variables as a loss. We now define the distortion risk measure, which is due to Wang et al. (1997):

Definition 2.1 *The distortion risk measure is given by*

$$\rho^g(Z) = \int_0^\infty g(S_Z(z))dz, \text{ for all } Z \in L^\infty, \tag{1}$$

where S_Z is the survival function of the loss Z and the probability distortion function $g : [0,1] \to [0,1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$.

Corresponding to the distortion risk measure, we have the distortion premium principle, which is defined as follows:

Definition 2.2 *The distortion premium principle is given by*

$$\pi^{\theta,g}(Z) = (1+\theta) \cdot \rho^g(Z), \text{ for all } Z \in L^\infty, \tag{2}$$

where $\theta \geq 0$ can be interpreted as the loading factor and g is a probability distortion function.

Note that when $g(x) = x$, the above distortion premium principle reduces to the (loaded) expected value premium principle. When the distortion function is concave, the distortion premium principle recovers Wang's premium principle. Wu and Wang (2003) and Wu and Zhou (2006) provide a characterization of the distortion premium principle based on additivity of comonotonic risks. The premium principle formulation (2) allows also for pricing formulas that include a risk component such as $\pi(Z) = E[Z] + \alpha \cdot \rho^{\hat{g}}(Z)$ for $\alpha \geq 0$, where $\rho^{\hat{g}}$ can be a VaR or a CVaR. See, for example, Acerbi and Tasche (2002) and (24) or (26) in Section 5.

2.2 Reinsurance model set-up

We now turn to our proposed optimal reinsurance model. We assume that an insurer faces a non-negative and bounded random loss $X \in L^\infty$ and that $M = \operatorname{esssup} X = \inf\{a \in R : \mathbb{P}(X > a) = 0\}$. We further assume that there are n reinsurers in this market and that each of these reinsurers is indexed by $i \in \{1, \ldots, n\}$. The probability distribution of the loss exposure X of the insurer is assumed to be a common knowledge to all the participating reinsures and let $f_i(X)$ represent the portion of the loss X that is ceded to reinsurer i. The problem of optimal reinsurance is therefore concerned with the optimal partitioning of X into $f_i(X), i = 1, \ldots, n$, and $X - \sum_{i=1}^n f_i(X)$. Note that $\sum_{i=1}^n f_i(X)$ represents the aggregate loss that is ceded to all n participating reinsurers so that $X - \sum_{i=1}^n f_i(X)$ captures the loss that is retained by the insurer.

Let us now provide some additional discussion on the shape of ceded loss functions. In particular, the set of ceded loss functions that is of interest to us is of the following:

$$\mathcal{F} = \Big\{ f : [0, M] \to [0, M] \Big| f(0) = 0,\ 0 \leqslant f(x) - f(y) \leqslant x - y,\ \forall\ 0 \leq y < x \leq M \Big\}. \tag{3}$$

While in the reinsurance market the ceded loss function admits a variety of forms, virtually all of these contracts satisfy the property as stipulated in $\mathcal{F}$. The importance for a ceded loss function to satisfy $\mathcal{F}$ is driven by the concern with moral hazard, which otherwise will exist for the insurer (see, e.g., Denuit and Vermandele, 1998; Young, 1999; Bernard and Tian, 2009). For this reason, we similarly require that $f_i(X), i = 1, \ldots, n$, and $X - \sum_{i=1}^{n} f_i(X)$ belong to $\mathcal{F}$ (see Asimit et al., 2013; Chi and Meng, 2014). See also Remark 4 at the end of the next section.

For any $f \in \mathcal{F}$, we have

$$\rho^g(f(X)) = \int_0^M g(S_X(z))df(z). \tag{4}$$

The above result follows from the definition of the distortion risk measure (1), $f(0) = 0$ and the fact that f is 1-Lipschitz. See Zhuang et al. (2016, Lemma 2.1 therein) for a detailed proof.

By partially transferring some of the losses to reinsurers, the insurer incurs an additional cost in the form of reinsurance premium that is payable to the reinsurers. The reinsurance premium depends on the ceded loss function $f_i(X)$, the loading factor θ_i, and the probability distortion function g_i of the reinsurer. We use $\pi^{\theta_i, g_i}(f_i(X))$ to denote the resulting reinsurance premium that is charged by the reinsurer i for assuming loss $f_i(X)$. Let W denote the future wealth of the insurer in the absence of insuring risk X. The future wealth W can be random or deterministic. By insuring and reinsuring X, the net worth of the insurer becomes $W - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\theta_i, g_i}(f_i(X))$. Given that the reinsurance premium increases with the ceded losses, this suggests that a conservative insurer could eliminate most of its risk at the expense of higher reinsurance premium. On the other hand, a more aggressive insurer could reduce its reinsurance premium but exposes itself to a higher potential loss. This demonstrates the trade-off between risk retention and risk transfer.

From a risk management point of view, the existence of such a trade-off also implies that it is important for the insurer to seek the best reinsurance strategy that optimally balances between risk retention and risk transfer. This can be accomplished by formulating the problem as an optimization problem. More specifically, let V capture the utility of an insurer's net worth. Here, V is a function that maps random variables to real numbers. We assume that $V(W) < \infty$, and V is strictly monotonic; i.e., for all $X > Y$ almost surely, we have $V(X) > V(Y)$. Moreover, for any $f^1, f^2 \in \mathcal{F}$, we define a metric

$$d(f^1, f^2) := \max_{t \in [0,M]} \mid f^1(t) - f^2(t) \mid . \tag{5}$$

In this paper, we further assume that V is continuous in $f \in \mathcal{F}$ under this metric. Then the optimal strategy for the insurer to cede its risk to n reinsures can be determined by solving the following optimization problem:

$$\begin{array}{ll} \max & V\Big(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\Big) \\ \text{s.t.} & f_i \in \mathcal{F}, \forall i = 1, \ldots, n,\ \sum_{i=1}^n f_i \in \mathcal{F}. \end{array} \tag{6}$$

Recall that the term inside the function V gives the net worth of the insurer in the presence of n multiple reinsurers. The optimal ceded loss functions $f_i, i = 1, \ldots, n$ therefore maximize the objective of the insurer while subject to the condition as stipulated by $\mathcal{F}$. In the special case with only one reinsurer, we omit the subscript i so that the above optimal reinsurance problem simplifies to

$$\begin{array}{ll} \max & V\Big(W - X + f(X) - \pi^{\theta, g}(f(X))\Big) \\ \text{s.t.} & f \in \mathcal{F}. \end{array} \tag{7}$$

The above special case of the optimal reinsurance problem has been studied extensively for various functional forms of the utility function V in the literature. For instance, if V is an expected utility of the insurer and the reinsurance premium principle is the expected value premium principle, the resulting problem is studied by Arrow (1963) who shows that the stop-loss contract is optimal. Moreover, if V is an expected utility function, the distortion function g is concave or piece-wise linear, and $\theta = 0$, then we recover the model of Young (1999). If V is a rank-dependent expected utility function and $g(s) = s$ for all $s \in [0, 1]$, the problem in (7) has been studied by Xu et al. (2015).

In this paper, our focus is the multiple reinsurance design problem (6). In particular, we will analyze the uniqueness and the construction of ceded loss contracts $f_i(X), i = 1, \ldots, n$.

3 The existence of a representative reinsurer - general case

We begin this section by first providing the following proposition which asserts the existence of the optimal solution to (7) with one reinsurer.

Proposition 3.1 *There exists a solution to the optimal reinsurance model* (7).

PROOF: By the Arzela-Ascoli's theorem (see Dunford and Schwartz, 1958), the set $\mathcal{F}$ is a compact set under the metric d defined in (5). Since V is strictly monotonic, we get from $V(W) < \infty$ that the utility of the insurer is bounded for any $f \in \mathcal{F}$. Moreover, combining this with the assumption that V is continuous in f under this metric d, an optimal solution to (7) exists by the Weierstrass extreme value theorem. $\square$

We now introduce an important concept known as the *representative reinsurer*. Consider a reinsurance market of n reinsurers and that the set of ceded loss functions $(f_i)_{i=1}^n$ is the solution to (6). Suppose further there exists a reinsurer that similarly uses distortion premium principle but with safety loading $\widetilde{\theta}$ and distortion function $\widetilde{g}$ such that

- $\sum_{i=1}^n f_i$ solves (7) with $\theta = \widetilde{\theta}$ and $g = \widetilde{g}$;
- $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\widetilde{\theta}, \widetilde{g}}(\sum_{i=1}^n f_i(X))$.

Then the reinsurer with safety loading $\widetilde{\theta}$ and distortion function $\widetilde{g}$ is denoted as the representative reinsurer. The fundamental role of representative reinsurer provides a powerful way of solving the multiple reinsurance problem (6). We can solve (6) by first focusing on the single reinsurer problem (7). Once we have established the representative reinsurer from (7), this in turn gives us the total losses that will be ceded to the n reinsurers, together with the aggregate reinsurance premium for solving (6). The remaining task is then to identify the optimal risk sharing among the n reinsurers while ensuring that the total reinsurance premium charged by these n reinsurers has the same amount as that charged by the representative reinsurer.

Not only the notion of representative reinsurer facilitates the solving of multiple reinsurance problem, it can, in part, capture some phenomenon that we observe in the reinsurance market. Instead of dealing with multiple reinsurers concurrently, an insurer may choose to deal only with one reinsurer but with the understanding that the reinsurer will in turn risk sharing the ceded risk among other reinsurers. This market practice is particularly common in reinsuring catastrophic risk where no single reinsurer is willing to reinsure the entire ceded risk. In this aspect, the reinsurer that the insurer is dealing with is facilitating as the role of representative reinsurer.

Remark 1 *The notion of representative reinsurer is also exploited in Cong and Tan (2016) for seeking optimal reinsurance strategies in the presence of multiple reinsurers. They apply pricing functions that are given by piecewise pricing principles. They, however, assume that the budget for reinsurance is fixed. Also, they assume that the preferences of the insurer are given by VaR.*

Before we show the existence of the representative reinsurer, we first specify its characteristics associate with $(\widetilde{\theta}, \widetilde{g})$, as shown in Lemma 3.2 below. For the remaining of the paper, we assume that the pair $(\widetilde{\theta}, \widetilde{g})$ is as specified in this lemma.

Lemma 3.2 *There is a unique $\widetilde{\theta} \geq 0$ and distortion function $\widetilde{g}$ on the range of S_X such that $(1+\widetilde{\theta})\widetilde{g}(S_X(z)) = \min_{1\leq i\leq n}(1+\theta_i)g_i(S_X(z))$ for all $z \in [0, M]$, and it is such that $\widetilde{\theta} = \min_{1\leq i\leq n}\theta_i$ and $\widetilde{g}(x) = \frac{\min_{1\leq i\leq n}(1+\theta_i)g_i(x)}{1+\widetilde{\theta}}$ for all x on the range of S_X.*

Proof: It follows from $\min_{1\leq i\leq n}(1+\theta_i)g_i(1) = (1+\widetilde{\theta})\widetilde{g}(1)$ and $g_i(1) = \widetilde{g}(1) = 1$ that $\widetilde{\theta} = \min_{1\leq i\leq n}\theta_i$. Moreover, $\min_{1\leq i\leq n}(1+\theta_i)g_i(x) = (1+\widetilde{\theta})\ \frac{\min_{1\leq i\leq n}(1+\theta_i)g_i(x)}{1+\widetilde{\theta}} = (1+\widetilde{\theta})\widetilde{g}(x)$ for all x on the range of S_X. Hence the desired result follows directly. □

Proposition 3.1 allows us to define the set

$$\mathcal{F}^* = \{f \in \mathcal{F} : f \text{ solves (7) with } \theta = \widetilde{\theta} \text{ and } g = \widetilde{g}\}.$$

For any given $f \in \mathcal{F}$, we define the following problem in which we minimize the total reinsurance premium given a total reinsurance contract $f(X)$:

$$\begin{array}{ll} \min & \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) \\ \text{s.t.} & f_i \in \mathcal{F}, \forall i = 1, \ldots, n, \ \sum_{i=1}^n f_i = f. \end{array} \tag{8}$$

Given the total risk f that will be ceded to n reinsurers, the above optimization problem optimally determines the least expensive reinsurance strategy of allocating the total risk to these reinsurers. This is collected in the following set:

$$\mathcal{F}(f) = \{f_i \in \mathcal{F}, i = 1, \ldots, n : (f_i)_{i=1}^n \text{ solves (8) for a given } f\},$$

for all $f \in \mathcal{F}$. Since the functions f and $(f_i)_{i=1}^n$ are absolutely continuous,[1] we can define h_i as the density of f_i for $i = 1, \ldots, n$, and h as the density of f, satisfying $f_i(z) = \int_0^z h_i(x)dx$ and $f(z) = \int_0^z h(x)dx$ for any $z \in [0, M]$. In the following theorem, we characterize all optimal solutions to (8).

[1] A function f is absolutely continuous (or in short a.c.) on $[0, M]$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of $[0, M]$ satisfies $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$.

Theorem 3.3 *Let $f \in \mathcal{F}$. We have $(f_i)_{i=1}^n \in \mathcal{F}(f)$ if and only if $(f_i)_{i=1}^n$ is such that*

$$h_i(z) = \begin{cases} \lambda_i(z) & \text{if } i \in \operatorname{argmin}_{1\le j\le n}(1+\theta_j)g_j(S_X(z)), \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

for all $i = 1, \dots, n$, and all $z \in [0, M]$ almost surely, where $\lambda_i(z), z \in [0, M]$ is such that

$$\sum_{i=1}^n h_i(z) = h(z).$$

Proof: We can rewrite the objective function of (8) as

$$\begin{aligned} \sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^M g_i(S_X(z))df_i(z) \\ &= \sum_{i=1}^n \int_0^M (1+\theta_i)g_i(S_X(z))h_i(z)dz. \end{aligned}$$

Hence, every h_i that minimizes the above expression such that $\sum_{i=1}^n h_i = h$ is optimal. It is easy to see that the set of optimal solutions is given by solutions satisfying (9) almost surely. □

From Theorem 3.3, we get that $\mathcal{F}(f)$ is non-empty for all $f \in \mathcal{F}$.

If $\theta_i = 0$ for all $i = 1, \dots, n$, and if the distortion functions $g_i, i = 1, \dots, n$ are all concave, the objective in (8) is the same as the objective to determine Pareto optimal risk redistributions as in Ludkovski and Young (2009). However, in the context of optimal reinsurance contracts, assuming $\theta_i = 0$ for all $i = 1, \dots, n$ is restrictive.

Corollary 3.4 *For any given $f \in \mathcal{F}$, the set $\mathcal{F}(f)$ is singleton if and only if the Lebesgue measure of the set*

$$\left\{ z \in [0, M] : h(z) > 0, |\operatorname*{argmin}_{1\le i\le n}(1+\theta_i)g_i(S_X(z))| > 1 \right\},$$

is zero, where h is the density of f, and $|A|$ denotes the cardinality of the set A.

We now state the main result of this paper.

Theorem 3.5 *It holds that $f_i(X), i = 1, \dots, n$, solve (6) if and only if $\sum_{i=1}^n f_i \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$.*

Before proving the above theorem, let us first prove the following two lemmas.

Lemma 3.6 *For any* $(f_i)_{i=1}^n$*, with* $\sum_{i=1}^n f_i \in \mathcal{F}$*, we have*

$$\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) \geq \pi^{\widetilde{\theta}, \widetilde{g}}\Big(\sum_{i=1}^n f_i(X)\Big).$$

Proof: This result follows from

$$\begin{aligned}
\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^M g_i(S_X(z)) df_i(z) && (10) \\
&\geq \sum_{i=1}^n \int_0^M \min_{1\leq i\leq n} (1+\theta_i) g_i(S_X(z)) df_i(z) && (11) \\
&= \sum_{i=1}^n \int_0^M (1+\widetilde{\theta})\widetilde{g}(S_X(z)) df_i(z) && (12) \\
&= \int_0^M (1+\widetilde{\theta})\widetilde{g}(S_X(z)) d \sum_{i=1}^n f_i(z) && (13) \\
&= \pi^{\widetilde{\theta}, \widetilde{g}}\Big(\sum_{i=1}^n f_i(X)\Big), && (14)
\end{aligned}$$

where (10) and (14) follow from (4), (12) follows from Lemma 3.2, and (13) follows from Fubini's theorem. This concludes the proof. □

Lemma 3.7 *Let* $f \in \mathcal{F}$*. If* $(f_i)_{i=1}^n \in \mathcal{F}(f)$*, we have* $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\widetilde{\theta}, \widetilde{g}}(\sum_{i=1}^n f_i(X)) = \pi^{\widetilde{\theta}, \widetilde{g}}(f(X))$.

Proof: Let $f \in \mathcal{F}$ and $(f_i)_{i=1}^n \in \mathcal{F}(f)$. Then, for $f_i(X), i = 1, \ldots, n$, as in Theorem 3.3, it follows that

$$\begin{aligned}
\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) &= \sum_{i=1}^n \int_0^M (1+\theta_i) g_i(S_X(z)) df_i(z) \\
&= \int_0^M \min_{1\leq i\leq n} (1+\theta_i) g_i(S_X(z)) df(z).
\end{aligned}$$

Then, the inequality (11) in the proof of Lemma 3.6 is an equality. Hence, $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\widetilde{\theta}, \widetilde{g}}(\sum_{i=1}^n f_i(X)) = \pi^{\widetilde{\theta}, \widetilde{g}}(f(X))$, as required. □

Proof of Theorem 3.5: First, we show the "if" part of the proof. We prove this by contradiction. Let $f \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(f)$, but assume that $f_i(X), i = 1, \dots, n$, do not solve the problem (6). It follows from Proposition 3.1 that there exist reinsurance contracts $\hat{f}_i(X), i = 1, \dots, n$, which solve (6). Hence

$$\begin{aligned} V\Big(W - X + \sum_{i=1}^n \hat{f}_i(X) - \pi^{\widetilde{\theta},\widetilde{g}}(\sum_{i=1}^n \hat{f}_i(X))\Big) &\geq V\Big(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(\hat{f}_i(X))\Big) \\ &> V\Big(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))\Big) \\ &= V\Big(W - X + \sum_{i=1}^n f_i(X) - \pi^{\widetilde{\theta},\widetilde{g}}(\sum_{i=1}^n f_i(X))\Big), \end{aligned}$$

where the first inequality is due to Lemma 3.6 and the equality follows from Lemma 3.7. This is a contradiction with the assumption that $f \in \mathcal{F}^*$. Hence, $f_i(X), i = 1, \dots, n$, solve (6).

We now continue with the "only if" part of the proof. Let $f_i(X), i = 1, \dots, n$, solve (6). Assume that we do not have $f = \sum_{j=1}^n f_j \in \mathcal{F}^*$ or $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$. First, suppose that $f \notin \mathcal{F}^*$. From Proposition 3.1 and Theorem 3.3, we get that there exists a $(\hat{f}_i)_{i=1}^n$ such that $\hat{f} \in \mathcal{F}^*$ and $(\hat{f}_i)_{i=1}^n \in \mathcal{F}(\hat{f})$. Let $\hat{f} \in \mathcal{F}^*$ and $(\hat{f}_i)_{i=1}^n \in \mathcal{F}(\hat{f})$. It follows from the conclusion for the "if" part that $\hat{f}_i(X), i = 1, \dots, n$ solve (6). We obtain

$$\begin{aligned} V\Big(W - X + \sum_{i=1}^n \hat{f}_i(X) - \pi^{\widetilde{\theta},\widetilde{g}}(\sum_{i=1}^n \hat{f}_i(X))\Big) &= V\Big(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(\hat{f}_i(X))\Big) \\ &= V\Big(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))\Big) \\ &\leq V\Big(W - X + \sum_{i=1}^n f_i(X) - \pi^{\widetilde{\theta},\widetilde{g}}(\sum_{i=1}^n f_i(X))\Big), \end{aligned}$$

where the first equality is due to Lemma 3.7, the second equality follows from the assumption that $f_i(X), i = 1, \dots, n$, solve (6), and the inequality follows from Lemma 3.6. However, this contradicts the assumption that $f \notin \mathcal{F}^*$.

Second, suppose that $(f_i)_{i=1}^n \notin \mathcal{F}(\sum_{j=1}^n f_j)$. Theorem 3.3, which implies there exists a $(\hat{f}_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$, and together with the strict monotonicity of V, lead to

$$V\Big(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(\hat{f}_i(X))\Big) > V\Big(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))\Big).$$

This is a contradiction with the assumption that $f_i(X), i = 1, \ldots, n$, solve (6). Consequently, we have $f \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$. This completes the proof. $\square$

The following proposition follows directly from Corollary 3.4 and Theorem 3.5. It provides a necessary and sufficient condition for verifying the uniqueness of the optimal solution to (6).

Proposition 3.8 *The optimal solution to* (6) *is unique if and only if the set $\mathcal{F}^*$ is single-valued and the Lebesgue measure of the set*

$$\left\{z \in [0, M] : h(z) > 0, |\operatorname*{argmin}_{1 \le i \le n}(1 + \theta_i)g_i(S_X(z)))| > 1\right\} \tag{15}$$

is zero, where h is the density of f and $f \in \mathcal{F}^*$.

The following proposition asserts that if it is optimal to cede f_i to reinsurer i with loading factor θ_i and distortion function g_i, then it is not possible to find another reinsurer that offers cheaper reinsurance premium for assuming the same ceded risk f_i. This also implies that in an optimal reinsurance arrangement, there is no incentive for the insurer to reinsure risks from one reinsurer to another.

Proposition 3.9 *Assume that $f_i(X), i = 1, \ldots, n$, solve* (6)*, then for all $i, j = 1, \ldots, n$, we have*

$$\pi^{\theta_i, g_i}(f_i(X)) \le \pi^{\theta_j, g_j}(f_i(X)).$$

Moreover, if the Lebesgue measure of the set in (15) *is zero, we have that there exist $i \ne j$ such that*

$$\pi^{\theta_i, g_i}(f_i(X)) < \pi^{\theta_j, g_j}(f_i(X)).$$

PROOF: We first prove the first result by contradiction. Suppose that $f_i(X), i = 1, \ldots, n$, solve (6) and that there exist i, j such that $\pi^{\theta_i, g_i}(f_i(X)) > \pi^{\theta_j, g_j}(f_i(X))$. Define the following contracts

$$\widetilde{f}_k(X) = \begin{cases} f_k(X) & \text{if } k \ne\ i \text{ or } j, \\ 0 & \text{if } k = i, \\ f_i(X) + f_j(X) & \text{if } k = j, \end{cases}$$

for $k = 1, \ldots, n,$. From the fact that $\pi^{\theta_i, g_i}(f_i(X)) + \pi^{\theta_j, g_j}(f_j(X)) > \pi^{\theta_j, g_j}(f_i(X)) + \pi^{\theta_j, g_j}(f_j(X)) = \pi^{\theta_j, g_j}(f_i(X)+f_j(X))$, we obtain $\sum_{i=1}^n \pi^{\theta_i, g_i}(\widetilde{f}_i(X)) < \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$. Moreover, we have $\sum_{i=1}^n \widetilde{f}_i(X)$

$= \sum_{i=1}^n f_i(X) = f(X)$. So, $f_i(X), i = 1, \dots, n$, is not minimizing (8). Hence, it follows from Theorem 3.5 that $f_i(X), i = 1, \dots, n$ do not solve (6). This is a contradiction.

The proof of the second result is analogue, where we note that from Corollary 3.4 there is a unique sequence $f_i(X), i = 1, \dots, n$, with $\sum_{i=1}^n f_i(X) = f(X)$ that minimizes $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$. Then, the result follows from Theorem 3.5. $\square$

Remark 2 *If there exists a reinsurer i such that $(1+\theta_i)g_i(S_X(z)) \geq \min_{1\leq j\leq n}(1+\theta_j)g_j(S_X(z))$ for all $z \in [0, M]$, then there exists an optimal solution such that $f_i(X) = 0$. This follows from Theorem 3.3, Theorem 3.5 and Proposition 3.9. Moreover, if there exists a reinsurer i such that $(1+\theta_i)g_i(S_X(z)) > \min_{1\leq j\leq n, j\neq i}(1+\theta_j)g_j(S_X(z))$ for all $z \in (0, M]$, then all optimal solutions are such that $f_i(X) = 0$. This remark is intuitive as rationally an insurer will not cede its risk to a reinsurer that does not offer competitive pricing.*

Remark 3 *If there exists another reinsurer which cannot be represented by the distortion premium principle, then our results still hold in the sense that we can find a representative reinsurer to represent the n reinsurers that use a distortion premium principle. The proof is similar to Theorem 3.5 and hence is omitted.*

We conclude this section by considering a more general class of ceded loss functions which we denote as $\mathcal{F}^N$.

Recall that a formal definition of a.c. was given in Footnote 1. It is a well-known result that, if f is a.c. on $[0, M]$, then there exists a Lebesgue integrable function h on $[0, M]$ such that $f(x) = f(0) + \int_0^x h(s)ds$ for all $x \in [0, M]$. Moreover, h is unique almost everywhere (a.e.). For a given upper bound $N > 0$, let us now consider the following set

$$\mathcal{F}^N = \left\{ f : [0, M] \to [0, N] \middle| f(0) = 0, f \text{ is non-decreasing and a.c.} \right\}. \tag{16}$$

Comparing to $\mathcal{F}$ in (3), the function in $\mathcal{F}^N$ is less restrictive. Despite that $\mathcal{F}^N$ is more general, it should be emphasized that even if we require that $f_i \in \mathcal{F}^N$ for all $i = 1, \dots, n$ and that $\sum_{i=1}^n f_i \in \mathcal{F}^N$, i.e., the feasible sets in (6) and (7) become $\mathcal{F}^N$, then all the results we have discussed so far still hold. The assumptions that reinsurance contracts are a.c. and bounded are mild technical conditions.

Remark 4 *We just pointed out that the results we have derived so far are equally applicable even if we consider a more general class of ceded loss functions $\mathcal{F}^N$. These results, unfortunately, have limited practical values. The key reason is that the assumptions $f_i \in \mathcal{F}^N$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n f_i \in \mathcal{F}^N$ could lead to the loss retained by the insurer, i.e., $x - \sum_{i=1}^n f_i(x)$, which is decreasing and even negative. The retained loss function that exhibits such phenomenon is perceived to be unreasonable and exposes the insurer to moral hazard. The retained loss should be non-decreasing in loss and that under no circumstances should the indemnified loss be greater than the insurer's incurred loss.*

4 Special case: preferences given by dual utility

In this section, we study the case where the preferences of the insurer are given by minimizing a distortion risk measure. This corresponds to dual utility, as introduced by Yaari (1987) via a modification of the independence axiom for expected utility. Preferences given by maximizing dual utility are equivalent to minimizing distortion risk measures. Hence we have $V(X) = -\rho^g(-X)$. These preferences have also been used as preferences of the insurer in bilateral reinsurance problems by Zheng and Cui (2014). Our preferences, however, are slightly less general than the law-invariant convex risk measures used by Cheung et al. (2014) in a bilateral reinsurance set-up. Nevertheless, both papers only show optimal reinsurance contracts for the expected value premium principle, whereas we use the more general distortion premium principle. For our problem, the optimal reinsurance problem is given by

$$\begin{array}{ll} \max & -\rho^g\Big(-W + X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\Big) \\ \text{s.t.} & f_i \in \mathcal{F}, \forall i = 1, \ldots, n, \ \sum_{i=1}^n f_i \in \mathcal{F}. \end{array} \tag{17}$$

where the wealth W is assumed to be deterministic.

As in Cui et al. (2013) and Assa (2015), we similarly assume that the premium is given by the general distortion premium principle. Unlike their works, we solve the optimal reinsurance problem in the context of multiple reinsurers. The distortion risk measure used by the insurer might be generated by any non-decreasing distortion function. A plausible function, which has recently gained popularity in behavioral finance, is the inverse-S shaped distortion function. See Bernard et al. (2015) and Xu et al. (2015) for more details. We will relegate the discussion of this particular distortion function to Section 5.

By the translation invariance property, the objective function in (17) can be simplified as follows:

$$-\rho^g\Big(-W+X-\sum_{i=1}^n f_i(X)+\sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))\Big)=W-\rho^g\Big(X-\sum_{i=1}^n f_i(X)\Big)-\sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X)), \tag{18}$$

since W and $\sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))$ are constants. Furthermore, we have

$$\rho^g\Big(X-\sum_{i=1}^n f_i(X)\Big)=\rho^g(X)-\sum_{i=1}^n \rho^g(f_i(X)), \tag{19}$$

as $f_i(X), i=1,\dots,n$, $\sum_{i=1}^n f_i(X)$ and $X-\sum_{i=1}^n f_i(X)$ are comonotonic random variables. We can ignore $\rho^g(X)$ in the objective function (19) since it is a constant and does not depend on $f_i(X), i=1,\dots,n$.

Finally, by using (4), the objective function (18) can be expressed by

$$\begin{aligned}\sum_{i=1}^n \rho^g(f_i(X))-\sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X) &= \sum_{i=1}^n \int_0^M g(S_X(z))df_i(z)-\sum_{i=1}^n(1+\theta_i)\int_0^M g_i(S_X(z))df_i(z)\\ &= \sum_{i=1}^n \int_0^M \Big[g(S_X(z))-(1+\theta_i)g_i(S_X(z))\Big]df_i(z). \end{aligned}\tag{20}$$

Consequently, an equivalent formulation of (17) can be written as follows

$$\begin{array}{ll}\min & \sum_{i=1}^n \int_0^M \Big[(1+\theta_i)g_i(S_X(z))-g(S_X(z))\Big]df_i(z)\\ \text{s.t.} & f_i\in\mathcal{F}, \forall i=1,\dots,n,\ \sum_{i=1}^n f_i\in\mathcal{F},\end{array}\tag{21}$$

By denoting $\mathcal{H}=\{h:[0,M]\to[0,1]\Big|0\leqslant h(z)\leqslant 1,\ a.s.\}$ and since $f_i\in\mathcal{F}, \forall i=1,\dots,n,\ \sum_{i=1}^n f_i\in\mathcal{F}$, this implies that (21) can be written as

$$\begin{array}{ll}\min & \sum_{i=1}^n \int_0^M \Big[(1+\theta_i)g_i(S_X(z))-g(S_X(z))\Big]h_i(z)dz\\ \text{s.t.} & h_i\in\mathcal{H}, \forall i=1,\dots,n,\ \sum_{i=1}^n h_i\in\mathcal{H}.\end{array}\tag{22}$$

Now let us define $A=\Big\{z\in[0,M]:\min_{1\le i\le n}\big\{(1+\theta_i)g_i(S_X(z))-g(S_X(z))\big\}<0\Big\}$, $B=\Big\{z\in[0,M]:\min_{1\le i\le n}\big\{(1+\theta_i)g_i(S_X(z))-g(S_X(z))\big\}=0\Big\}$ and $C=\Big\{z\in[0,M]:\min_{1\le i\le n}\big\{(1+\theta_i)g_i(S_X(z))-g(S_X(z))\big\}>0\Big\}$. Then, Theorem 3.5 and Theorem 3.3 lead to the following proposition.

Proposition 4.1 *Every optimal solution to* (22) *satisfies*

$$h_i(z) = \begin{cases} \lambda_i(z) & \text{if } z \in A \text{ and } i \in \operatorname{argmin}_{1\le j\le n}\left\{(1+\theta_j)g_j(S_X(z)) - g(S_X(z))\right\}, \\ \phi_i(z) & \text{if } z \in B \text{ and } i \in \operatorname{argmin}_{1\le j\le n}\left\{(1+\theta_j)g_j(S_X(z)) - g(S_X(z))\right\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $z \in [0, M]$*, where* $\lambda_i(z), \phi_i(z) \in [0, 1]$ *are such that*

$$\sum_{i=1}^{n} h_i(z) = \begin{cases} 1 & \text{if } z \in A, \\ \phi(z) & \text{if } z \in B, \\ 0 & \text{if } z \in C, \end{cases}$$

and $\phi(z) \in [0, 1]$ *for any* $z \in B$.

It follows immediately from Proposition 4.1 that there is a unique optimal solution to (22) if and only if the Lebesgue measure of B is zero and $\operatorname{argmin}_{1\le j\le n}\left\{(1+\theta_j)g_j(S_X(z)) - g(S_X(z))\right\}$ is unique almost surely for $z \in A$.

In the special case with only one reinsurer, the optimal reinsurance model (17) simplifies to

$$\begin{aligned} \min \quad & \rho^g\Big(X - f(X) + \pi^{\widetilde{\theta},\widetilde{g}}(f(X))\Big) \\ \text{s.t.} \quad & f \in \mathcal{F}. \end{aligned} \tag{23}$$

Using similar arguments as in (19) and (20) show that the optimal solution to (23) satisfies

$$h(z) = \begin{cases} 1 & \text{if } z \in A, \\ \phi(z) & \text{if } z \in B, \\ 0 & \text{if } z \in C, \end{cases}$$

where $\phi(z) \in [0, 1]$ for all $z \in B$ almost surely. This result with one reinsurer is alternatively established by Cui et al. (2013) and Assa (2015).

5 Optimal reinsurance with conditional value-at-risk (CVaR)

In this section, we illustrate the concept of a representative reinsurer by means of an example involving the conditional value-at-risk. First, let us recall the definitions of VaR and CVaR risk measures. The VaR of a non-negative random variable Z at a confidence level $1-\alpha$ where $0 < \alpha < 1$ is given by $VaR_\alpha(Z) = \inf\{z \geq 0 : \mathbb{P}(Z > z) \leq \alpha\}$. The CVaR of Z at a confidence level $1-\alpha$ is given by $CVaR_\alpha(Z) = \frac{1}{\alpha}\int_0^\alpha VaR_s(Z)ds$. It is well-known that $CVaR_\alpha$ is a distortion premium principle with $\theta = 0$ and $g(s) = \min\{\frac{s}{\alpha}, 1\}$ for all $s \in [0,1]$ (see, e.g., Dhaene et al., 2006).

In this example we assume that reinsurer i adopts the following premium principle:

$$\pi_i(X) = E[X] + \alpha_i(CVaR_{\beta_i}(X) - E[X]) \text{ for all } i = 1, \dots, n, \tag{24}$$

where $\alpha_i, \beta_i \in (0,1)$. This is an example of a distortion premium principle, as can be seen by setting $\theta_i = 0$ and g_i to

$$g_i(s) = \begin{cases} (1-\alpha_i + \frac{\alpha_i}{\beta_i})s & \text{if } 0 \leq s \leq \beta_i, \\ (1-\alpha_i)s + \alpha_i & \text{if } \beta_i < s \leq 1. \end{cases}$$

By defining $i^* \in \mathrm{argmax}_{1 \leq i \leq n}\, \alpha_i \left(1 - \frac{1}{\beta_i}\right)$, $j^* \in \mathrm{argmin}_{1 \leq j \leq n}\, \alpha_j$, and setting

$$\hat{s} = \frac{\alpha_{j^*}}{\alpha_{j^*} + \alpha_{i^*}(\frac{1}{\beta_{i^*}} - 1)} \in (0,1), \tag{25}$$

it is easy to show that $g_{i^*}(s) \leq g_{j^*}(s)$ for $s \in [0, \hat{s}]$ and $g_{i^*}(s) \geq g_{j^*}(s)$ for $s \in [\hat{s}, 1]$. It can also be shown that the pricing principle of the representative reinsurer is such that $\widetilde{\theta} = 0$, and the distortion function $\widetilde{g}$ is given by

$$\widetilde{g}(s) = \begin{cases} (1-\alpha_{i^*} + \frac{\alpha_{i^*}}{\beta_{i^*}})s & \text{if } 0 \leq s \leq \hat{s}, \\ (1-\alpha_{j^*})s + \alpha_{j^*} & \text{if } \hat{s} < s \leq 1. \end{cases}$$

Therefore, if $i^* = j^*$, we have $\widetilde{g} = g_{i^*}$.

In this example, we wish to study an insurer that is endowed with an inverse-S shaped distortion function. Formally, an inverse-S shaped distortion function is defined as follows:

Definition 5.1 *A distortion function g is inverse-S shaped if:*

- *it is continuously differentiable;*
- *there exists $b \in (0,1)$ such that g is strictly concave on the domain $(0,b)$ and strictly convex on the domain $(b,1)$;*
- *it holds that $g'(0) = \lim_{s\downarrow 0} g'(s) > 1$ and $g'(1) = \lim_{s\uparrow 1} g'(s) > 1$.*

Tversky and Kahneman (1992) propose an inverse-S distortion function, which is given by $g^{\gamma}(s) = s^{\gamma}/(s^{\gamma} + (1-s)^{\gamma})^{\frac{1}{\gamma}}$, for all $s \in [0,1]$, where $\gamma > 0$. As noted by Rieger and Wang (2006) and Ingersoll (2008), this probability distortion function is increasing and exhibiting inverse-S shaped for any $\gamma \in (0.279, 1)$. For inverse-S shaped distortion functions, the next property of the function

$$p(s) = \frac{1-g(s)}{1-s}, \text{ for all } s \in [0,1),$$

follows from Xu et al. (2015).

Lemma 5.1 *The function p is continuous. Moreover, there exists a point $a \in (0,b)$ such that p is strictly decreasing on $[0,a]$ and strictly increasing on $[a,1]$.*

According to the results established in the previous section, we need to study the sign of the difference between the distortion functions g and $\widetilde{g}$. It turns out that in our example, there are six cases to consider. Each case corresponds to a specific structure of the reinsurance contracts. We first split these cases depending on the relative magnitude of $\widetilde{g}'(0)$ and $g'(0)$. In particular, the condition $\widetilde{g}'(0) = 1 - \alpha_{i^*} + \frac{\alpha_{i^*}}{\beta_{i^*}} < g'(0)$ leads to four cases for locations of $(\hat{s}, \widetilde{g}(\hat{s}))$, which are labeled as $1, 2, \ldots, 4$ in Figure 1. In what follows, we determine $f_{i^*}(X)$ and $f_{j^*}(X)$ separately. If $i^* = j^*$, then this reinsurer reinsures the risk $f_{i^*}(X) + f_{j^*}(X)$.

Case 5.1 If $\widetilde{g}(\hat{s}) \leq g(\hat{s})$, then there exists a $c \in [\hat{s}, 1)$ such that $\widetilde{g}(s) < g(s)$ for $s \in (0,c)$ and $\widetilde{g}(s) > g(s)$ for $z \in (c,1)$. Therefore, it follows from Proposition 4.1 that the optimal solution to (23) is given by

$$h(z) = \begin{cases} 1 & \text{if } 0 \leq S_X(z) \leq c, \\ 0 & \text{if } c < S_X(z) \leq 1, \end{cases}$$

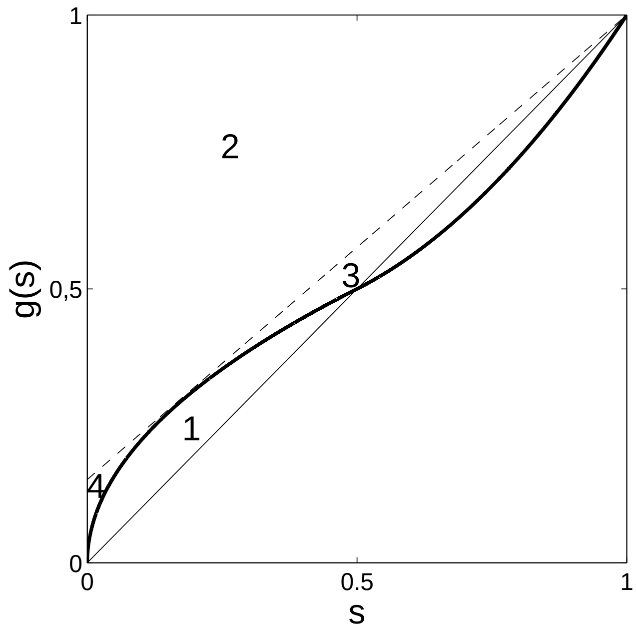

Figure 1: This figure displays an inverse-S shaped probability distortion function. Moreover, it shows the first four cases for any location of $(\hat{s}, \widetilde{g}(\hat{s}))$, where $1 \leq \widetilde{g}'(0) < g'(0)$.

for all $z \in [0, M]$ almost surely, or

$$h(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_c(X), \\ 1 & \text{if } VaR_c(X) < z \leq M. \end{cases}$$

This leads to

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_c(X), \\ z - VaR_c(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = (X - VaR_c(X))^+$, where $(Y)^+ = \max\{Y, 0\}$. Moreover, Proposition 4.1 implies that an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = (X - VaR_{\hat{s}}(X))^+$, $f_{j^*}(X) = \min\{(X - VaR_c(X))^+, VaR_{\hat{s}}(X) - VaR_c(X)\}$, and $f_i(X) = 0$ for all $i \neq i^*, j^*$.

Case 5.2 If $\widetilde{g}(\hat{s}) > g(\hat{s})$ and $p(a) \geq 1 - \alpha_{j^*}$, then there exists a point $c \in (0, \hat{s})$ such that $\widetilde{g}(s) < g(s)$ for $s \in (0, c)$ and $\widetilde{g}(s) > g(s)$ for $s \in (c, 1)$. The optimal solution to (23) is given by $f(X) = (X - VaR_c(X))^+$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = f(X)$, and $f_i(X) = 0$ for all $i \neq i^*$.

Case 5.3 If $\widetilde{g}(\hat{s}) > g(\hat{s})$, $\hat{s} \geq a$ and $p(a) < 1 - \alpha_{j^*}$, the solution coincides with the solution of Case 5.2.

Case 5.4 If $\widetilde{g}(\hat{s}) > g(\hat{s})$, $\hat{s} < a$ and $p(a) < 1 - \alpha_{j^*}$, then there exist three points $c \in (0, \hat{s})$, $d \in (\hat{s}, a)$ and $e \in (a, 1)$ such that $\widetilde{g}(s) < g(s)$ for $s \in (0, c)$, $\widetilde{g}(s) > g(s)$ for $s \in (c, d)$, $\widetilde{g}(s) < g(s)$ for $s \in (d, e)$ and $\widetilde{g}(s) > g(s)$ for $s \in (e, 1)$. The optimal solution to (23) is given by

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_e(X), \\ z - VaR_e(X) & \text{if } VaR_e(X) < z \leq VaR_d(X), \\ VaR_d(X) - VaR_e(X) & \text{if } VaR_d(X) < z \leq VaR_c(X), \\ z - VaR_c(X) + VaR_d(X) - VaR_e(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = \min\{(X - VaR_e(X))^+, VaR_d(X) - VaR_e(X)\} + (X - VaR_c(X))^+$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = (X - VaR_c(X))^+$, $f_{j^*}(X) = \min\{(X - VaR_e(X))^+, VaR_d(X) - VaR_e(X)\}$, and $f_i(X) = 0$ for all $i \neq i^*, j^*$.

We now consider the remaining two cases, as stipulated by the condition $\widetilde{g}'(0) = 1 - \alpha_{i^*} + \frac{\alpha_{i^*}}{\beta_{i^*}} \geq g'(0)$. These two cases are labeled as 5 and 6 in Figure 2.

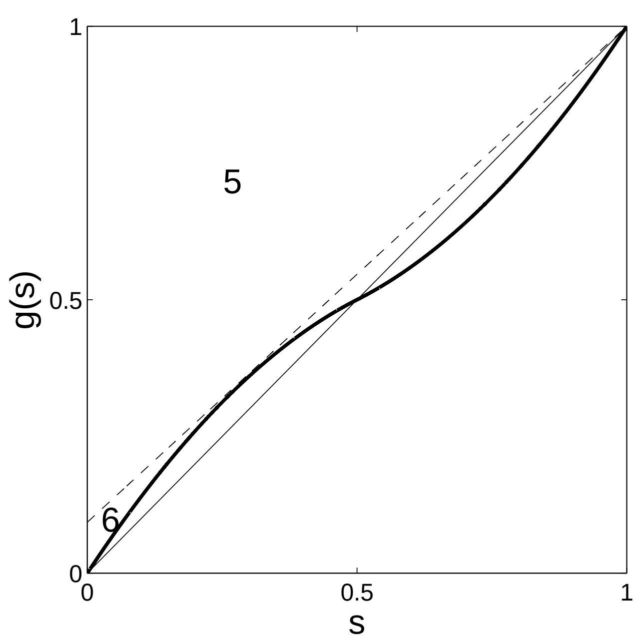

Figure 2: This figure displays the two different cases where $\widetilde{g}'(0) \geq g'(0)$ holds.

Case 5.5 If $p(a) \leq 1-\alpha_{j^*}$, then $\widetilde{g}(s) > g(s)$ for $s \in (0,1)$. The optimal solution to (23) is $f(z) = 0$ for all $z \in [0, M]$, i.e., no reinsurance is optimal.

Case 5.6 If $p(a) > 1-\alpha_{j^*}$, then there exist two points $c \in (0,a)$ and $d \in (a,1)$ such that $\widetilde{g}(s) > g(s)$ for $s \in (0,c)$, $\widetilde{g}(s) < g(s)$ for $s \in (c,d)$, and $\widetilde{g}(s) > g(s)$ for $s \in (d,1)$. The optimal solution to (23) is given by

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_d(X), \\ z - VaR_d(X) & \text{if } VaR_d(X) < z \leq VaR_c(X), \\ VaR_c(X) - VaR_d(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = \min\{(X - VaR_d(X))^+, VaR_c(X) - VaR_d(X)\}$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{j^*}(X) = f(X)$, and $f_i(X) = 0$ for all $i \neq j^*$.

We now provide another example for which the n reinsurers adopt the following premium principles:

$$\pi_i(X) = E[X] + \alpha_i CVaR_{\beta_i}(X), \text{ for all } i = 1, \ldots, n, \tag{26}$$

where $\alpha_i \geq 0$ and $\beta_i \in (0,1)$. Note that this is another example of a distortion premium principle with $\theta_i = \alpha_i$ and g_i of the form

$$g_i(s) = \begin{cases} \frac{\beta_i + \alpha_i}{\beta_i(1+\alpha_i)} s & \text{if } 0 \leq s \leq \beta_i, \\ \frac{s+\alpha_i}{1+\alpha_i} & \text{if } \beta_i < s \leq 1. \end{cases}$$

Let $i^* \in \operatorname{argmin}_{1\leq i\leq n} \frac{\alpha_i}{\beta_i}$ and $j^* \in \operatorname{argmin}_{1\leq j \leq n} \alpha_j$. For $\hat{s} = \frac{\alpha_{j^*}\beta_{i^*}}{\alpha_{i^*}} \in (0,1)$, we have $(1+\alpha_{i^*})g_{i^*}(s) \leq (1+\alpha_{j^*})g_{j^*}(s)$ for $s \in [0, \hat{s}]$, and $(1+\alpha_{i^*})g_{i^*}(s) \geq (1+\alpha_{j^*})g_{j^*}(s)$ for $s \in [\hat{s}, 1]$. We derive that the pricing principle of the representative reinsurer is such that $\widetilde{\theta} = \alpha_{j^*}$, and the distortion function $\widetilde{g}$ is given by

$$\widetilde{g}(s) = \begin{cases} \frac{\beta_{i^*} + \alpha_{i^*}}{\beta_{i^*}(1+\alpha_{j^*})} s & \text{if } 0 \leq s \leq \hat{s}, \\ \frac{s+\alpha_{j^*}}{1+\alpha_{j^*}} & \text{if } \hat{s} < s \leq 1. \end{cases}$$

Therefore, if $i^* = j^*$ we have $\widetilde{g} = g_{i^*}$. The remaining analysis of deriving the optimal ceded loss function is similar to the previous example and hence is omitted for brevity.

We conclude this section by stating an interesting result that is consequence of the representations of $\widetilde{g}$ that we have considered in this section. This finding is somewhat surprising in that it asserts that it is never optimal for the insurer to cede its risk to more than two reinsurers, even in a well functioning reinsurance market with multiple reinsurers. This result also follows directly from Theorem 3.5 and is formally stated in the proposition below.

Proposition 5.2 *If* $\pi_i, i = 1, \ldots, n$, *are all of the form of either* (24) *or* (26)*, then there exist reinsurance contracts* $f_i, i = 1, \ldots, n$, *solving* (17) *such that* $f_i(X) \neq 0$ *for at most two reinsurers.*

6 Conclusion

In this paper, we study the problem of optimal reinsurance in the presence of multiple reinsurers. When all reinsurers use a generalized distortion premium principle, we derive that there exists a representative reinsurer in the market with a specified pricing principle. This pricing principle is also a distortion pricing principle and it is used to determine the optimal aggregate reinsurance contract and its price.

If the insurer minimizes a distortion risk measure of its own risk, the optimal reinsurance contract is such that there exists "tranching" of the insurance risk. The insurance risk will be partitioned in layers, and any layer will be either retained, or reinsured by a particular reinsurer. The optimal ceded loss functions among multiple reinsurers are derived explicitly under the additional assumptions that the insurer's preferences are given by an inverse-S shaped distortion risk measure and that the reinsurer's premium principles are some functions of the CVaR. An interesting result of our analysis is that for our prescribed example, it is never optimal for the insurer to cede its risk to more than two reinsurers.

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