Competitive Insurance Pricing Strategies for Multiple Lines of Business: A Game-Theoretic Approach

Fotios Mourdoukoutas[1,2], Tim J. Boonen[1,*], Athanasios A. Pantelous[2], Greg Taylor[3]

[1] *Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam, Hong Kong, Hong Kong SAR, China*
[2] *Department of Econometrics and Business Statistics, Monash Business School, Monash University, 20 Chancellors walk, Wellington Road, Clayton Campus, Victoria, Australia*
[3] *School of Risk and Actuarial Studies, UNSW Business School, UNSW Sydney NSW 2052, Australia*

Abstract

This paper explores an insurance market with insurers competing in multiple lines of business over a finite time horizon. They set premiums for single lines of business, whereas in a multi-policy contract, a discount is applied to the total sum of the single premiums. To illustrate the combinatorial components of the weighted average market premium, Young tableaux are used. We use the collective risk model to define insurers' aggregate loss, with claim amounts independent of distinct lines of business but dependent on the state of the environment. Our dynamic insurance game falls into the category of multi-stage games with observed actions. In this framework, we characterize the open-loop and closed-loop equilibrium premium profiles.
Keywords: Game theory; Open- and closed-loop equilibrium; Competitive markets; Multiple lines of business.

JEL classification: G22; C61; C72.

1 Introduction

1.1 Motivation and literature review

Previous research on premium strategy and competitiveness has focused on a single *line of business* (LoB). In practice, insurers' portfolios include multiple LoBs, and they typically offer both single-policy contracts and multi-policy packages with package/loyalty discounts. This suggests that different LoBs should not be addressed separately throughout the pricing process. The amount of competition is currently expanding rapidly, as premium rates for one LoB have an impact on premium strategies and exposure volumes for other LoBs. To our understanding, there is still opportunity in the existing literature for more research into how complex competition affects insurance prices for single and multiple LoBs. As a result, our goal is to create a mathematical model and study the interactions between competitors as well as the interdependence of the various LoBs.

Standard actuarial premium principles are merely a risk-assessment exercise (see Kaas et al., 2008), and they do not take into account both the competition that exists within the insurance

[*] Corresponding Author

market as well as the interdependence that exists between various LoBs. This implies that premium values are calculated based on higher order moments of the underlying distribution of the claim amounts and/or using utility theory, ignoring the underwriting strategies (Emms and Haberman, 2005).[1] However, according to empirical evidence, the actual premiums charged by insurers deviate from the actuarial premiums. This arises from the strong competition between the various insurers in the market, who try to optimally respond to competitors' premium choices (Taylor, 1986). Moreover, the constant interaction between the insurers leads to premium cycles, i.e., insurers' premium policies fluctuate below and above the actuarial price during a finite time period (Emms, 2012; Boonen et al., 2018). All these complexities of price competition impose several modelling and pricing challenges which further motivate us.

1.1.1 Multiple lines of business and environment's effect on claims

The concept of multiple LoBs has been extensively studied within the context of risk theory. In that framework, models of the number of claims, the claims amounts, and the aggregate loss are built in order to evaluate risk measures such as ruin probability, ruin time, and the joint distribution of the surplus before and after ruin. Furthermore, a constant premium rate is assumed. Cossette and Marceau (2000) assume that the number of claims arising from distinct LoBs is dependent, and that this dependence is described by a Poisson common shock model and a negative binomial component model. They then assess the effects of dependence on the ruin probability and the adjustment coefficient. Picard et al. (2003) focus on insurance portfolios that cover several interdependent risks and study the probability of ruin for one or more of those risks. Instead of focusing on the unidimensional risk process representing the total wealth of the company, as in Cossette and Marceau (2000), they develop a multi-risk model with a multivariate risk process to represent the surplus of each risk separately. Furthermore, rather than implementing the risk dependence with respect to claim frequencies, they apply it to claim amounts, and they find that more positively dependent claim amounts yield higher non-ruin probabilities. Loisel (2004, 2005) defines risk measures based on a multivariate risk process that models the evolution of the wealth of different LoBs and evaluates the optimal reserve allocation of the global reserve to the various LoBs in order to minimize those risk measures. The risk measures proposed are the multidimensional ruin probability (i.e., the probability that at least one LoB will be ruined), the sum of the expected cost of ruin for each LoB until a time horizon T, and the aforementioned sum if the company's total wealth is positive. There are two types of dependencies between the aggregated claim amounts. The claims for different LoBs may arise from a common event (for example, the Poisson common shock model) or may arise independently for each LoB but depend on an external factor such as weather conditions.

Ng and Yang (2006), Li and Lu (2008) and Li et al. (2014) all use Markov-modulated risk models to calculate various risk measures for a single LoB. The basic idea behind this formulation is that the frequency of claim arrivals and the distribution of claim amounts are influenced by an external environment process, which is represented by a Markov process.

[1] Such calculations would sometimes be referred to as supply-side as they take no account of the demand side (see Canto et al. (2014)).

1.1.2 Premium competition

In a single-person optimization problem, Taylor (1986) addresses the subjective and competitive nature of insurance underwriting in a deterministic discrete-time framework. He introduces two demand functions, the negative exponential and the constant elasticity, to determine the number of policies sold by the insurer as a result of a premium choice versus the premiums of other competitors. The insurer's optimal premiums are those that maximize the expected discounted profit over a finite time horizon, given the demand function and a projection of the average market premium rates. Taylor (1987) extends his previous model by considering the effect of expenses on optimal underwriting strategy. It has been proven that significant changes in the optimal premiums may occur when non-constant expense rates occur. Emms and Haberman (2005), Emms et al. (2007) and Emms (2007) further develop the literature by employing a stochastic continuous-time framework. Using a demand law and optimal control theory, they calculate the optimal premium strategy that maximizes an insurer's expected terminal wealth.

There is a more recent literature that studies competition and its effect on competitors' premiums. Non-cooperative game theory provides tools for moving away from individual optimization procedures and towards simultaneous optimization solutions for all market participants. Two well-studied non-cooperative game models with applications to non-life insurance markets are the Bertrand oligopoly, in which insurers set premiums, and the Cournot oligopoly, in which insurers choose the volume of business. Emms (2012) studies the Nash equilibrium of an n-player non-cooperative differential game in which insurers seek to maximize their expected utility of terminal wealth under a demand law. Depending on whether the break-even premium is considered uncertain, a deterministic or stochastic differential game is proposed. In a one-period non-cooperative game, Dutang et al. (2013) prove the existence and uniqueness of the Nash equilibrium as well as the existence of the Stackelberg equilibrium. In their game, insurers solve a constrained maximisation problem with a deterministic quadratic objective function and a solvency constraint function. A multinomial logit model is used to describe the probabilistic behaviour of policyholders given insurer premiums, and the aggregate claim amount follows a compound Poisson or negative binomial distribution. Wu and Pantelous (2017) show the existence of Nash equilibria in an n-player potential game with non-linear aggregation to find the optimal premium strategy. The insurance market competition is assessed by aggregating all of the market's paired competitions. By including the solvency ratio in the demand function, Boonen et al. (2018) apply optimal control theory to determine the open-loop Nash equilibrium premium strategies in an n-player differential game. Asmussen et al. (2019) consider the customer's problem with market frictions and develop a stochastic differential game between two insurance companies of varying sizes. Mourdoukoutas et al. (2021) study a one-period stochastic game to determine the optimal premiums of n non-life insurers in a competitive market who are expected to be exponential utility maximizers. The total loss of each insurer is described by the collective risk model, where the number of policies follows either a Poisson or negative binomial distribution. They use two distinct exponential demand functions with opposite curvature to capture insurer competition. In the case of a concave-shaped demand function, the existence and uniqueness of a pure strategy Nash equilibrium are demonstrated. It should be noted that all existing literature, including this paper, is based on a complete-information insurance market. However, an insurance market with incomplete information has recently emerged, in which insurers' returns are determined not only by premium selections but also by risk profiles, see Mourdoukoutas et al. (2024).[2] Table 1 classifies the papers with non-cooperative game-theoretic

[2] Moreover, Boonen et al. (2024) show the existence of Stackelberg and Nash equilibria in a single-period stochastic

orientation into static/dynamic and deterministic/stochastic settings.[3]

Papers	static	dynamic	deterministic	stochastic	multi-LoB
Emms (2012)		√	√	√	
Dutang et al. (2013)	√			√	
Wu and Pantelous (2017)	√		√		
Boonen et al. (2018)		√	√		
Asmussen et al. (2019)		√		√	
Mourdoukoutas et al. (2021)	√			√	
This paper		√		√	√

Table 1: Non-cooperative papers' segmentation regarding time frame (single- or multi-period time horizon), randomness (deterministic or stochastic claims/break-even premium) and number of lines of business underwritten by insurers.

1.2 Our contribution

The present paper formulates a stochastic competitive insurance market over a finite time horizon to determine insurers' optimal premium strategies. Similar to Dutang et al. (2013) and Mourdoukoutas et al. (2021), the randomness arises from the fact that the aggregate loss amount is characterized by the collective risk model, which depends on the premiums in the market. We differ from Dutang et al. (2013) and Mourdoukoutas et al. (2021) in the following ways. First, this paper assumes that insurers compete for more than one LoB over a finite time horizon, and thus LoBs are not treated in isolation, as has been the case in previous literature. Second, it is assumed that insurers offer multi-policy discounts when underwriting multiple LoBs at the same time. Particular attention is given to the definition of the market-average competitors' premium, in which all the components are described by Young diagrams. Third, the complexity is present in the loss model too, where a Poisson model with a common shock is applied, as in Cossette and Marceau (2000), in order to decompose the total number of contracts in a LoB into single- and multi-policy contracts. A dependence structure between the number of policies in successive periods is assumed, in which the relative change of the expected number of policies is from the negative exponential family as proposed by Taylor (1986). Fourth, we assume that the severity of claims are affected by the environmental state at the claim level. Thus, we adopt a discrete-time Markov process with a finite state space to represent the environment's status at every period. Moreover, expenses such as operational and underwriting costs, as well as premium reductions such as commission, are taken into consideration. Finally, we classify our insurance game as a multi-stage game with observed actions. We depart significantly from the current literature on dynamic insurance games by considering a closed-loop information structure for our insurance game. That is, we define premium strategies, which are mappings from histories to premium selections, and allow insurers to observe and condition their current actions on previous game actions. This allows for the characterization of both open- and closed-loop equilibrium premium profiles, where the former is more tractable and can be used to analyse insurers' strategic incentives in the closed-loop information structure.

insurance duopoly, and they define a decision game to determine under which conditions both insurers prefer sequential over simultaneous premium setting in terms of utility.

[3]Game-theoretic approaches appear in reinsurance contracting problems as well. Cao et al. (2023) apply Stackelberg differential games to solve the reinsurance contracting problem in a market of a single insurer and two reinsurers. Additionally to one-reinsurer two-insurer market, Bai et al. (2022) consider an investment component in risk-free and risky assets. They characterize optimal reinsurance-investment strategies using a Stackelberg differential subgame and a non-zero-sum stochastic differential subgame.

In summary, our model could accommodate a number of realistic factors that affect insurers' pricing decision. These factors include the discounts on premiums when insureds purchase more than one policy, the great uncertainty of climate changes from one period to the other, and various expenses such as operational costs, underwriting costs and commissions. Finally, in our multi-period setting, we provide characterizations for equilibrium premiums which are optimal to every part of the entire time horizon that starts from any period of time to the end.

This paper is set out as follows. Section 2 starts with an illustrative presentation of the set of all possible combinations of single- and multi-policy contracts offered by insurers. Next, we define the premium profile of insurers and how they apply discounts to multi-policy contracts. At the end of this section, the market-average competitors' premium is thoroughly analysed via an application of Young diagrams. In Section 3, the number of an insurer's contracts per LoB is decomposed into single- and multi-policy contracts, and the aggregate loss is defined by the collective risk model. The claim amounts are independent between distinct LoBs, but dependent on the state of the environment. Section 4 classifies the insurance game into the class of multi-stage games with observed actions, and open- and closed-loop equilibria are characterized. In Section 5, there is a numerical application of the model constructed in this paper. Section 6 concludes and provides directions for further research. The proofs are delegated to Appendix A. In the online supplementary appendix, we provide a table of notation, and three examples of the pure strategy Nash equilibrium corresponding to Section 5.1.

2 Model formulation

In the insurance industry, the term LoB is used to characterize a set of policies that cover a particular range of risks. In contractual terms, the risks associated with a LoB are understood as "all monetary risks aggregated", although we will group them together and simply refer to them as "risk". Therefore, an insurance policy is related to a single LoB and protects the policyholder against the underlying risk of the LoB. When an individual purchases insurance coverage from an insurer against different risks that belong to distinct LoBs, then a policy contract is underwritten for each LoB, and all these policies are bundled into a single contract (multi-policy contract) for pricing purposes. The strategy of insurers is to set a premium for each policy, and when an individual purchases multiple policies, insurers reward them by offering a discount on the total sum.

In this paper, we are looking for the optimal premium strategies for insurers who compete with each other in more than one LoB. Attention needs to be given to the interdependence among distinct LoBs and the interpretation of competition for these lines. Particularly in this section, we define the set of all the possible policy contracts, single and multiple, that can be offered by insurers. Moreover, we set premiums only for the single LoBs, whereas the premiums of multi-policy contracts are given as a discount on the total sum of their subcomponents. Finally, we analyse and provide the market competitor premium by invoking a combinatorial procedure to describe the competitive counterparts.

2.1 Sets of single- and multi-policy contracts

Let $N = \{1, 2, \ldots, n\}$ and $L = \{1, 2, \ldots, l\}$ be the sets of all insurers and LoBs in the market, respectively. Moreover, $L_i \subseteq L$ represents all LoBs offered by insurer i. We further assume that the population of insurers is fixed, i.e., there is no entrance of new insurers or withdrawal of already existing insurers, during the interval up to a finite time horizon of T periods.

Since more than one LoB is offered by some insurers, an opportunity is created for not only single-policy contracts, but also multi-policy ones. The latter consist of policies for different LoBs that remain distinct (i.e., there is not a single policy covering all LoBs), but they are bundled just for the purpose of pricing. Note that a policyholder seeking insurance in several LoBs might select a different insurer for each LoB, or separate insurers for some single LoBs and also some multi-LoB insurers. Let $\mathcal{M}_i(1)$ be the set of all the single-policy contracts in L_i, and in general we define $\mathcal{M}_i(j)$ as the set of all the multi-policy contracts to exactly j LoBs, $j = 2, \ldots, |L_i|$, where $|L_i|$ is the cardinality of the set L_i. Now, all the possible combinations of contracts underwritten by insurer i are given by the union

$$\mathcal{M}_i = \bigcup_{j=1}^{|L_i|} \mathcal{M}_i(j). \tag{1}$$

An element of $\mathcal{M}_i$, denoted by $m \in \mathcal{M}_i$, is then interpreted as a contract. A convention of the paper is that m is represented by an ordered collection of elements in L_i. Without any loss of generality, if $m \in \mathcal{M}_i(k)$ with $m = (\lambda_1, \ldots, \lambda_k)$ and $\lambda_\sigma \in L_i$, we assume that $\lambda_\sigma < \lambda_{\sigma+1}$, $\sigma = 1, \ldots, k-1$, so that any permutation of m's components does not lead to a new contract. Let us illustrate those definitions using an example.

Example 1 *Let $i \in N$ and $L_i = \{1, 2, 3\}$. Then, the single-policy contracts are given by $\mathcal{M}_i(1) = \{(1), (2), (3)\}$, combinations of two distinct LoBs are in the set $\mathcal{M}_i(2) = \{(1, 2), (1, 3), (2, 3)\}$, and finally $\mathcal{M}_i(3) = \{(1, 2, 3)\}$ is the set of the triple-policy contracts. Figure 1 displays all the contracts of every possible combination of LoBs.*

$\mathcal{M}_i(3)$**:**	$(1, 2, 3)$		
$\mathcal{M}_i(2)$**:**	$(1, 2)$	$(1, 3)$	$(2, 3)$
$\mathcal{M}_i(1)$**:**	(1)	(2)	(3)

Figure 1: *Contracts underwritten by insurer i who offers three LoBs, $L_i = \{1, 2, 3\}$. The bottom row illustrates the set of single-policy contracts, the middle row illustrates the set of double-policy contracts, and the top row illustrates the triple-policy contract.*

2.2 Premiums of single- and multi-policy contracts

It is assumed that every insurer i forms a strategy for the next T periods by setting premiums for all the single LoBs. In the case of a combination of LoBs, insurers usually apply a (bundle-)discount to the total sum of the single premiums.

For a contract $m \in \mathcal{M}_i$, we define $\mathcal{I}(m)$ as the index set of the LoB(s) in contract m. Then, the total premium of contract m charged by insurer i in period t is defined as

$$p_i^{(m)}(t) = [1 - d_i(|\mathcal{I}(m)|)] \sum_{v \in \mathcal{I}(m)} p_i^{[v]}(t), \tag{2}$$

where $p_i^{[v]}(t)$ is the single pre-discount premium associated with the single LoB $v \in L_i$, and $d_i : \{1, \ldots, |L_i|\} \to [0, 1]$ is the discount function applied by insurer i. It is assumed that d_i is an increasing function and normalized such that $d_i(1) = 0$. Particularly, when $|\mathcal{I}(m)| \geq 2$, we deem equal discounts for all single policies $v \in \mathcal{I}(m)$ of one insured, and consequently we regard $p_i^{(m)}(t)$

as the premium visible to the competitors. It is important to observe that when there is no discount applied to single-policy contracts, the total premium of a single-policy contract in a LoB is equal to the pre-discount premium associated with that LoB. In other words, $p_i^{(v)}(t) = p_i^{[v]}(t)$ for $v \in L_i$. Hereafter, a contract that covers only one insurance policy and the corresponding LoB will be referred to interchangeably as a single-policy contract.

Example 2 *Consider the double-policy contract* $(1,3) \in \mathcal{M}_i(2)$, *where* $\mathcal{I}((1,3)) = \{1,3\}$ *and* $|\mathcal{I}((1,3))| = 2$. *Then, the total premium is equal to* $p_i^{(1,3)}(t) = [1 - d_i(2)][p_i^{[1]}(t) + p_i^{[3]}(t)]$.

2.3 Competing premium rates

Competition for any combination of LoBs is an important point that requires our attention. We should not only consider competitors who offer a specific combination of LoBs. For instance, let $(1), (2) \in \mathcal{M}(1)$ be the single-policy contracts of LoB 1 and 2, respectively, whereas $(1,2) \in \mathcal{M}(2)$ denotes the multi-policy contract of LoBs 1 and 2. Also, consider a market with two insurers who offer both LoBs. Is it adequate to define the insurer 1's competing premium rate for $(1,2)$ as only the $p_2^{(1,2)}$? We expect that there is a deeper interrelation between competing premiums for $(1,2)$, that leads us to take into account the combinations $(p_1^{(1)} + p_2^{(2)})$ and $(p_1^{(2)} + p_2^{(1)})$, as well. In the following paragraphs, we attempt to unravel and describe the complexity of the components of the market's average competitor premium.

Consider insurer i who underwrites the (possibly multi-policy) contract $m \in \mathcal{M}_i$. Then, $\bar{p}_{-i}^{(m)}(t)$ denotes the market average of competing premium rates for m at period $t \in \{1, 2, \dots, T\}$. Now, a competing premium rate will be the aggregate of rates for all LoBs included in m, some of which may themselves be multi-LoB. Before providing a formal definition, we first illustrate the market-average competitors' premium with an example.

Example 3 *Let* $N = \{1,2\}$ *and* $L_1 = L_2 = \{1,2,3\}$. *The sets* $\mathcal{M}_i(s)$, $s = 1,2,3$, *are described in Example 1. For the single-policy contracts* $m' \in \mathcal{M}_i(1)$, *we have* $\bar{p}_{-i}^{(m')}(t) = p_j^{(m')}(t)$, $j \neq i$. *Now, let* $(x,z) \in \mathcal{M}_i(2)$ *be a double-policy contract. The competing premium rates to* $p_i^{(x,z)}(t)$ *are the premium rate* $p_j^{(x,z)}(t)$ *and the combinations of rates* $p_i^{(x)}(t) + p_j^{(z)}(t)$ *and* $p_i^{(z)}(t) + p_j^{(x)}(t)$, *for* $j \neq i$. *Thus, for* $j \neq i$, *we obtain*

$$\bar{p}_{-i}^{(x,z)}(t) = \frac{1}{\sum_{k=1}^{3} w_{ik}^{(x,z)}} \left[w_{i1}^{(x,z)} \left(p_i^{(x)}(t) + p_j^{(z)}(t) \right) + w_{i2}^{(x,z)} \left(p_i^{(z)}(t) + p_j^{(x)}(t) \right) + w_{i3}^{(x,z)} p_j^{(x,z)}(t) \right]$$

and assuming unit weights for brevity for the triple-policy contract $(1,2,3) \in \mathcal{M}_i(3)$, *we have*

$$\bar{p}_{-i}^{(1,2,3)}(t) = \frac{1}{7} \left[\sum_{v=1}^{3} \left(p_i^{(v)}(t) + p_j^{(v)}(t) \right) + \sum_{x<z} \left(p_i^{(x,z)}(t) + p_j^{(x,z)}(t) \right) + p_j^{(1,2,3)}(t) \right].$$

Our formal definition of $\bar{p}_{-i}^{(m)}(t)$ depends on identifying all the possible partitions of m over all insurers. Each competing premium rate to $\bar{p}_{-i}^{(m)}(t)$ corresponds to a specific partition of the set $\mathcal{I}(m)$ over the insurers. Now, each partition is represented by a so-called *Young diagram*, which is a figure via boxes. A Young diagram, denoted by $\mathcal{Y}$, consists of exactly $|\mathcal{I}(m)|$ boxes, arranged in $s(\mathcal{Y})$ $(\leq |\mathcal{I}(m)|)$ rows containing $k_1, k_2, \dots, k_{s(\mathcal{Y})}$ boxes respectively, subject to the restrictions

$$k_\rho \geq k_{\rho+1}, \text{ for all } \rho = 1, 2, \dots, s(\mathcal{Y}) - 1,$$

$$\sum_{\rho=1}^{s(\mathcal{Y})} k_\rho = |\mathcal{I}(m)| \,, \tag{3}$$

where the rows are labeled by $\rho \in \{1, 2, \ldots, s(\mathcal{Y})\}$, and k_ρ denotes the number of boxes in the ρth row. Interested readers can find more details on Young diagrams at Fulton (1997).

Our key convention is that in each diagram, each row represents a single insurer, and the boxes in the row represent the LoBs underwritten by that insurer. In the following, a three-step procedure is used to illustrate how LoBs are assigned to boxes and insurers are assigned to rows. Ultimately, we will develop a sophisticated structure of Young diagrams known as Young tableau, which encompasses all the competing premiums included in $\bar{p}_{-i}^{(m)}(t)$.

Step 1. We have to find all the possible partitions of $\mathcal{I}(m)$, i.e., to create all the Young diagrams of order $|\mathcal{I}(m)|$. The Young diagrams can be obtained using a recursive process based on the number of boxes in the first row. Starting with $k_1 = 1$, there is only one feasible Young diagram that consists of $|\mathcal{I}(m)|$ rows, each containing a single box. Extending the sequence to $k_1 = 2$, we arrange below the first row all possible Young subdiagrams of size $|\mathcal{I}(m)| - 2$, where the first row (i.e., k_2 of the main Young diagram) contains either one or two boxes. The Young subdiagrams attached to the main Young diagram are created recursively based on the number of boxes in their first row, which corresponds to the second row of the main diagram. The recursion starts with $k_2 = 1$ and ends with $k_2 = k_1$. The recursion terminates when k_1 reaches the cardinality of $\mathcal{I}(m)$, resulting in the formation of a single-row Young diagram with $|\mathcal{I}(m)|$ boxes.

Step 2. Having constructed all the possible Young diagrams of order $|\mathcal{I}(m)|$, we label the boxes in each row of each diagram such that they represent the LoBs. Consider a diagram $\mathcal{Y}$ and let $h_{\rho\sigma}(\mathcal{Y})$ denote the number in the σth box (counting from the left) of $\mathcal{Y}$'s ρth row, and let $H_\rho(\mathcal{Y}) = \{h_{\rho\sigma}(\mathcal{Y}) : \sigma = 1, \ldots, k_\rho\}$ for $\rho = 1, \ldots, s(\mathcal{Y})$. The requirements in the present case are that $h_{\rho\sigma}(\mathcal{Y}) \in \mathcal{I}(m)$, $h_{\rho\sigma}(\mathcal{Y}) < h_{\rho,\sigma+1}(\mathcal{Y})$, $H_{\rho_1}(\mathcal{Y}) \cap H_{\rho_2}(\mathcal{Y}) = \emptyset$ for $\rho_1 \neq \rho_2$, and $\cup_{\rho=1}^{s(\mathcal{Y})} H_\rho(\mathcal{Y}) = \mathcal{I}(m)$. The second requirement, that the boxes of any row are numbered in increasing order from left to right, recognizes that permutation of the numbers in a row does not yield any new case. Considering all the potential combinations of numbering boxes for each diagram, we observe that the collection $\mathcal{Y}$ of order $|\mathcal{I}(m)|$ obtained in Step 1 is enlarged to include some duplicated Young diagrams with different numbering in their boxes. This collection will undergo additional modifications once all potential insurers have been considered in the subsequent step.

Step 3. The last piece of our construction is the different selections of insurers. Since a row represents an insurer, the rows are labeled with some of the insurers. In Step 2, we select only the Young diagrams $\mathcal{Y}$ from our collection that satisfy the condition $1 \leq s(\mathcal{Y}) \leq n$. A scenario where $s(\mathcal{Y}) > n$ cannot be insured in the current market due to insufficient insurers to provide coverage for this case. Let $i_\rho(\mathcal{Y})$ be the insurer index corresponding to the ρth row of the Young diagram $\mathcal{Y}$. The insurers will assess the premium options for all selections $i_1(\mathcal{Y}), i_2(\mathcal{Y}), \ldots, i_{s(\mathcal{Y})}(\mathcal{Y})$ where $i_\rho(\mathcal{Y}) \in N$, $i_{\rho_1}(\mathcal{Y}) \neq i_{\rho_2}(\mathcal{Y})$ for $\rho_1 \neq \rho_2$, and $i_\rho(\mathcal{Y}) < i_{\rho+1}(\mathcal{Y})$ when $k_{\rho+1} = k_\rho$. In this scenario, the last condition prevents rearrangements of insurers that are linked to rows of the same length from resulting in new cases.

The resulting Young tableau, denoted by $\mathfrak{D}$, is the set of all Young diagrams $\mathcal{Y}$ of order $|\mathcal{I}(m)|$ with all admissible assignments of LoBs to their boxes and all admissible selections of insurers for labeling their rows. Figure 2 illustrates the three-step procedure for generating the Young tableau that provides the competing premium rates for a triple-policy contract in a market of two insurers. Now, we are ready to proceed with the formal definition of the market-average competitors' premium.

Definition 1 *Let $m \in \mathcal{M}_i$ and $t \in \{1, \ldots, T\}$. Insurer i's market-average competitors' premium for m in period t is denoted by $\bar{p}_{-i}^{(m)}(t)$ and defined as*

$$\bar{p}_{-i}^{(m)}(t) = \frac{\sum_{\mathcal{Y} \in \mathfrak{D} \setminus \mathcal{Y}_i^{(m)}} \left(w_{\mathcal{Y}}(t) \sum_{\rho=1}^{s(\mathcal{Y})} p_{i_\rho(\mathcal{Y})}^{(m_\rho(\mathcal{Y}))}(t) \right)}{\sum_{\mathcal{Y} \in \mathfrak{D} \setminus \mathcal{Y}_i^{(m)}} w_{\mathcal{Y}}(t)}, \tag{4}$$

where the terms in the expression have the following definitions:

$\mathfrak{D}$	*Young tableau of all Young diagrams $\mathcal{Y}$ of order $\lvert\mathcal{I}(m)\rvert$ with all admissible assignments of LoBs to their boxes and all admissible selections of insurers for labeling their rows*
$\mathcal{Y}_i^{(m)}$	*the single-row Young diagram of $\lvert\mathcal{I}(m)\rvert$ boxes and the assignment of insurer i to its row*
$i_\rho(\mathcal{Y})$	*insurer index associated with the ρth row of $\mathcal{Y}$*
$m_\rho(\mathcal{Y})$	*the single- or multi-policy contract represented by the boxes in the ρth row of $\mathcal{Y}$*
$p_{i_\rho(\mathcal{Y})}^{(m_\rho(\mathcal{Y}))}(t)$	*the total premium of insurer $i_\rho(\mathcal{Y})$ for the contract $m_\rho(\mathcal{Y})$*
$w_{\mathcal{Y}}(t)$	*weight associated with the combined premium rate.*

The weights are positive, and may be chosen based on insurers' previous experience with client movement within the insurance market. They may reflect the importance assigned to the respective combined premium rates, e.g., the weights may be equal to last year's written premium for $\mathcal{Y}$.

3 Stochastic loss model

The goal of this section is to define the total cost that insurers face. This cost is stochastic because both the number of insureds per LoB and the severity of the claims are random. To begin with, we divide the total number of policyholders per LoB into those who have single-policy contracts and those who have multi-policy ones. Furthermore, given the environmental conditions, we assume that the claim amounts per policy are conditionally independent. Other fixed costs associated with the underwriting procedure are considered.

3.1 Number of policies

The quantity directly linked to premium strategies and affected by the competition is the exposure volume. In a deterministic setting, Taylor (1986, 1987) suggests that the exposure volume of insurer i in period t is proportional to the exposure volume of previous period $t-1$ and depends not only on the premium selection of insurer i, but also on the premiums charged by competitors. In this section, the number of policies sold by insurers is considered stochastic whose expectation is referred to as exposure volume.

Recall that L_i is the set of all LoBs offered by insurer i. For every individual LoB $v \in L_i$, let us denote

$N_i^v(t)$	the total number of policyholders who have purchased policies for LoB v from insurer i within period t.

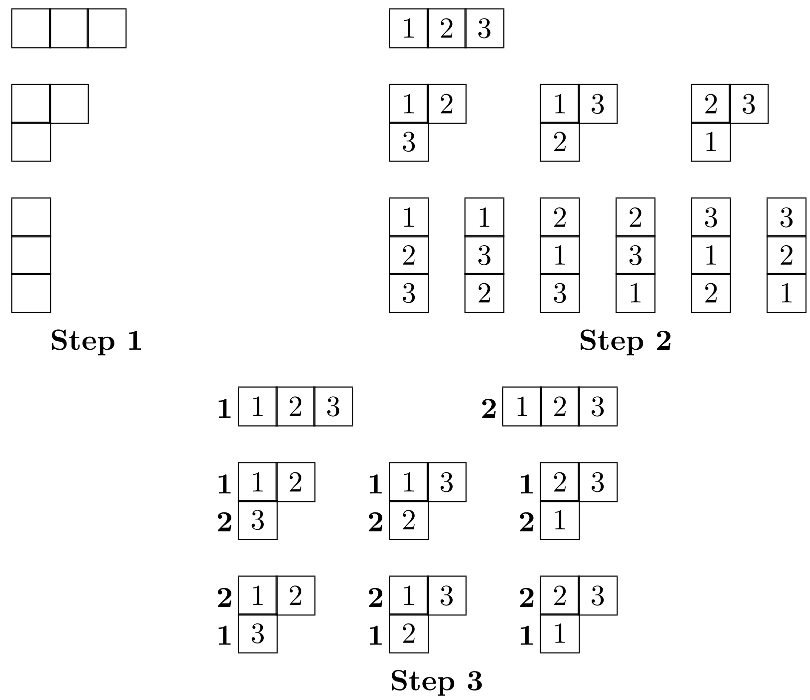


Figure 2: *Consider the triple-policy contract* $(1,2,3)$ *and an insurance market with two insurers, i.e.,* $N = \{1,2\}$*, who both offer three LoBs, i.e.,* $L_1 = L_2 = \{1,2,3\}$*. Then,* ***Step 1*** *depicts the Young diagrams of order* 3 $(= |\mathcal{I}((1,2,3))|)$ *and* ***Step 2*** *illustrates the collection of Young diagrams of order* 3 *with all possible numbering for their boxes. Finally,* ***Step 3*** *provides the Young tableau* $\mathfrak{D}$ *with all the competing premium rates for* $(1,2,3)$ *(notice that the triple-row, single-box Young diagrams in Step 2 are not possible to be assigned to insurers in the two-insurer market). The top-left and top-right Young diagrams in Step 3 are denoted by* $\mathcal{Y}_1^{(1,2,3)}$ *and* $\mathcal{Y}_2^{(1,2,3)}$*, respectively.*

For a given period t, the random variables $N_i^v(t)$, $v \in L_i$, are considered dependent. This interdependence arises from the fact that some insureds might have purchased more than one LoB from insurer i. Therefore, every random variable $N_i^v(t)$ consists of insureds who possess contracts either insuring only LoB v (single-policy contracts), or LoB v with a LoB other than v (multi-policy contracts). This leads us to decompose the random variables $N_i^v(t)$, $v \in L_i$, into a sum of random variables which distinguish the single-policy holders from the multi-policy ones who have purchased LoB v from insurer i in period t.

For all $v \in L_i$, we define the set $\mathcal{D}_i(v) = \{m \in \mathcal{M}_i : v \in \mathcal{I}(m)\}$, which consists of contracts, single- or multi-policy, that contain LoB v. If

$N_i^{(m)}(t)$ is the number of policyholders of insurer i with contract m in period t,

then the total number of policyholders of LoB $v \in L_i$ is written as

$$N_i^v(t) = \sum_{m \in \mathcal{D}_i(v)} N_i^{(m)}(t),$$

which aggregates the policyholders of insurer i that hold the single-LoB v and the policyholders that hold multi-policy contracts that include LoB v.

Assumption 1 *A policyholder with a multi-policy contract* $m \in \mathcal{M}_i$, $|\mathcal{I}(m)| \geq 2$, *from insurer* i *does not affect* $N_i^{(m')}(t)$, *where* $m' \in \mathcal{M}_i$ *and* $\mathcal{I}(m')$ *is a strict subset of* $\mathcal{I}(m)$. *Those policyholders are accounted for only by the random variable* $N_i^{(m)}(t)$. *Therefore, given a period* t, $\{N_i^{(m)}(t)\}_{m \in \mathcal{M}_i}$ *are independent random variables.*

Example 4 *Let us illustrate the decomposition of* $N_i^v(t)$ *when* $v \in L_i = \{1, 2, 3\}$. *In this special case we obtain*

$$\begin{aligned} N_i^1(t) &= N_i^{(1)}(t) + N_i^{(1,2)}(t) + N_i^{(1,3)}(t) + N_i^{(1,2,3)}(t), \\ N_i^2(t) &= N_i^{(2)}(t) + N_i^{(1,2)}(t) + N_i^{(2,3)}(t) + N_i^{(1,2,3)}(t), \\ N_i^3(t) &= N_i^{(3)}(t) + N_i^{(1,3)}(t) + N_i^{(2,3)}(t) + N_i^{(1,2,3)}(t). \end{aligned}$$

The above decomposition has been applied to Poisson models with common shocks and can be found in Cossette and Marceau (2000). Their work is within the context of discrete-time ruin theory, in which N represents the number of claims instead of policies, and a constant premium income rate is assumed throughout the entire time horizon.

At this point, let us clarify the following issues regarding the notation. First, the random variables $N_i^v(t)$ and $N_i^{(v)}(t)$, $v \in L_i$, are not the same. The former refers to the total number of insured individuals who hold a policy in LoB v. This policy can be either a single-policy contract or combined with policies from other LoBs to form a multi-policy contract. On the other hand, the random variable $N_i^{(v)}(t)$ denotes the number of insureds who possess strictly single-policy contracts in LoB v. Second, as mentioned at the beginning of this section, the number of single- or multi-policy contracts $m \in \mathcal{M}_i$ depends on premium strategies of insurers. Therefore, a more precise notation is $N_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t))$, which states clearly that the number of policyholders of insurer i with contract m in period t is affected by the competition between insurer i's total premium $p_i^{(m)}(t)$ and insurer i's market-average competitors' premium $\bar{p}_{-i}^{(m)}(t)$. For brevity, we suppress the premium dependence and keep only the time reference in parentheses, i.e., $N_i^{(m)}(t)$.

Now, we will discuss the distribution of the number of policyholders per contract and per period. It is assumed that the number of policies sold by insurers in a period, conditional on the number of policyholders in the previous period, follows a Poisson distribution with intensity equal to the number of policyholders in the previous period, adjusted by a price elasticity factor. Given a contract $m \in \mathcal{M}_i$, a hierarchical structure is implied for the random variables $N_i^{(m)}(t)$, $t \in \{1, 2, \ldots, T\}$, due to the assumption that the expected number of insureds in a period is proportional to the actual number in the previous period. Specifically, we assume that for all $m \in \mathcal{M}_i$ it holds that

$$N_i^{(m)}(t)|N_i^{(m)}(t-1) \sim \text{Poisson}\left(f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t))N_i^{(m)}(t-1)\right), \tag{5}$$

where $N_i^{(m)}(0) = q_i^{(m)}(0)$ is the initial exposure volume of insurer i for the line(s) of business indicated by m. Additionally, let $f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t))$ in the intensity parameter of the Poisson distribution denote the relative change in the expected number of single- or multi-policy contracts $m \in \mathcal{M}_i$ sold by insurer i within period t. It is assumed that $f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t))$ is a positive real-valued function and twice continuously differentiable with respect to $p_i^{(v)}(t)$, for $v \in \mathcal{I}(m)$.

From now on, and for the rest of this paper, we simplify our notation by writing $f_i^{(m)}(t)$ instead of $f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t))$, retaining only the time reference as in the case of $N_i^{(m)}(t)$.

3.2 Aggregate loss

The collective risk model is used in this paper to show how much it costs insurers as a whole. Not only are the claim sizes of each distinct LoB considered, but so are other expenses related to insurers' operational and underwriting costs.

Furthermore, attempting to capture the impact of climate instability on policy pricing,[4] we consider a discrete-time Markov process that represents the state of the environment at each period in our finite horizon. The amount of the claim is determined by the state of the environment. In our analysis, we consider a finite state space for the environment process, and the claim amounts will range from relatively low to extremely high depending on the state. In this way, we aim to model various probabilistic scenarios for the climate change and quantify their impact on premium rates. Thus, let $\{J(t)\}_{t=0}^{T}$ be a discrete-time Markov process, with a finite state space $\{j_1, j_2, \ldots, j_s\}$. This process represents the state of the environment in every period. Then, we define $Y_{ik}^{v}|J(t) = j_z$ as the individual claim amount under LoB v for policy k of insurer i in period t when the environment is in state j_z.

Now, consider the single- or multi-policy contracts $m \in \mathcal{M}_i$ from insurer i. Given the state of the environment, the aggregate loss for insurer i in period t, regarding contracts m, is given by

$$S_i^{(m)}\left(N_i^{(m)}(t)\right) = \left(1 + r_i^{(m)}(t)\right)\left(\sum_{v \in \mathcal{I}(m)} \sum_{k=1}^{N_i^{(m)}(t)} Y_{ik}^{v} + c_i^{(m)}(t) N_i^{(m)}(t)\right), \tag{6}$$

where $r_i^{(m)}(t) \in (0,1)$ is the cost per unit outgoing revenue, and $c_i^{(m)}(t) > 0$ is the cost per contract m.

In the present paper the following assumptions are made.

Assumption 2 *(i) Insureds are considered independent and hence, the claims which arise from policy k_1 are independent of the claims which arise from policy k_2, $k_1 \neq k_2$.*

(ii) Using the past experience, Insurer i assumes that, within a specific state of the environment, the claim size associated with LoB v, $v \in L_i$, is drawn from a distribution common for all policies k.

Based on these assumptions, given the state of the environment, we derive that $\{Y_{ik}^{v}\}_k$ are independent and identically distributed random variables. Let Y_i^v be a random variable which follows the same distribution as them. Finally, it is worth mentioning that a multi-LoB insurance consists simply of a number of single-LoB policies that are distinct in all respects except premium rate and hence, given the state of the environment, $\{Y_i^v\}_{v \in L_i}$ are independent random variables.

[4] Our motivation is a technical report commissioned by the Actuaries Institute, available at `https://www.actuaries.asn.au/public-policy-and-media/our-thought-leadership/reports/home-insurance-affordability-and-socioeconomic-equity-in-a-changing-climate`, which addresses the effects of climate change on the insurance industry. The so-called Green Paper raises the concerns around insurance affordability and availability, and examines the impact of the changing climate on home insurance premium particularly. They find that climate change is expected to materially increase the risk of extreme weather events, and their various weather scenarios reveal that vulnerable population (i.e., households that are already struggling to pay home insurance premiums) will suffer most from the impacts of climate change on home insurance premiums.

4 Insurance game

This section is devoted to present the game theoretical approach followed to analyze the strategic incentives of the insurers that lead to an equilibrium state.

A game theoretical analysis begins with setting down the information structure of the game. In this paper, we assume that insurers select their pre-discount single-policy premium values at the start of each period simultaneously. That is, when they are determining their premium values for each period, they are unaware of their competitors' premium choices for that period. On the other hand, it is assumed that all insurers can track the past premium choices of all insurance market participants. Based on the aforementioned assumptions, our insurance game belongs to the category of multi-stage games with observed actions. Hereafter, we refer to our insurance game as G.

In what follows, we provide first the premium-choice sets and premium histories in order to conclude with the insurers' payoff function. Then, we define the premium strategies in game G. Finally, we demonstrate two possible, distinct, equilibrium notions for game G.

4.1 Premium histories and payoff function

The players of the insurance game G are the insurers denoted by the set N. Here, the payoff function will be the insurers' expected, discounted, net wealth which, in turn, will be defined on the set of all insurers' premium rates over the entire time horizon.

Let $p_i(t) = (p_i^{(m')}(t))_{m' \in \mathcal{M}_i(1)}$ be a period-t premiums choice for insurer i, which is actually a vector of premium rates of all the single-policy contracts of insurer i in period t. We assume that $P_i^{(m')}(t) = [\underline{\pi}_i^{(m')}(t), \overline{\pi}_i^{(m')}(t)]$ is the set of available premium rates of the single-policy contracts $m' \in \mathcal{M}_i(1)$ offered by insurer i in period t, and these sets are only time dependent. The Cartesian product $P_i(t) = \prod_{m' \in \mathcal{M}_i(1)} P_i^{(m')}(t)$ consists of possible period-t premiums choices for insurer i. The positive lower and upper premium bounds, $\underline{\pi}_i^{(m')}(t)$ and $\overline{\pi}_i^{(m')}(t)$ respectively, might be viewed as fixed limits imposed by government regulators, or even by the insurers themselves.[5] Now, let the premium vector $p(t) = (p_i(t))_{i \in N}$ be a period-t premium profile of all insurers in the market, which belongs to the set $P(t) = \prod_{i \in N} P_i(t)$.

At the start of each period, insurers know the rates of all premiums in previous periods. Let the premium vector $h(t) = (p(k))_{k=1}^{t-1}$, for $t \in \{2, \dots, T+1\}$, represent this knowledge, which is available to all insurers at the beginning of period t and called the history of the insurance game at period t. Let $H(t) = \prod_{k=1}^{t-1} P(k)$, for $t \in \{2, \dots, T+1\}$, denote the set of all period-t histories, and $H(1) = \{\emptyset\}$. Moreover, we observe that $h(T+1)$ is a complete premium profile of the insurance game, which describes all the premium choices for all insurers over the entire time horizon T. Let us denote any such terminal history as $p := h(T+1) = (p(t))_{t=1}^{T}$, which belongs to $P = \prod_{t=1}^{T} P(t)$. It is common in game theory to decompose the premium profile into $p = (p_i, p_{-i})$, where $p_i = (p_i(t))_{t=1}^{T}$ and $p_{-i} = ((p_j(t))_{t=1}^{T})_{j \in N \setminus \{i\}}$ are the premium profiles of insurer i and the insurer i's competitors, respectively. Let $P_i = \prod_{t=1}^{T} P_i(t)$ be the set of premium profiles for insurer i, while the counterpart set of insurer i's competitors is $P_{-i} = \prod_{j \in N \setminus \{i\}} \prod_{t=1}^{T} P_j(t)$. Next, we define the wealth of insurers and relate it to their payoff function.

Let $g_i^{(m)}(p_i^{(m)}(t))$, $m \in \mathcal{M}_i$, denote the reduced premium associated with the total premium

[5] In this paper, we consider only the net income of the insurers, excluding reinsurance costs and investment income. However, the boundaries on the premium values set by an insurer partially reflect how reliant it is on reinsurance and investment income.

$p_i^{(m)}(t)$ of insurer i, for the line(s) of business indicated by m, in period t. This premium reduction may be viewed as expenses related to premium income, such as commission. It is assumed to be an increasing function of $p_i^{(v)}(t)$, $v \in \mathcal{I}(m)$, and defined by

$$g_i^{(m)}(p_i^{(m)}(t)) = \delta_i^{(m)} p_i^{(m)}(t), \tag{7}$$

for some $\delta_i^{(m)} \in (0, 1)$. Letting $W_{i,0}$ be the initial wealth of insurer i and $u \in [0, 1]$ the time-discount factor, the net present value of future wealth of insurer i over the entire time horizon is obtained by

$$W_i\left(p_i, p_{-i}\right) \quad = \quad W_{i,0} + \sum_{t=1}^{T} u^t \sum_{m \in \mathcal{M}_i} \left[g_i^{(m)}(t) N_i^{(m)}(t) - S_i^{(m)}\left(N_i^{(m)}(t)\right)\right]. \tag{8}$$

The notation $g_i^{(m)}(t)$ is used instead of $g_i^{(m)}(p_i^{(m)}(t))$, retaining once more the time reference for brevity.

The insurer i's payoff function is defined on the set of possible premium profiles and denoted by $O_i : P \to \mathbb{R}$. It is equal to the insurer i's expected net present value of future wealth $\mathbb{E}\left[W_i\left(p_i, p_{-i}\right)\right]$, defined in (8). The following proposition demonstrates how the insurers' payoff function is derived from the wealth function.

Proposition 1 *The insurer i's payoff function can be written as:*

$$O_i(p_i, p_{-i}) = W_{i,0} + \sum_{t=1}^{T} u^t \sum_{m \in \mathcal{M}_i} \prod_{k=1}^{t} f_i^{(m)}(k) q_i^{(m)}(0) \left(\delta_i^{(m)} p_i^{(m)}(t) - \mu_i^{(m)}(t)\right), \tag{9}$$

where

$$\mu_i^{(m)}(t) = \left(1 + r_i^{(m)}(t)\right) \left[\sum_{k=1}^{s} \Pr[J(t) = j_k] \sum_{v \in \mathcal{I}(m)} \mathbb{E}\left[Y_i^v | J(t) = j_k\right] + c_i^{(m)}(t)\right]$$

is the total expected loss per contract m.

Proof. See Appendix A.1. ■

4.2 Equilibrium concepts

The information structure of game G enables insurers, in every period $t \in \{2, \dots, T\}$, to condition the period-t premiums choice on the history $h(t)$ of the game (we exclude the first period since they could not make such a condition due to $H(1) = \{\emptyset\}$). Thus, an insurer i's period-t premium choice for a single-policy $m' \in \mathcal{M}_i(1)$, i.e., $p_i^{(m')}(t)$, is actually the result of a mapping from the set $H(t)$ to the set $P_i^{(m')}(t)$. With a slight abuse of notation, it will be implied that $p_i^{(m')}(t)$ depends not only on the time but also on the history $h(t)$, i.e., $p_i^{(m')}(t) \equiv p_i^{(m')}(t; h(t))$, and hence insurer i's premium strategy $p_i = (p_i(t))_{t=1}^{T}$ is actually a sequence of mappings from histories to premiums choices. In game theory, this information structure and the induced strategies are termed as *closed-loop.*

In the following two sections, we present two solution concepts to our insurance game G. In Section 4.2.1, we restrict our analysis to a particular class of premium strategies referred to as open-loop. These strategies lead to a more mathematically tractable equilibrium which can be used to explain the insurers' strategic incentives in a closed-loop equilibrium presented in Section 4.2.2.

4.2.1 Open-loop equilibrium

In this section, we specifically focus on a premium strategy profile p on game G where, for all t, all $i \in N$, and all $m' \in \mathcal{M}_i(1)$, we have $p_i^{(m')}(t; h(t)) = p_i^{(m')}(t)$ for every period-t history $h(t) \in H(t)$. This means that the insurers' premium choices for period t are constant and do not depend on the game history. This strategy can be interpreted as each insurer i choosing its entire premium profile p_i at the start of the game, without considering the opponents' strategies. Under this specific strategic interaction, the following theorem provides the *open-loop*[6] pure-strategy Nash equilibrium of G.

Theorem 1 *Let $G = \langle N, (P_i)_{i\in N}, (O_i)_{i\in N} \rangle$. A premium strategy profile $p^* = (p_i^*, p_{-i}^*)$ is an open-loop pure-strategy Nash equilibrium (PSNE) of G if, for all $i \in N$, p_i^* is a solution to maximization problem*

$$\max_{p_i \in P_i} \left\{ O_i(p_i, p_{-i}^*) \right\}, \tag{10}$$

and it exists if $O_i(p_i, p_{-i})$ is concave in p_i for each fixed feasible value of p_{-i}.

Proof. See Appendix A.2. ■

Our interest lies in the interior points of P_i that maximize insurer i's payoff function and are characterized by the first- and second-order conditions. Specifically, since $O_i(p_i, p_{-i})$ is twice continuously differentiable for all $p_i \in P_i$, an interior optimal point p_i^* is one that makes the gradient of $O_i(p_i, p_{-i}^*)$ vanish, i.e., it satisfies the first-order condition (FOC) $\nabla_{p_i} O_i(p_i^*, p_{-i}^*) = 0$. Moreover, it is a local optimum if the Hessian matrix, $\nabla_{p_i}^2 O_i(p_i^*, p_{-i}^*)$, is negative definite.

4.2.2 Closed-loop equilibrium

In this section, we revisit closed-loop premium strategies, which are mappings from histories to premium choices. The finite nature of the time horizon enables us to evaluate the closed-loop subgame perfect (Nash) equilibrium premium profile in the insurance market by using the backwards induction principle.

Consider a period-t history $h(t) = (p(k))_{k=1}^{t-1}$ of G. All insurers know $h(t)$ at the beginning of that period. Consequently, the game that occurs from period t to the end of the time horizon T is a separate game, denoted by $G(h(t))$. This game is a subgame of G. Its strategies and payoffs are inherited from the original game G and are simply the restrictions of the original strategies and payoffs within the subgame $G(h(t))$. It is important to note that the payoffs in subgame $G(h(t))$ are still defined on the entire premium profile P, but only on its subset that is consistent with the history $h(t)$. Given $h(t)$ and p, let $p|h(t)$ represent the restriction of p to $G(h(t))$. This restriction is a complete premium profile that aligns with $h(t)$ in the first $t-1$ periods. The following definition is adapted from Fudenberg and Tirole (1991).

Definition 2 *A premium strategy profile p in the entire insurance game G is a subgame perfect equilibrium (SPE) if, for each period $t \in \{1, \ldots, T\}$ and every period-t history $h(t)$, the restriction $p|h(t)$ to subgame $G(h(t))$ is a Nash equilibrium of $G(h(t))$, i.e., for all $i \in N$ and for all $\tilde{p}_i$ it holds $O_i(p_i, p_{-i}|h(t)) \geq O_i(\tilde{p}_i, p_{-i}|h(t))$.*

[6]In game theory, the term *open loop* characterizes an information structure in which players do not observe the competitors' actions in past periods. Consequently, open-loop strategies cannot depend on action histories of the game, and players choose their strategies at the start of the game, permitting no revision of the strategies as the game proceeds.

An SPE premium strategy profile ensures that each insurer's premium strategy is optimal not only at the start of the insurance game, but also after each period and for any premium history. The next theorem provides a characterization for the closed-loop SPE premiums and illustrates the insurers' strategic incentives, i.e., how an insurer's current premium choices can affect the opponents' future premium strategies.

Theorem 2 *Let $G = \langle N, (P_i)_{i \in N}, (O_i)_{i \in N} \rangle$. Assume that $O_i(p_i, p_{-i})$ is concave in $p_i(T)$ for each fixed value of period-T history $h(T)$ and $p_{-i}(T)$. Moreover, we assume that for all t, $O_i(h(t), p_i(t), p_{-i}(t), p^*(t+1), \ldots, p^*(T))$ is concave in $p_i(t)$ for each fixed value of $h(t)$ and $p_{-i}(t)$, whereas $p^*(k) \equiv p^*(k; h(k))$, for $k \in \{t+1, \ldots, T\}$, are the Nash equilibria of the subgames $G(h(k))$. Then, a closed-loop subgame perfect equilibrium exists and is given by solving recursively for $t = T, T-1, \ldots, 1$ the following system of equations: for all $i \in N$ and all $m' \in \mathcal{M}_i(1)$*

$$\frac{\partial O_i}{\partial p_i^{(m')}(t)} + \sum_{j \neq i} \sum_{k=t+1}^{T} \sum_{\tilde{m} \in \mathcal{M}_j(1)} \frac{\partial O_i}{\partial p_j^{(\tilde{m})}(k)} \frac{\partial p_j^{(\tilde{m})*}(k)}{\partial p_i^{(m')}(t)} = 0. \tag{11}$$

Proof. See Appendix A.3. ■

Let $p^{*N} = \left(p_i^{*N}, p_{-i}^{*N}\right)$ represent the corresponding open-loop PSNE. In contrast to the corresponding open-loop FOC, $\nabla_{p_i} O_i(p_i, p_{-i}^{*N}) = 0$, Eq. (11) includes additional terms $\partial p_j^{(\tilde{m})*}(k)/\partial p_i^{(m')}(t)$ that correspond to insurer i's strategic incentive to modify $p_i^{(m')}(t)$ in order to influence competitor j's future strategy $p_j^{(\tilde{m})}(k)$. For instance, if insurer i prefers an increase in $p_j^{(\tilde{m})}(k)$, and $\partial p_j^{(\tilde{m})*}(k)/\partial p_i^{(m')}(t)$ is positive at $p^{*N}(t)$, then insurer i's strategic incentive is to increase $p_i^{(m')}(t)$ beyond $p_i^{(m')*N}(t)$.

5 Numerical Applications

In this section, we consider a competitive insurance market consisting of two LoBs and two insurers offering both LoBs, i.e., $N = \{1, 2\}$ and $L_1 = L_2 = L = \{1, 2\}$. Let Insurer 1's initial wealth be \$1,000,000 and Insurer 2's \$1,500,000. The current exposure volumes of the insurers are given in Table 2, and we assume that Insurer 2 has greater market power in total than Insurer 1. In particular, their market power per LoB is parametrically reflected in the price-sensitivity parameters, as we will discuss in the following subsections.

As far as the environment process $\{J(t)\}_{t=1}^{T}$ is concerned, we assume three possible states, $\{j_1, j_2, j_3\}$, where j_1 results in low severity claim amounts per individual, j_2 results in moderate claim amounts, and j_3 results in very high claims. The one-step transition matrix of the process is denoted by Π and its initial distribution by the row vector A, i.e., the zth element of A is $A_z = \Pr[J(0) = j_z]$. Then, it holds that $\Pr[J(t) = j_k] = (A\Pi^t)_k$, which is the probability of the environment process being in state j_k at period t. The initial distribution is set equal to $A = [0.15, 0.80, 0.05]$ and the transition matrix is given by

$$\Pi = \begin{bmatrix} 0.30 & 0.65 & 0.05 \\ 0.15 & 0.70 & 0.15 \\ 0.05 & 0.80 & 0.15 \end{bmatrix}.$$

This selection of probabilities implies that the environment's state is more likely to remain in the moderate state j_2 for the upcoming periods. Given the state of the environment in period t, the conditional expected claims per individual in LoB v for the insurer i, i.e., $\mathbb{E}\left[Y_i^v | J(t) = j_k\right]$, are given in Table 2. It is assumed for the feasible region that the lower and upper premium bounds satisfy $[\underline{\pi}_i^{(v)}(t), \overline{\pi}_i^{(v)}(t)] = [0.7\mathbb{E}[Y_i^v], 3\mathbb{E}[Y_i^v]]$ for $i \in N$, $v \in L$ and $t \in \{1, \ldots, T\}$.

The Figure 3 displays the Young tableaux that represent the market-average competitors' premium for both single-policy and double-policy contracts. In our case of a market with two insurers, we observe that the average premium charged by a competitor for a single-policy contract is equal to the premium rate charged by that competitor for the same contract. This can be expressed as $\bar{p}_{-i}^{(m)}(t) = p_j^{(m)}(t)$ for all time periods t and contract types m belonging to the set $\{(1), (2)\}$. Here, i and j represent the insurers in the market, and they are not the same. Regarding the selection of weights in the market-average competitors' premium $\bar{p}_{-i}^{(1,2)}(t)$, Table 3 provides the competing premium components of $\bar{p}_{-i}^{(1,2)}(t)$ with the respective weights. Our selection of weights represents the belief that the weaker insurer puts more weight on the cross selection of double-policy contracts, fearing that policyholders might find it preferable to keep only one LoB and purchase the other one from the stronger insurer.

The time-discount factor is given as $u = \frac{1}{1.07} = 0.9346$, while the remaining model parameters correspond to the values provided in Tables 2 and 3. At this point, it is important to note that the weights given to competing premium rates and the discount functions used for multi-policy contracts are determined arbitrarily. The former can be contingent on the history of the game in each period, but the computational cost is increasing dramatically. The discount on multi-policy contracts can be treated as a decision variable in a future research endeavour, but at the time being we avoid further complexities in our model.

In the following two subsections, we find the PSNE premium profiles using a model of more than two periods (Section 5.1), and investigate insurers' strategic incentives through a two-period time horizon model (Section 5.2).

	LoB 1		LoB 2		LoB (1,2)	
	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
$q_i^{(m)}(0)$	1000	1300	1700	1500	1000	1100
$c_i^{(m)}(t)$ for all t	5	5	5	5	10	10
$r_i^{(m)}(t)$ for all t	10%	10%	10%	10%	10%	10%
$\delta_i^{(m)}$	90%	90%	90%	90%	90%	90%
$d_i(\|\mathcal{I}(m)\|)$	0	0	0	0	5%	8%
$\mathbb{E}\left[Y_i^v \| J(t) = j_1\right]$ for all t	90		130		220	
$\mathbb{E}\left[Y_i^v \| J(t) = j_2\right]$ for all t	100		150		250	
$\mathbb{E}\left[Y_i^v \| J(t) = j_3\right]$ for all t	200		300		500	

Table 2: This table reports the basic model parameters for a competitive insurance market consisting of two LoBs and two insurers offering both LoBs, i.e., $N = \{1, 2\}$ and $L_1 = L_2 = L = \{1, 2\}$.

5.1 Pure strategy Nash equilibrium in a four-period time horizon

In this subsection, we assume the relative change in the expected number of policies, which appears in the argument of the Poisson distribution in (5), is from the exponential family, first introduced

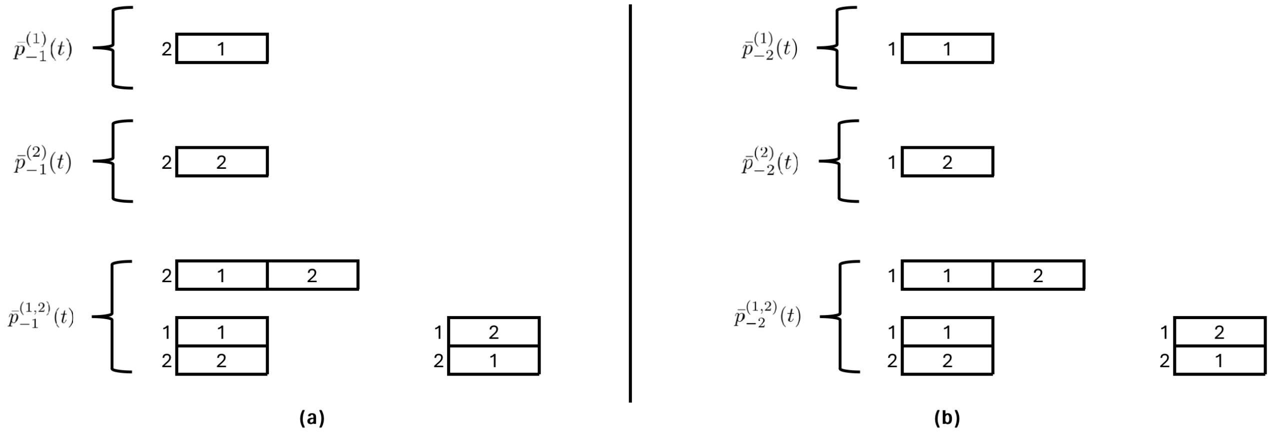


Figure 3: Illustration of insurers' competing premium rates in the market-average competitors' premium $\bar{p}_{-i}^{(m)}(t)$ for $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t \in \{1,\dots,T\}$. Particularly, Figures (a) and (b) display all the Young tableau without the single-row Young diagrams $\mathcal{Y}_1^{(m)}$ and $\mathcal{Y}_2^{(m)}$, respectively, for each $m \in \{(1),(2),(1,2)\}$.

Premium Combinations	Weights $w_{\mathcal{Y}}(t)$ for all t	
	Insurer 1	Insurer 2
$p_1^{(1)}(t) + p_2^{(2)}(t)$	1/3	0.28
$p_1^{(2)}(t) + p_2^{(1)}(t)$	1/3	0.30
$p_{-i}^{(1,2)}(t)$	1/3	0.42

Table 3: Weight selection for the market-average competitors' premium $\bar{p}_{-i}^{(1,2)}(t)$; the notation $p_{-i}^{(1,2)}(t)$ stands for the competitor j's total premium rate $p_j^{(1,2)}(t)$.

by Taylor (1986), and defined by

$$f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t)) = \exp\left\{-a_i^{(m)} \frac{p_i^{(m)}(t) - \bar{p}_{-i}^{(m)}(t)}{\bar{p}_{-i}^{(m)}(t)}\right\}, \quad (12)$$

where $a_i^{(m)} > 0$ is the price-sensitivity parameter for the contracts $m \in \mathcal{M}_i$.

Given a premium rate 20% above the average market premium, it is assumed that the higher the market power for a particular LoB an insurer possesses, the higher the proportion of previous exposure volume the insurer maintains. Letting $b_i^{(m)}$ denote the proportion of previous exposure volume of contracts m maintained by the insurer i, the price-sensitivity parameters are determined by the equations

$$\exp\left\{-a_i^{(m)}(1.2\bar{p}_{-i}^{(m)}(t) - \bar{p}_{-i}^{(m)}(t))/\bar{p}_{-i}^{(m)}(t)\right\} = b_i^{(m)},$$

$i \in N$ and $m \in \{(1),(2),(1,2)\}$. The proportions of exposure volume with the respective price-sensitivity parameters are provided by Table 4. The relative change in exposure volume of the double-policy contracts is considered comparatively lower than that of the single-policy ones, since withdrawal from even one of the two LoBs results in the termination of the double policy.

Assuming a four-period time horizon, the open-loop PSNE premiums are evaluated for four distinct scenarios, where Scenario 1 serves as our baseline case. In all four scenarios, the PSNE premium profile $p^* = (p_i^*, p_{-i}^*)$ is the solution to maximization problem (10), and consists of the insurers' single-LoB, pre-discount premiums that belong to the interior of the feasible region. That

	LoB 1		LoB 2		LoB (1,2)	
	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
$b_i^{(m)}$	0.552	0.560	0.565	0.563	0.465	0.470
$a_i^{(m)}$	2.971	2.899	2.855	2.872	3.829	3.775

Table 4: Proportions of previous-period exposure volume of contracts (1), (2) and $(1,2)$ maintained by insurers 1 and 2, with the associated values of the price-sensitivity parameters regarding the relative change in the expected number of policyholders as defined in Equation (12).

is, in all scenarios, p^* satisfies the first-order condition $\nabla_{p_i} O_i(p_i^*, p_{-i}^*) = 0$ and the second-order condition $\nabla_{p_i}^2 O_i(p_i^*, p_{-i}^*)$ (the Hessian matrix) is negative definite for all $i \in N$. In fact, we find that the Hessian matrices of both insurers, evaluated at the critical point p^*, are symmetric with all their eigenvalues negative.

We also evaluate the PSNE premium profile using the classical characterization of the market-average competitors' premium introduced by Taylor (1986). In this case, the market-average competitors' premium for contract m is given by

$$\bar{p}_{-i}^{(m)}(t) = \frac{1}{|\{j \in N : j \neq i,\ m \in \mathcal{M}_j\}|} \sum_{j \in N: j \neq i,\ m \in \mathcal{M}_j} p_j^{(m)}(t),$$

where $|\{j \in N : j \neq i,\ m \in \mathcal{M}_j\}|$ denotes the cardinality of the set of the insurer i's competitors who underwrite the contract m. This paper's characterization of the market-average competitors' premium is briefly referred to as *combinatorial market premium*, whereas Taylor's as *traditional market premium*. Notice that the traditional market premium for contract m (single- or multi-policy) is the average of competitors' total premium for this only particular contract m. That is, competitors who cannot offer all LoBs under m as well as cross selections between distinct insurers (that all together cover the LoBs of m) are not accounted for as competing premium rates.[7]

By illustrating the PSNE results under both characterizations of the market-average competitors' premium, we intend to capture the differences in insurers' optimal premium strategies. Section C.4, in the online supplementary appendix, provides the percentage differences in the PSNE premiums and exposure volumes for the two characterizations in all four scenarios that we study. Our findings show that under the combinatorial market premium, both insurers' optimal premium strategies focus on attracting double-policy holders, who might possess the double-policy contract from an insurer or possess two distinct policies, one from each insurer. For this reason, both insurers' exposure volume of $(1,2)$ displays an increasing total trend, with Insurer 1's growth being larger than Insurer 2's in all scenarios except Scenario 3. This is not the case under the traditional market premium which treats the multi-LoB $(1,2)$ in isolation and hence, when an insurer gains market share, the competitor has to lose. Specifically, in all scenarios, the overall trend in Insurer 2's exposure volume of $(1,2)$ displays a growth whereas Insurer 1's a decline (in Scenarios 2 and 3 particularly, these trends are immaterial).

The differences between the two characterizations mentioned above arise from the fact that the combinatorial market premium for a multi-policy contract m assigns positive weights to any cross selection between distinct insurers (all together covering the LoBs under m) and captures the

[7] Table 4 provides the values of the price-sensitivity parameters for both characterizations of the marker-average competitors' premium. This arises from the fact that the price-sensitivity parameters are always given by the equation $\exp\{-0.2a_i^{(m)}\} = b_i^{(m)}$, which is independent of $\bar{p}_{-i}^{(m)}(t)$.

possibility for insurers to accommodate the entire account of those policyholders who contemplate being covered for m by various insurers. If the weights of all such cross selections between distinct insurers are set equal to zero, then we have the case of the traditionally evaluated market premium, which cannot capture the inflows (or outflows) of policyholders with contracts from various insurers.

Scenario 1: Baseline case

In this scenario, all of the model parameters have been defined thus far. For both characterizations of the market-average competitors' premium, Table 5 shows the PSNE premium values, whereas Figure 4 displays the PSNE premiums and exposure volumes.

First, let us focus on the single-policy contracts in LoBs 1 and 2. When competitors' premiums are aggregated by the combinatorial market premium, Insurer 1's equilibrium premium profiles are lower than Insurer 2's over the entire time horizon, with the highest difference being observed in the first-period premium values. As a result, Insurer 1's exposure volumes are growing exponentially, whereas Insurer 2's are declining exponentially, over the four periods. Moreover, the premiums of both insurers display a cyclical behaviour, i.e., low values in a period are followed by higher values in the next period, and vice versa. Due to the finite time horizon and termination of the optimization algorithm at the end of the fourth period, the insurers' last-period premium values are the highest ones in the entire horizon in order to generate substantial premium margins and offset losses from the lowest premium values (over the entire horizon) taking place in the penultimate period.

In the case of the traditional market premium, we observe similar equilibrium premium strategies for the insurers' single-policy contracts to the case of the combinatorial market premium. That is, Insurer 1's equilibrium premium profiles are below Insurer 2's over the entire time horizon yielding an increase in Insurer 1's exposure volumes and a decrease in Insurer 2's. Noticeably, under the traditional market premium, Insurer 1's equilibrium premium profile in LoB 2 is slightly lower than Insurer 2's, mainly because the insurers' price-sensitivity parameters are closer to each other for contract (2) than for contract (1). Furthermore, the premium values of insurers in subsequent periods continue to exhibit cyclical patterns. However, when comparing the traditional market premium to the combinatorial market premium, the differences in insurers' equilibrium premium values in successive periods are reduced. Comparing the equilibrium premium profiles of insurers for single-policy contracts in the traditional market premium case to those in the combinatorial market premium case, the former exhibits a slower rate of growth for Insurer 1's exposure volume and the latter a higher rate of decline for Insurer 2. For LoB 2, insurers 1 and 2 see low changes in exposure volume over successive periods because of the small differences in their equilibrium premium profile for that LoB. This is reflected in the low gains and losses in exposure volume under the traditional market premium. The differences in the PSNE outcomes for the single-policy contracts (1) and (2) are due to the fact that, under the combinatorial market premium, the single-LoB premiums $p_i^{(m)}(t)$, $i \in N$, $m \in \{(1), (2)\}$ and $t \in \{1, \ldots, 4\}$, also play a role in the competition for the double-policy contract $(1, 2)$, whereas under the traditional market premium, they only contribute to $p_i^{(1,2)}(t)$.

Now, we shall turn our attention to the double-policy contract $(1, 2)$ in relation to the combinatorial market premium. The insurers' total premium for $(1, 2)$ is calculated as the discounted sum of the single premiums for policies (1) and (2). As expected, the equilibrium premium profile for $(1, 2)$ follows a similar pattern to that of the single-policy contracts. It is worth noting that the total premium of Insurer 1 has slightly surpassed that of Insurer 2 in the last period. This is due to Insurer 2 applying a higher discount to the policy (1,2), and the fact that the premiums charged by both insurers for a single LoB in the last period are very similar to each other, especially when compared to previous periods. Interestingly, the volume of exposure for both insurers displays an overall increase over the whole time period, except for the first period. In the first period, Insurer 2 experiences a small decrease due to the significant difference in equilibrium premiums for a single LoB between the two insurers. The increase in Insurer 1's exposure volume may be related to its lower equilibrium single-policy premium values compared to Insurer 2. On the other hand, Insurer 2's growth in exposure volume after the first period is mostly due to its bigger discount for the contract $(1, 2)$.

Now, we will examine the traditional market premium. In this case, Insurer 1's total premiums for (1,2) are higher than Insurer 2's for the whole time period. It is worth noting that the total premiums for both insurers in the first period are roughly equal, resulting in no change in exposure volume for either insurer during that period. The fact that Insurer 1's total premium for $(1, 2)$ is greater than Insurer 2' in all periods can be due to the following factors: The difference in single-policy premiums is smaller in the traditional market premium compared to the combinatorial market. Insurer 2 offers a larger discount for the $(1, 2)$ contract compared to Insurer 1. The competition is solely based on the insurers' total premiums, without taking into account cross selections between them for the $(1, 2)$ contract. Therefore, when Insurer 2 increases its exposure volume for (1,2), Insurer 1, its competitor, must decrease its exposure volume. This is not the case with the combinatorial market premium, in which the premium $p_i^{(1,2)}(t)$ "competes" with the cross selections $(p_i^{(1)}(t) + p_j^{(2)}(t))$ and $(p_i^{(2)}(t) + p_j^{(1)}(t))$, $j \neq i$, as well.

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	117.98	130.93	173.46	200.08	276.86	304.53
2	178.64	186.68	254.18	270.71	411.18	420.80
3	85.40	89.45	130.95	139.42	205.53	210.56
4	238.08	243.02	343.44	352.13	552.44	547.54

(a)

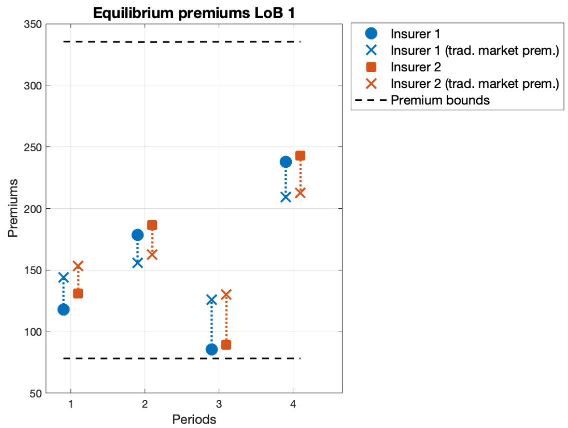
	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	144.13	153.15	211.40	213.30	337.75	337.13
2	155.90	162.70	238.00	240.09	374.21	370.57
3	125.94	129.99	177.98	179.54	288.73	284.76
4	209.20	212.55	314.76	317.47	497.77	487.62

(b)

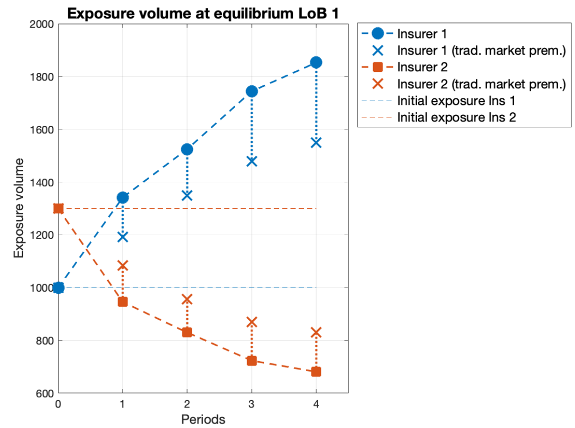
Table 5: Scenario 1: PSNE premium profiles in a two-insurer two-LoB four-period insurance market. Table (a) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the combinatorial market premium. Table (b) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the traditional market premium.

Scenarios 2 to 4: Overview (see Online supplementary material for more details)

In this section, we will examine three more scenarios. The related tables and figures for these scenarios can be found in Section C of the online supplementary appendix. In Scenario 2, Insurer 2 is assumed to retain a greater percentage of the past exposure volume for contracts (1), (2), and $(1, 2)$ compared to Scenario 1, when it charges a premium that is 20% more than the average market



(a)



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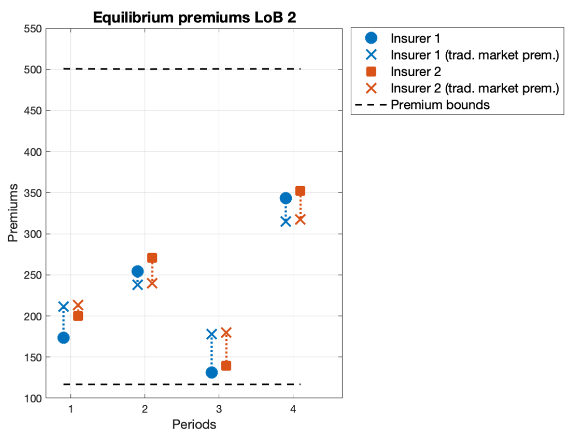

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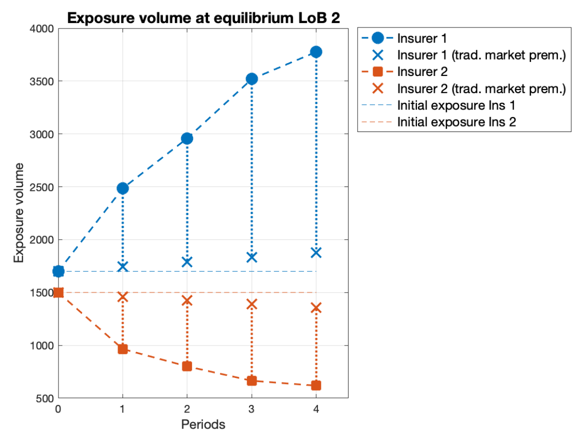

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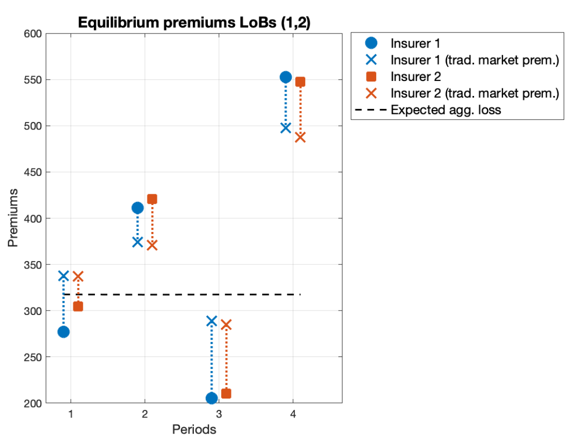
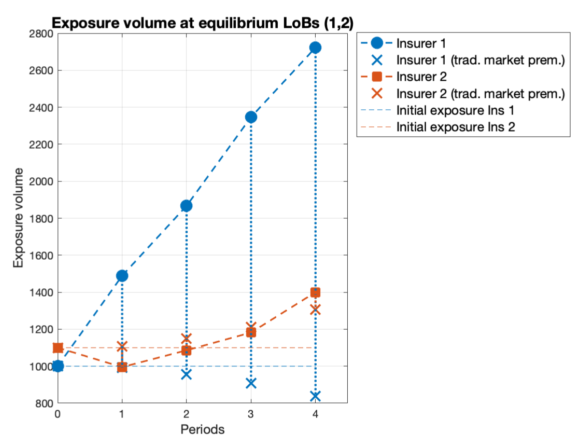


(e)



(f)

Figure 4: Scenario 1: Figures (a), (c) and (e) show the PSNE premium profiles per contracts (1), (2) and $(1,2)$, respectively, in a two-insurer two-LoB four-period insurance market, and Figures (b), (d) and (f) show the associated exposure volumes. The solid markers represent the results obtained using the combinatorial market premium, and the x markers represent the results obtained using the traditional market premium. The dashed lines in Figures (a) and (c) delimit the feasible premium region of the single-policy contracts, and the dashed line in Figure (e) illustrates the total expected loss per contract $(1,2)$. The horizontal dashed lines in Figures (b), (d) and (f) illustrate the initial exposure volumes.

premium. The insurers' PSNE premium strategies follow similar patterns to those in Scenario 1. However, due to the reduced price-sensitivity characteristics in this scenario, Insurer 2 charges higher premiums in its equilibrium premium profile compared to Scenario 1. In contrast, Insurer 1 charges reduced premiums in Scenario 2 compared to Scenario 1, making it far more affordable than its competitor. Insurer 1 experiences a greater increase in the number of single- and double-policy contracts compared to Scenario 1.

In Scenario 3, we reduce the discount provided by Insurer 2 for the double-policy contracts (1,2) to be just 1% greater than the discount given by Insurer 1. Insurer 2's premium profile is generally lower than in Scenario 1, except for the penultimate period. On the other hand, Insurer 1's premiums are generally higher than those in Scenario 1, except for the last period. Currently, the equilibrium premium values for the insurers' single policies are similar, resulting in little change in their exposure volumes in the single LoBs. Furthermore, the exposure volume of Insurer 2 for $(1,2)$ is larger than that of Insurer 1 and greater than its exposure volume for $(1,2)$ in Scenario 1.

Claims with increasing frequency and severity are anticipated to occur during the four periods in Scenario 4. We may incorporate this into our model by giving greater probability to the third state of the environment. As expected, the equilibrium pricing strategies of both insurers are the same as those in Scenario 1, leading to similar changes in respective exposure volumes. The main distinction between Scenario 1 and this scenario is in the extent of the equilibrium premiums, which are influenced by the anticipated claim amounts being larger in this particular scenario.

5.2 Open-loop and closed-loop equilibrium premiums in a two-period time horizon

In this subsection, we assume that the relative change in the expected number of policies is from the exponential family, but now we consider premium differences in the exponent, as in Dutang et al. (2013), rather than premium ratios. This is defined as

$$f_i^{(m)}(p_i^{(m)}(t), \bar{p}_{-i}^{(m)}(t)) = \exp\left\{-a_i^{(m)}(p_i^{(m)}(t) - \bar{p}_{-i}^{(m)}(t))\right\}, \tag{13}$$

where $a_i^{(m)}$ is the price-sensitivity parameter for the contracts $m \in \mathcal{M}_i$. Intending to focus our analysis on insurers' strategic incentives, we only use the combinatorial market premium $\bar{p}_{-i}^{(m)}(t)$, as given by Definition 1, to characterize the market-average competitors' premium.

Consider a premium rate 20% above the average market premium, while the market on average sets a premium rate equal to $\mu_i^{(m)}(1)$ (i.e., the current expected aggregate loss per policy). It is assumed that the higher the market power for a particular LoB an insurer possesses, the higher the proportion of previous exposure volume the insurer maintains. Therefore, the price-sensitivity parameters of both insurers are determined by the equations $\exp\left\{-0.2a_i^{(m)}\mu_i^{(m)}(1)\right\} = b_i^{(m)}$, for $i = 1, 2$ and $m = (1), (2), (1, 2)$, and the parameters are provided in Table 6. We assume once more that the relative change in the exposure volume of the double-policy contracts is comparatively lower than the single-policy ones since withdrawal from even one of the two LoBs results in the termination of the double policy.

In this example, a premium strategy profile of our insurance game, written in time order, is $p = (p_1, p_2) = (p(1), p(2)) = (p_1(1), p_2(1), p_1(2), p_2(2))$, where $p_i(t) = (p_i^{(1)}(t), p_i^{(2)}(t))$ for $i = 1, 2$ and $t = 1, 2$.

First, we evaluate the open-loop PSNE premium profile $p^* = (p_i^*, p_{-i}^*)$, which is the solution to the maximization problem (10), and is calculated by solving the first-order conditions

	LoB 1		LoB 2		LoB (1,2)	
	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
$b_i^{(m)}$	0.6520	0.6900	0.6650	0.6930	0.5650	0.6000
$a_i^{(m)}$	0.0166	0.0144	0.0108	0.0097	0.0090	0.0080

Table 6: Proportions of previous-period exposure volume of contracts (1), (2) and $(1,2)$ maintained by insurers 1 and 2, with the associated values of the price-sensitivity parameters regarding the relative change in the expected number of policyholders as defined in Equation (13).

$\nabla_{p_i} O_i(p_i, p^*_{-i}) = 0$ for $i = 1, 2$. Then, we observe that the second-order condition is satisfied, i.e., $\nabla^2_{p_i} O_i(p^*_i, p^*_{-i})$ (the Hessian matrix) is negative definite for all $i = 1, 2$. The PSNE is shown in Table 7 and Figure 5.

Next, the closed-loop SPE premiums are given by solving the system of equations in Eq. (11). Namely, we solve:

$$\nabla_{p_i(2)} O_i(p_i, p_{-i}) = 0 \tag{14}$$

and

$$\frac{\partial O_i(p_1(1), p_2(1), p_1^{**}(2), p_2^{**}(2))}{\partial p_i^{(v)}(1)} + \sum_{\tilde{v}=1}^{2} \frac{\partial O_i(p_1(1), p_2(1), p_1^{**}(2), p_2^{**}(2))}{\partial p_j^{(\tilde{v})}(2)} \frac{\partial p_j^{(\tilde{v})**}(2)}{\partial p_i^{(v)}(1)} = 0, \tag{15}$$

for $i = 1, 2$, $v = 1, 2$ and $j \neq i$. The system of equations in (14) does not yield explicit expressions for the $p^{**}(2; p(1)) = (p_1^{**}(2; p(1)), p_2^{**}(2; p(1)))$ with respect to the first-period premium profile. Thus, we use the implicit function theorem in order to evaluate the rate of change of the second-period equilibrium premiums with respect to the first-period premiums.

Let us define the left-hand side of Equation (14) as $F_i^{(v)}(p_i, p_{-i}) = \partial O_i(p_i, p_{-i}) / \partial p_i^{(v)}(2)$, for $v = 1, 2$ and $i = 1, 2$. Moreover, let

$$\nabla_{(p_1(t), p_2(t))} F_i^{(v)}(p_1, p_2) = \left[\partial F_i^{(v)} / \partial p_1^{(1)}(t), \partial F_i^{(v)} / \partial p_1^{(2)}(t), \partial F_i^{(v)} / \partial p_2^{(1)}(t), \partial F_i^{(v)} / \partial p_2^{(2)}(t) \right]$$

be the row vector of the partial derivatives of $F_i^{(v)}$ with respect to $(p_1(t), p_2(t))$. Now,

$$J_{F,t}(p_1, p_2) = \begin{bmatrix} \nabla_{(p_1(t), p_2(t))} F_1^{(1)}(p_1, p_2) \\ \nabla_{(p_1(t), p_2(t))} F_1^{(2)}(p_1, p_2) \\ \nabla_{(p_1(t), p_2(t))} F_2^{(1)}(p_1, p_2) \\ \nabla_{(p_1(t), p_2(t))} F_2^{(2)}(p_1, p_2) \end{bmatrix},$$

for $t = 1, 2$, is the Jacobian matrix of F with respect to the vector $(p_1(t), p_2(t))$. The matrix of the partial derivatives of the second-period equilibrium premiums with respect to the first-period premiums is obtained by

$$\left[\nabla_{(p_1(1), p_2(1))} p_i^{(v)**}(2) \right]_{1 \leq i, v \leq 2} = - \left(J_{F,2}(p_1, p_2) \right)^{-1} \cdot J_{F,1}(p_1, p_2). \tag{16}$$

Therefore, Equations (14), (15) and (16) yield the closed-loop SPE premiums $p^{**} = (p_1^{**}, p_2^{**})$ shown in Table 7 and Figure 5. Now, let us examine how the open-loop PSNE p^* can be used to explain the strategic incentives of insurers, i.e., how they change the first-period premiums in order

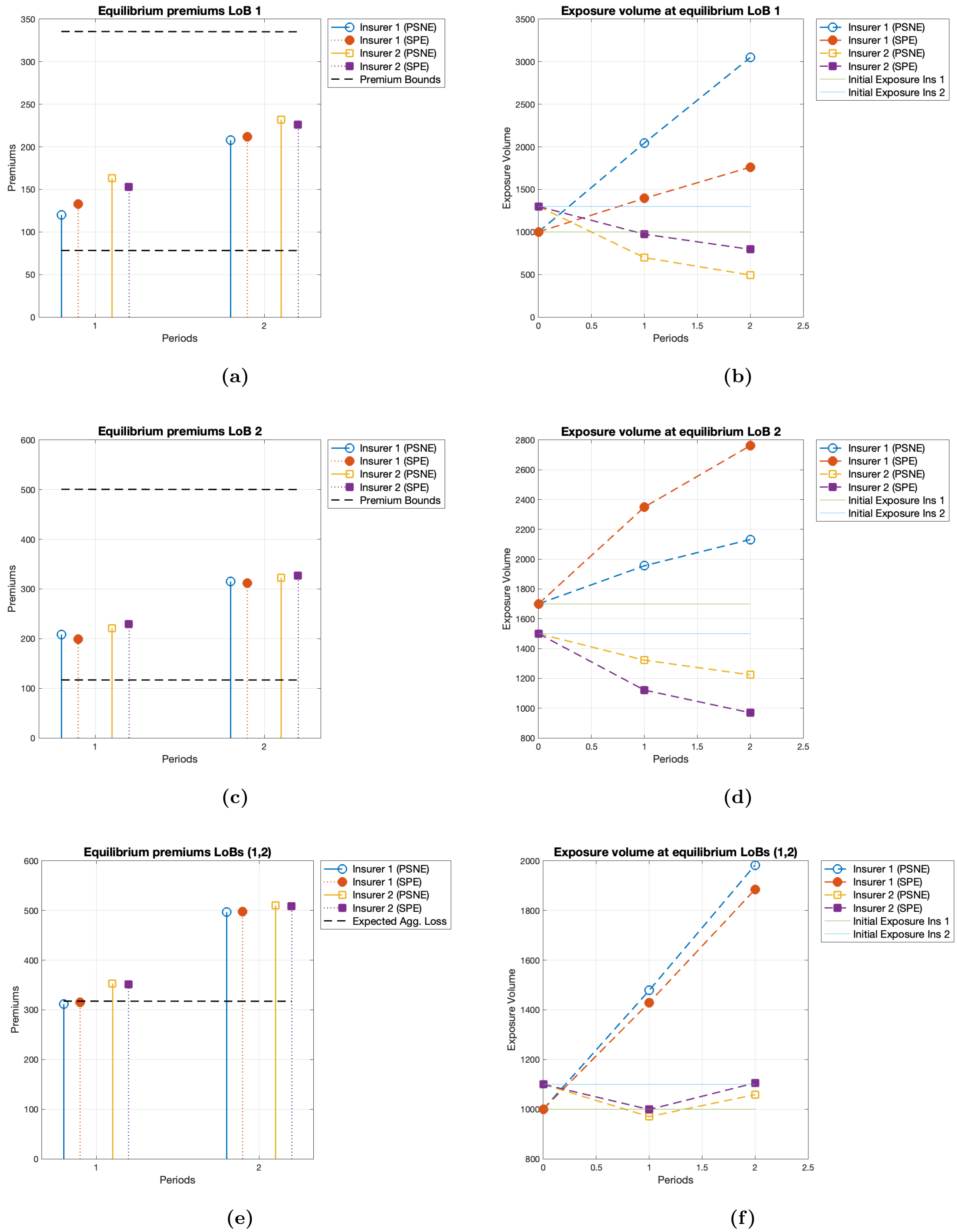


Figure 5: Figures (a), (c) and (e) show the (open-loop) PSNE and (closed-loop) SPE premium profiles per contracts (1), (2) and $(1, 2)$, respectively, in a two-insurer two-LoB two-period insurance market, and Figures (b), (d) and (f) show the associated exposure volumes. The filled solid markers represent the results related to the SPE characterization, and the empty-filled markers represent the results related to the PSNE characterization. The dashed lines in Figures (a) and (c) delimit the feasible premium region of the single-policy contracts, and the dashed line in Figure (e) illustrates the total expected loss per contract $(1, 2)$. The horizontal dashed lines in Figures (b), (d) and (f) illustrate the initial exposure volumes.

Insurer	Period	LoB 1		LoB 2		LoB (1,2)	
		PSNE	SPE	PSNE	SPE	PSNE	SPE
1	1	120	133	208	199	312	315
	2	208	212	315	312	497	498
2	1	163	153	221	229	353	351
	2	232	226	323	327	511	509

Table 7: Open-loop PSNE premium profiles and closed-loop SPE premium profiles, per contracts (1), (2) and $(1,2)$, in a two-insurer two-LoB two-period insurance market.

to affect the second-period premium choices of their rivals. We substitute the first-period PSNE premiums $p^*(1)$ into Eq. (14), and we solve the system of equations $\nabla_{p_i(2)} O_i(p^*(1), p_i(2), p_{-i}(2)) = 0$, for $i = 1, 2$, with respect to $p(2)$. Let $\tilde{p}^*(2)$ denote this solution. Now, we evaluate the matrix $\left[\nabla_{(p_1(1),p_2(1))} p_i^{(v)**}(2)\right]_{1 \le i, v \le 2}$ in Eq. (16) at the point $(p^*(1), \tilde{p}^*(2))$ to get the following values:

$$\begin{bmatrix} 0.0806 & -0.0519 & -0.0802 & 0.0523 \\ -0.0974 & 0.0741 & 0.0995 & -0.0721 \\ -0.1989 & 0.1274 & 0.2045 & -0.1218 \\ 0.1118 & -0.0791 & -0.1105 & 0.0804 \end{bmatrix}.$$

Consider Insurer 2, which has greater market power in LoB 1 than Insurer 1. Insurer 2 expects that Insurer 1 will charge low premium values in LoB 1 in order to gain exposure. Therefore, Insurer 2's strategic incentive is to increase Insurer 1's premium values in LoB 1 during period 2. Based on the element $(1,3)$ of the above matrix, we have that $\partial p_1^{(1)**}(2)/\partial p_2^{(1)}(1) < 0$, and hence Insurer 2 sets the first-period SPE premium value $p_2^{(1)**}(1)$ lower than the respective first-period PSNE premium value $p_2^{(1)*}(1)$, i.e., $p_2^{(1)**}(1) < p_2^{(1)*}(1)$. Similarly, since Insurer 1 holds a greater market share in LoB 2 than Insurer 2, Insurer 1 anticipates that Insurer 2 will charge low premium values in LoB 2 to gain exposure. Therefore, Insurer 1's strategic incentive is to increase Insurer 2's premium values in LoB 2 during period 2. Based on the element $(4,2)$ of the above matrix, we have that $\partial p_2^{(2)**}(2)/\partial p_1^{(2)}(1) < 0$, and hence Insurer 1 sets the first-period SPE premium value $p_1^{(2)**}(1)$ lower than the respective first-period PSNE premium value $p_1^{(2)*}(1)$, i.e., $p_1^{(2)**}(1) < p_1^{(2)*}(1)$.

6 Conclusion

This paper studies a stochastic competitive insurance market over a finite time horizon to determine the optimal premium strategies of insurers competing in multiple Lines of Business (LoBs). Multi-policy discounts are considered when underwriting multiple LoBs. Young diagrams are utilized to determine the market-average competitors' premiums. A Poisson model with a common shock divides the total number of policies in a LoB into single- and multi-policy contracts. The expected number of policies follows a negative exponential demand function with a dependence structure between successive periods. We assume that claim severity depends on the environmental state described by a discrete-time Markov process with a finite state space. Operational and underwriting costs, and commission reductions are considered. Finally, our insurance game assumes a closed-loop information structure where premium strategies map histories to premium strategies. We define open- and closed-loop equilibrium premium profiles. The former can be used as a benchmark for the latter when analyzing insurers' strategic incentives. The numerical examples capture premium cycles and loss-leading LoBs.

As long as we model the interdependence of the number of policies within a LoB and the risks of the LoBs under a multi-policy contract are considered independent in all respects, the assumption that these risks are dependent could be an immediate future direction for further research. Furthermore, we are aware of the limitations of the negative exponential demand function, particularly the fact that this approach does not control for the total number of insureds in the market. For further research, it is recommended to investigate a closed insurance market, where customer migrations between insurers follow a Markov chain, and the migration intensity is directly proportional to the price difference between the insurers. Finally, since the problem can be NP-hard, computational issues may arise as the number of insureds and LoBs grows.

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A Proofs

A.1 Proof of Proposition 1

Taking expectation on Eq. (8) requires the evaluation of the expected values of $N_i^{(m)}(t)$ and $S_i^{(m)}\left(N_i^{(m)}(t)\right)$ for all m and all t. Consider a contract $m \in \mathcal{M}_i$ and a time period $t \in \{1, \ldots, T\}$. First, in order to find the insurer i's expected number of policyholders of contract m in period t, we use induction on time t. For $t = 1$, the hierarchical structure and Poisson distribution in Eq. (5) imply that

$$\mathbb{E}\left[N_i^{(m)}(1)\right] = f_i^{(m)}(1)q_i^{(m)}(0).$$

For time $t-1$, let us consider the induction assumption

$$\mathbb{E}\left[N_i^{(m)}(t-1)\right] = \prod_{k=1}^{t-1} f_i^{(m)}(k)q_i^{(m)}(0).$$

For time t, the law of iterated expectations and Eq. (5) yield

$$\mathbb{E}\left[N_i^{(m)}(t)\right] = \mathbb{E}\left[\mathbb{E}\left[N_i^{(m)}(t) \mid N_i^{(m)}(t-1)\right]\right] = \mathbb{E}\left[f_i^{(m)}(t)N_i^{(m)}(t-1)\right],$$

and using the induction hypothesis we obtain

$$\mathbb{E}\left[N_i^{(m)}(t)\right] = f_i^{(m)}(t)\mathbb{E}\left[N_i^{(m)}(t-1)\right] = f_i^{(m)}(t)\prod_{k=1}^{t-1} f_i^{(m)}(k)q_i^{(m)}(0) = \prod_{k=1}^{t} f_i^{(m)}(k)q_i^{(m)}(0). \quad \text{(A.1)}$$

From Eq. (6), the insurer i's expected aggregated loss for contracts m in period t is given by

$$\begin{aligned}
\mathbb{E}\left[S_i^{(m)}\left(N_i^{(m)}(t)\right)\right] &= \left(1+r_i^{(m)}(t)\right)\left(\sum_{v\in\mathcal{I}(m)} \mathbb{E}\left[\sum_{k=1}^{N_i^{(m)}(t)} Y_{ik}^v\right] + c_i^{(m)}(t)\mathbb{E}\left[N_i^{(m)}(t)\right]\right) \\
&= \left(1+r_i^{(m)}(t)\right)\mathbb{E}\left[N_i^{(m)}(t)\right]\left(\sum_{v\in\mathcal{I}(m)} \mathbb{E}\left[Y_i^v\right] + c_i^{(m)}(t)\right).
\end{aligned}$$

Since the individual claim amount depends on the state of the environment process $\{J(t)\}$,[8] we finally get

$$\mathbb{E}\left[S_i^{(m)}\left(N_i^{(m)}(t)\right)\right] = \left(1+r_i^{(m)}(t)\right)\mathbb{E}\left[N_i^{(m)}(t)\right]\left[\sum_{k=1}^{s} \Pr[J(t)=j_k] \sum_{v\in\mathcal{I}(m)} \mathbb{E}\left[Y_i^v | J(t)=j_k\right] + c_i^{(m)}(t)\right]. \quad \text{(A.2)}$$

Thus, the insurer i's expected net present value of future wealth is given by

$$\mathbb{E}\left[W_i\left(p_i, p_{-i}\right)\right] = W_{i,0} + \sum_{t=1}^{T} u^t \sum_{m\in\mathcal{M}_i}\left(g_i^{(m)}(t)\mathbb{E}\left[N_i^{(m)}(t)\right] - \mathbb{E}\left[S_i^{(m)}\left(N_i^{(m)}(t)\right)\right]\right), \quad \text{(A.3)}$$

and substituting (A.1) and (A.2) in (A.3) yields Eq. (9).

[8]The state of the environment and the number of policyholders of contract m in period t are independent random variables.

A.2 Proof of Theorem 1

By construction, insurer i's set of premium strategies, P_i, is compact and convex. Moreover, insurer i's payoff function $O_i(p_i, p_{-i})$ is continuous in $p = (p_i, p_{-i})$. Therefore, if $O_i(p_i, p_{-i})$ is concave in p_i, Theorem 1 in Rosen (1965) guarantees the existence of a PSNE.

A.3 Proof of Theorem 2

By definition, the set of period-t premiums strategies $P_i(t)$ for insurer i is a time-dependent, compact, and convex set for all $t \in \{1, 2, \ldots, T\}$. Now, consider the last period T with history $h(T)$. We have that $O_i(p_i, p_{-i})$ is continuous in $p(T) = (p_i(T), p_{-i}(T))$ and concave in $p_i(T)$ by assumption. Then, the conditions in Theorem 1 of Rosen (1965) are satisfied and there exists a Nash equilibrium of the subgame $G(h(T))$, denoted by $p^*(T) \equiv p^*(T; h(T))$, and given by solving the system of equations $\nabla_{p_i(T)} O_i(p_i, p_{-i}) = 0$ for all $i \in N$.

Let us move backwards to period $T-1$ with current premium history $h(T-1)$. We have that $O_i(h(T-1), p_i(T-1), p_{-i}(T-1), p^*(T))$ is continuous in $p(T-1) = (p_i(T-1), p_{-i}(T-1))$ and concave in $p_i(T-1)$ by assumption, and hence Theorem 1 of Rosen (1965) guarantees the existence of a Nash equilibrium of the subgame $G(h(T-1))$. This Nash equilibrium is obtained by solving the system of equations $\nabla_{p_i(T-1)} O_i(h(T-1), p_i(T-1), p_{-i}(T-1), p^*(T)) = 0$ for all $i \in N$. At this point, we should pay attention to the fact that insurers can condition their premium strategies in period T on past premium choices including period $T-1$. As a result, an application of the chain rule and the envelope theorem lead to the following first-order condition: for all $i \in N$ and all $m' \in \mathcal{M}_i(1)$

$$\frac{\partial O_i}{\partial p_i^{(m')}(T-1)} + \sum_{j \neq i} \sum_{\tilde{m} \in \mathcal{M}_j(1)} \frac{\partial O_i}{\partial p_j^{(\tilde{m})}(T)} \frac{\partial p_j^{(\tilde{m})*}(T)}{\partial p_i^{(m')}(T-1)} = 0. \tag{A.4}$$

Following similar arguments we keep moving backwards to the first period, and at the end of this process we will have constructed a closed-loop subgame perfect equilibrium premium profile.

Remark 1 *1. It is worth pointing out two properties related to Eq.* (A.4)*. First, the Envelope Theorem and FOC at period T yield the partial derivatives $\partial O_i / \partial p_i^{(\tilde{m})}(T)$ to vanish for all $\tilde{m} \in \mathcal{M}_i(1)$ in period $T-1$. Second, the second factor of the summands in Eq.* (A.4) *represents mathematically the strategic incentive of insurer i to change the current premium choice in order to influence future actions of the opponents (see Fudenberg and Tirole, 1991). In the numerical illustration, we show how the sign of the partial derivatives $\partial p_j^{(\tilde{m})*}(T)/\partial p_i^{(m')}(T-1)$, evaluated at the open-loop PSNE, can indicate the strategic incentives of insurer i.*

2. In the numerical illustration, we see that, when we cannot have explicit expressions of $p^*(T) \equiv p^*(T; h(T))$, we invoke the implicit function theorem, since we only need the partial derivatives of the form $\partial p_j^{(\tilde{m})*}(T)/\partial p_i^{(m')}(T-1)$.

Online Supplementary Material: Appendices

"Competitive Insurance Pricing Strategies for Multiple Lines of Business: A Game-Theoretic Approach"

Fotios Mourdoukoutas[1,2], Tim J. Boonen[2], Athanasios A. Pantelous[1], Greg Taylor[3]

[1]Department of Econometrics and Business Statistics, Monash Business School, Monash University, 20 Chancellors walk, Wellington Road, Clayton Campus, Victoria, Australia

[2]Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam, Hong Kong, Hong Kong SAR

[3]School of Risk and Actuarial Studies, UNSW Business School, UNSW Sydney NSW 2052, Australia

Abstract

This paper explores an insurance market with insurers competing in multiple lines of business (LoBs) over a finite time horizon. They set premiums for single LoBs, whereas in a multi-policy contract, a discount is applied to the total sum of the single premiums. To illustrate the combinatorial components of the weighted average market premium, Young tableaux are used. We use the collective risk model to define insurers' aggregate loss, with claim amounts independent of distinct LoBs but dependent on the state of the environment. Our dynamic insurance game falls into the category of multi-stage games with observed actions. In this framework, we characterize the open-loop and closed-loop equilibrium premium profiles.

Keywords: Decision analysis; Game theory; Open- and closed-loop equilibrium; Competitive markets; Multiple lines of business.

JEL classification: G22; C61; C72.

B Table of notation

The paper makes use of a great deal of notation. For readers' convenience, we provide a list of the mathematical notation used throughout the paper, accompanied by a short explanation.

N	subset of positive integers that represents all insurers in the market,
L	subset of positive integers that represents all LoBs in the market,
L_i	subset of L that represents all LoBs offered by insurer i,
m	an ordered collection of elements in L_i that represents an insurance contract,
$\mathcal{I}(m)$	index set of the LoB(s) in contract m,
$\mathcal{M}_i(k)$	set of all contracts m such that $\lvert\mathcal{I}(m)\rvert = k$,
$\mathcal{M}_i$	all the possible single- and multi-policy contracts offered by insurer i,
$d_i(k)$	discount function applied to contracts of k policies by insurer i; it is considered an increasing function with $d_i(1) = 0$ and range of values $[0, 1]$,
$p_i^{[v]}(t)$	insurer i's single pre-discount premium associated with the single LoB $v \in L_i$ in period t,
$p_i^{(m)}(t)$	total premium value for contract m charged by insurer i in period t,
$\bar{p}_{-i}^{(m)}(t)$	insurer i's market-average competitors' premium related to contract m in period t,
$\mathcal{Y}$	Young diagram; a structure of enumerated rows and enumerated boxes that represents a competing premium rate,
$\mathcal{Y}_i^{(m)}$	Young diagram associated with insurer i's total premium for contract m,
$\mathfrak{D}$	Young tableau; a collection of Young diagrams that represents all the possible competing premium rates for a contract,
$N_i^v(t)$	total number of policyholders who have purchased policies (single or multiple) for LoB v from insurer i within period t,
$\mathcal{D}_i(v)$	set of contracts from insurer i, single or multiple, that contain LoB v,
$N_i^{(m)}(t)$	number of policyholders of insurer i with contract m in period t,
$f_i^{(m)}(t)$	relative change in the insurer i's expected number of contracts m in period t,
$q_i^{(m)}(0)$	insurer i's initial exposure volume for contract m,
$J(t)$	the state of the environment in period t; it is considered a discrete-time Markov process with a finite state space,
Π	one-step transition matrix of the environment process,
A	initial distribution of the environment process; it is a row vector whose zth element is $A_z = \Pr[J(0) = j_z]$,
$Y_{ik}^v \lvert J(t) = j_z$	individual claim amount under LoB v for policyholder k of insurer i in period t when the environment is in state j_z,
$r_i^{(m)}(t)$	per unit cost of outgoing revenue,
$c_i^{(m)}(t)$	cost per contract m,
$S_i^{(m)}(t)$	aggregate loss for insurer i in period t, regarding policies m,
$p_i(t)$	period-t premiums choice of insurer i; premium vector with insurer i's period-t premium rates for all the single-policy contracts,
$P_i^{(m')}(t)$	set of available premium rates for the single-policy contract $m' \in \mathcal{M}_i(1)$ offered by insurer i in period t; it is equal to $[\underline{\pi}_i^{(m')}(t), \overline{\pi}_i^{(m')}(t)]$,
$P_i(t)$	set of all insurer i's period-t premiums choices; it is the Cartesian product of $P_i^{(m')}(t)$ for all $m' \in \mathcal{M}_i(1)$,

$p(t)$	period-t premium profile of all insurers in the market; premium vector with the insurers' period-t premiums choices,
$P(t)$	set of all period-t premium profiles,
$h(t)$	history of the insurance game at period t; it is a premium vector that consists of the period-k premium profiles, $k = 1, \ldots, t-1$,
$H(t)$	set of all period-t histories; we set $H(1) = \{\emptyset\}$,
T	terminal time period of our analysis,
p	(complete) premium profile of the insurance game; a premium vector with the premium choices for all insurers over the entire time horizon,
P	set of feasible premium profiles of the insurance game,
$p = (p_i, p_{-i})$	decomposition of premium profile into the premium profile of insurer i and the premium profiles of insurer i's competitors,
P_i	set of all premium profiles for insurer i,
P_{-i}	set of all premium profiles for insurer i's competitors,
$g_i^{(m)}(p_i^{(m)}(t))$	reduced premium associated with insurer i's total premium $p_i^{(m)}(t)$,
$W_{i,0}$	initial wealth of insurer i,
u	time-discount factor,
$W_i\left(p_i, p_{-i}\right)$	net present value of future wealth of insurer i over the total time horizon,
$O_i(p_i, p_{-i})$	insurer i's payoff function; $O_i : P \to \mathbb{R}$ and is equal to insurer i's expected net present value of future wealth,
$\mu_i^{(m)}(t)$	total expected loss per contract m.

C Pure strategy Nash equilibrium: Scenarios 2 to 4 corresponding to Section 5.1

C.1 Scenario 2

In this scenario, we keep all the model parameters as in Scenario 1, except for Insurer 2's price-sensitivity parameters for the single- and double-policy contracts. We assume that the market dynamics allow Insurer 2 to maintain a slightly higher proportion of previous exposure volume, compared to Scenario 1, when it charges 20% above the average market premium. Namely, when $p_2^{(m)}(t) = 1.2\bar{p}_{-2}^{(m)}(t)$ for $m = (1), (2), (1,2)$, Insurer 2 maintains 56.6%, 56.8% and 47.3% of previous exposure volume in LoB 1, LoB 2 and LoBs $(1,2)$, respectively. This yields the following values for Insurer 2's price-sensitivity parameters: $(\tilde{a}_2^{(1)}, \tilde{a}_2^{(2)}, \tilde{a}_2^{(1,2)}) = (2.846, 2.828, 3.743)$.

Table 1 provides the PSNE premium values under both the combinatorial and traditional market premium. Figure 1 illustrates the PSNE premiums and exposure volumes under the combinatorial and traditional market premium, and it also displays the PSNE premiums and exposure volumes from Scenario 1 for the combinatorial market premium.

In relation to the combinatorial market premium, insurers' premium strategies in this scenario portray the patterns observed in Scenario 1. Insurer 1 has lower equilibrium single-policy premium values than Insurer 2 in all periods, with the greatest difference occurring in period one. As a result, Insurer 1 experiences a substantial increase in market share for contracts (1) and (2), whereas Insurer 2 faces a decrease in exposure volume for these contracts. Insurer 1's total premium for policy (1,2) is lower than Insurer 2's total premium in all periods except period four. This is because the total premium for policy (1,2) is calculated by discounting the sum of the individual premiums for each policy. During the latest time frame, Insurer 2's total premium is lower than Insurer 1's due to the mix of a small difference in single-policy premiums among insurers and the larger discount provided by Insurer 2. However, the exposure volumes for the double-policy contract $(1,2)$ are increasing for both insurers. This is because their total premium not only contends with the competitor's total premium (as in the traditional market premium), but also with the cross selections $p_i^{(1)}(t)+p_j^{(2)}(t)$ and $p_i^{(2)}(t)+p_j^{(1)}(t)$, where $i \neq j$. Insurer 2 only experiences a decrease in exposure volume during the first period, specifically for the contracts $(1,2)$, due to the occurrence of the biggest differences in premiums between insurers in that time.

In comparison to Scenario 1, the decrease in Insurer 2's price-sensitivity parameters for all contracts, whether single or double-policy, results in bigger differences in the equilibrium premium prices of insurers. Consequently, these bigger differences in premiums result in more significant fluctuations in the number of policies that insurers are exposed to compared to Scenario 1. Specifically, Insurer 1 is experiencing a significant decrease in cost compared to Insurer 2, leading to a bigger increase in the number of single- and double-policy contracts, in contrast to Scenario 1. Insurer 2 is experiencing a higher rate of decrease in the number of single-policy contracts. It is observed that the exposure volume of Insurer 2 for the double-policy contracts (1,2) is increasing after period one, but at a slower rate compared to Scenario 1. However, it remains below the starting exposure level during the entire time period, whereas in Scenario 1, it surpasses the initial level after period two.

In regards to the traditional market premium, Insurer 1 has smaller equilibrium premium values for single-LoB policies compared to Insurer 2. As a result, Insurer 1 is able to capture a larger portion of the market share for single-policy contracts, whereas Insurer 2 experiences a decline in market share. Nevertheless, the differences in insurers' equilibrium, single-LoB premium values are less significant compared to those observed in the combinatorial market premium case. Therefore, the observed fluctuations in insurers' exposure levels occur at a reduced pace when compared to those in the case of the combinatorial market premium. Concerning the double-policy contract $(1, 2)$, in the setting of the traditional market premium, when one insurer increases their exposure volume, it results in a corresponding decrease in the competitor's exposure volume. Therefore, due to Insurer 2's larger discount in price for the $(1, 2)$ policy, its overall cost for this policy becomes lower than that of Insurer 1 after the first period of insurance. As a result, there is a little rise in the number of policies that Insurer 2 is exposed to over these periods, while the number of policies that Insurer 1 is exposed to decreases.

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	114.03	132.57	166.51	202.73	266.51	308.48
2	177.98	189.12	252.65	274.70	409.10	426.72
3	83.70	89.06	127.76	138.80	200.89	209.63
4	238.43	244.35	344.02	354.75	553.33	551.17

(a)

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	142.83	154.12	209.59	214.71	334.79	339.32
2	155.39	163.60	237.49	241.60	373.23	372.79
3	125.25	130.08	176.87	179.56	287.02	284.87
4	209.36	213.42	315.03	318.78	498.17	489.62

(b)

Table 1: Scenario 2: PSNE premium profiles in a two-insurer two-LoB four-period insurance market, when Insurer 2's price-sensitivity parameters are modified. Table (a) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the combinatorial market premium. Table (b) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the traditional market premium.

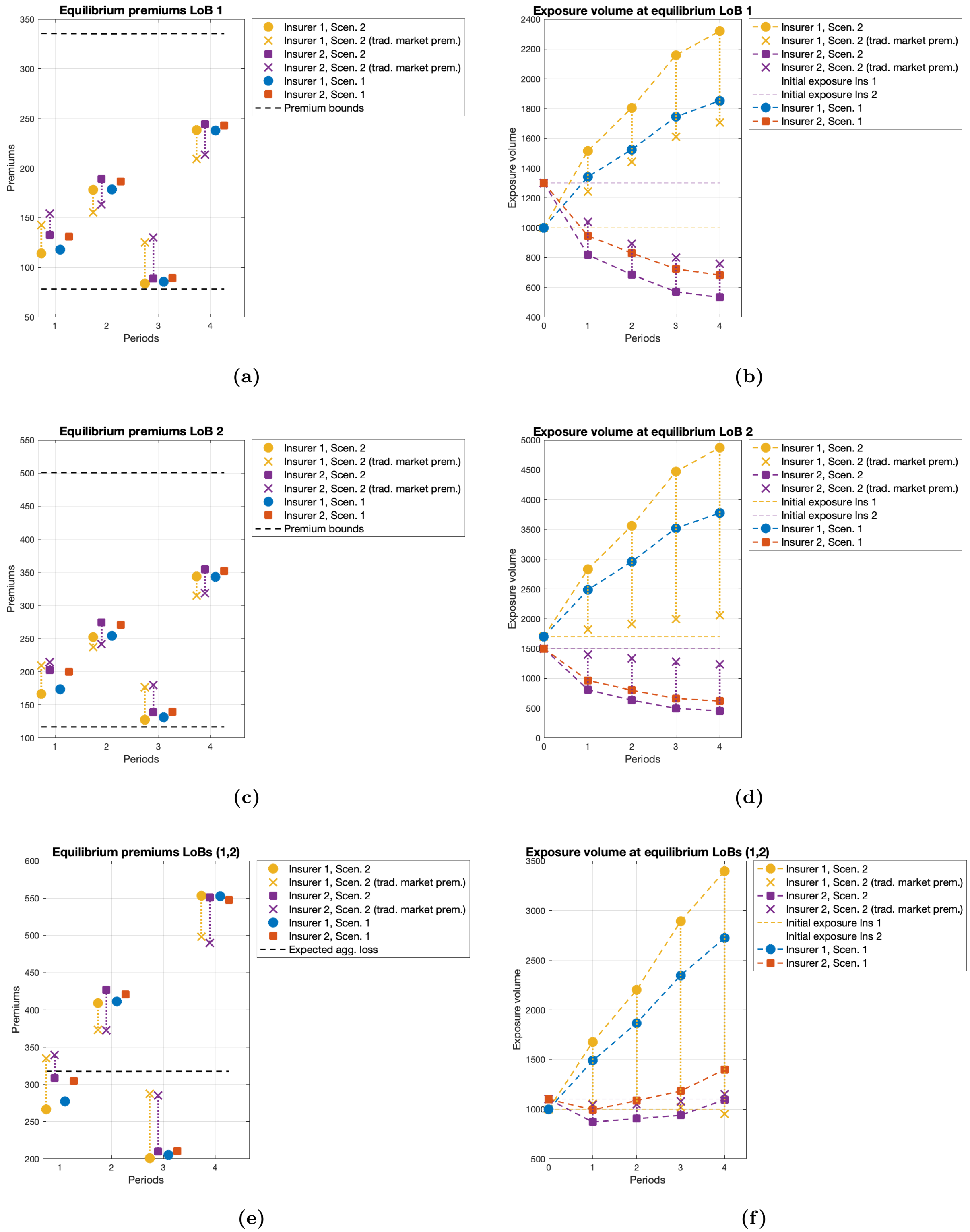


Figure 1: Scenario 2: Figs. (a), (c) and (e) show the PSNE premium profiles per contracts (1), (2) and $(1,2)$, respectively, in a two-insurer two-LoB four-period insurance market, and Figs. (b), (d) and (f) show the associated exposure volumes, for Insurer 2's modified price-sensitivity parameters corresponding to Scenario 2. The solid markers represent the results obtained using the combinatorial market premium, and the x markers represent the results obtained using the traditional market premium. The PSNE results from Scenario 1 are also presented. The dashed lines in Figs. (a) and (c) delimit the feasible premium region of the single-policy contracts, and the dashed line in Fig. (e) illustrates the total expected loss per contract $(1,2)$. The horizontal dashed lines in Figs. (b), (d) and (f) illustrate the initial exposure volumes.

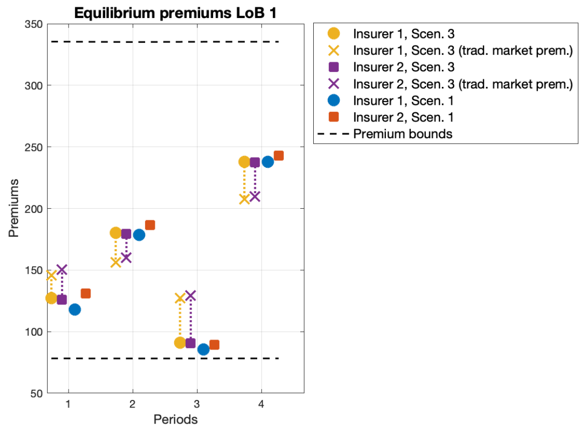
C.2 Scenario 3

In this scenario, we decrease the discount offered by Insurer 2 related to the double-policy contracts $(1,2)$. We assume that $\tilde{d}_2(2) = 6\%$, just 1% above Insurer 1's.

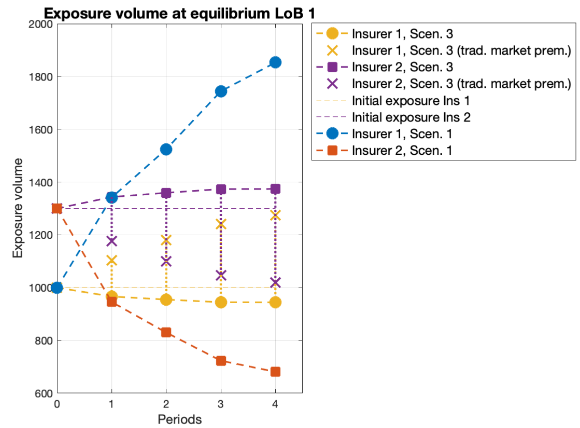
For the cases of the combinatorial and traditional market premium, Table 2 provides the PSNE premiums, while Figure 2 additionally displays the associated exposure volumes. Only for the case of the combinatorial market premium, the same figure contains the PSNE premiums and exposure volumes from Scenario 1.

Under the combinatorial market premium, in contrast to Scenario 1, Insurer 2's equilibrium premium for contract (1) is now slightly lower than the competitor's premium in all periods. As a result, Insurer 1 loses exposure volume of contract (1), while Insurer 2 gains an advantage over the entire time horizon. Nevertheless, the differences in premium prices for contract (1) among insurers are far smaller compared to those in Scenario 1. Consequently, the rate of change in insurers' exposure volume for contracts (1) is also relatively low, in contrast to the rate of change in Scenario 1. Specifically, in the case of LoB 2, the premiums charged by both insurers for single-policy contracts remain almost the same throughout all periods, resulting in no fluctuations in their volume of contracts (2). This can be mostly attributed to the lower difference in insurers' price-sensitivity parameters for contracts (2) compared to contracts (1) and $(1,2)$. Insurer 2 continues to provide a greater discount on double-policy contracts (1,2) compared to Insurer 1, with a difference of 1%. As a result, Insurer 2's total premium for contracts (1,2) is consistently lower than that of Insurer 1 across all time periods. When comparing to Scenario 1, the exposure volume of contracts $(1,2)$ for Insurer 2 is currently increasing in all periods and significantly above the level of Scenario 1. Simultaneously, Insurer 1 is experiencing an increase in the number of contracts (1,2) over all time periods, although at a slower rate and smaller overall magnitude compared to Scenario 1.

Let us now examine the case of the traditional market premium. Unlike the combinatorial market premium, Insurer 2's equilibrium premium values for contract (1) are higher than Insurer 1's in all periods. As a result, Insurer 1 gains exposure volume of contracts (1) while Insurer 2 loses. Regarding the single-policy contracts in LoB 2, we continue to observe that insurers' equilibrium premium values remain nearly similar, resulting in no major changes in insurers' exposure volume. In the case of the traditional market premium, Insurer 2's total premium for contracts (1,2) is slightly lower than Insurer 1's in all periods except period one. As a result, Insurer 2's exposure volume of contracts (1,2) is increasing relative to Insurer 1's in periods two to four, while Insurer 1 is experiencing a decrease in exposure volume during those periods. It is worth noting that the changes in insurers' exposure volumes are insignificant due to the small differences in their premiums.



(a)



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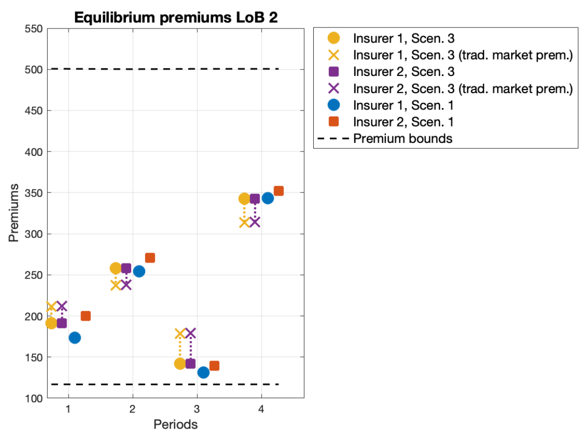

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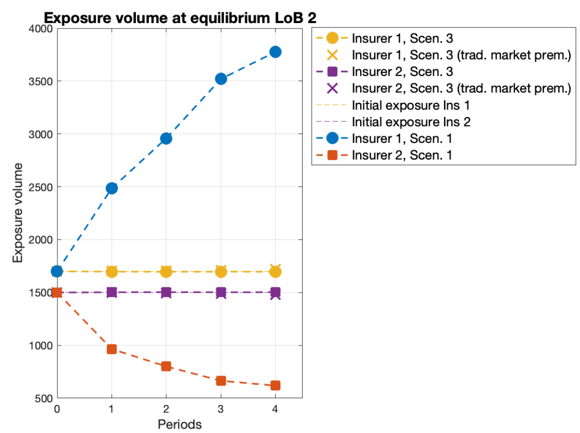

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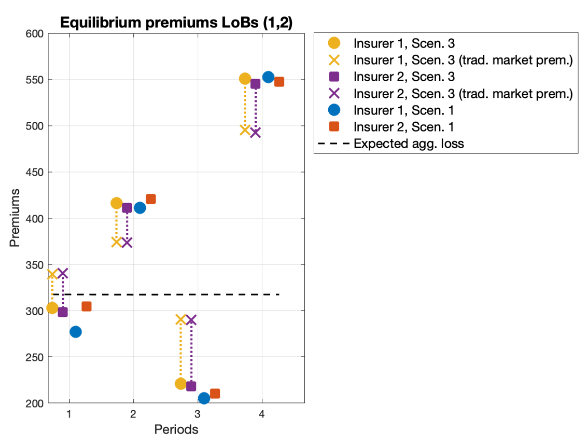
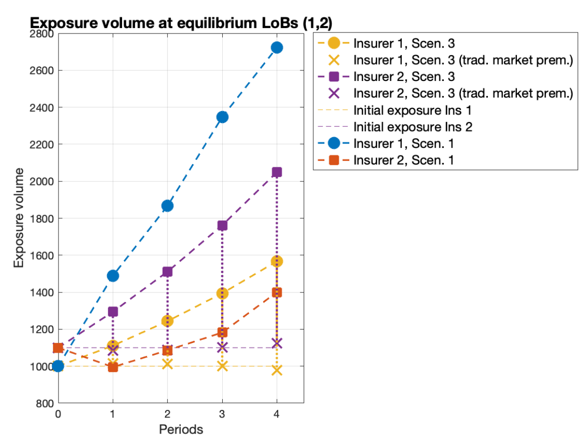


(e)



(f)

Figure 2: Scenario 3: Figs. (a), (c) and (e) show the PSNE premium profiles per contracts (1), (2) and (1, 2), respectively, in a two-insurer two-LoB four-period insurance market, and Figs. (b), (d) and (f) show the associated exposure volumes, when Insurer 2's discount function is modified as in Scenario 3. The solid markers represent the results obtained using the combinatorial market premium, and the x markers represent the results obtained using the traditional market premium. The PSNE results from Scenario 1 are also presented. The dashed lines in Figs. (a) and (c) delimit the feasible premium region of the single-policy contracts, and the dashed line in Fig. (e) illustrates the total expected loss per contract (1, 2). The horizontal dashed lines in Figs. (b), (d) and (f) illustrate the initial exposure volumes.

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	127.40	125.95	191.51	191.40	302.96	298.31
2	180.30	179.57	257.94	257.88	416.32	411.20
3	90.95	90.63	141.75	141.78	221.07	218.46
4	237.66	237.63	342.52	342.50	551.18	545.33

(a)

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	145.52	150.51	211.67	211.81	339.33	340.58
2	156.35	160.01	237.40	237.60	374.07	373.75
3	127.18	129.35	178.85	179.04	290.73	289.89
4	207.72	209.61	313.72	314.27	495.37	492.45

(b)

Table 2: Scenario 3: PSNE premium profiles in a two-insurer two-LoB four-period insurance market, when Insurer 2's discount function is modified. Table (a) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the combinatorial market premium. Table (b) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the traditional market premium.

C.3 Scenario 4

This scenario assumes that the insurance market will face adverse conditions during the following four periods, as it is anticipated that there will be an increase in both the frequency and severity of claims. Incorporating this into our model involves increasing the frequency of the environment process being in the third state, while keeping the other model parameters the same as in the baseline instance (Scenario 1). Therefore, we make the assumption that the initial distribution of the environment process is equal to $\tilde{A} = [0.10, 0.20, 0.70]$ and the transition matrix is provided as follows:

$$\tilde{\Pi} = \begin{bmatrix} 0.15 & 0.35 & 0.50 \\ 0.15 & 0.30 & 0.55 \\ 0.02 & 0.18 & 0.80 \end{bmatrix}.$$

For the combinatorial and traditional market premium, the PSNE premiums are given in Table 3. Figure 3 illustrates the PSNE premiums and exposure volumes for both characterizations of the market premium, as well as the PSNE results from Scenario 1 only for the combinatorial market premium.

Given that all the model inputs for both insurers remain unchanged from Scenario 1, except for the claim sizes, the equilibrium premium strategies of both insurers exhibit identical patterns as those seen in Scenario 1, regardless of how the market premium is characterized. The primary difference between them is in their respective magnitudes, with the premiums in this particular scenario being greater than those in Scenario 1 because of the increased anticipated payouts to those making claims. Due to differences in insurers' premium strategies, the insurers' trends in exposure volume per LoB are affected. However, when comparing these strategies to Scenario 1, we find that the insurers' exposure volumes align with those in Scenario 1.

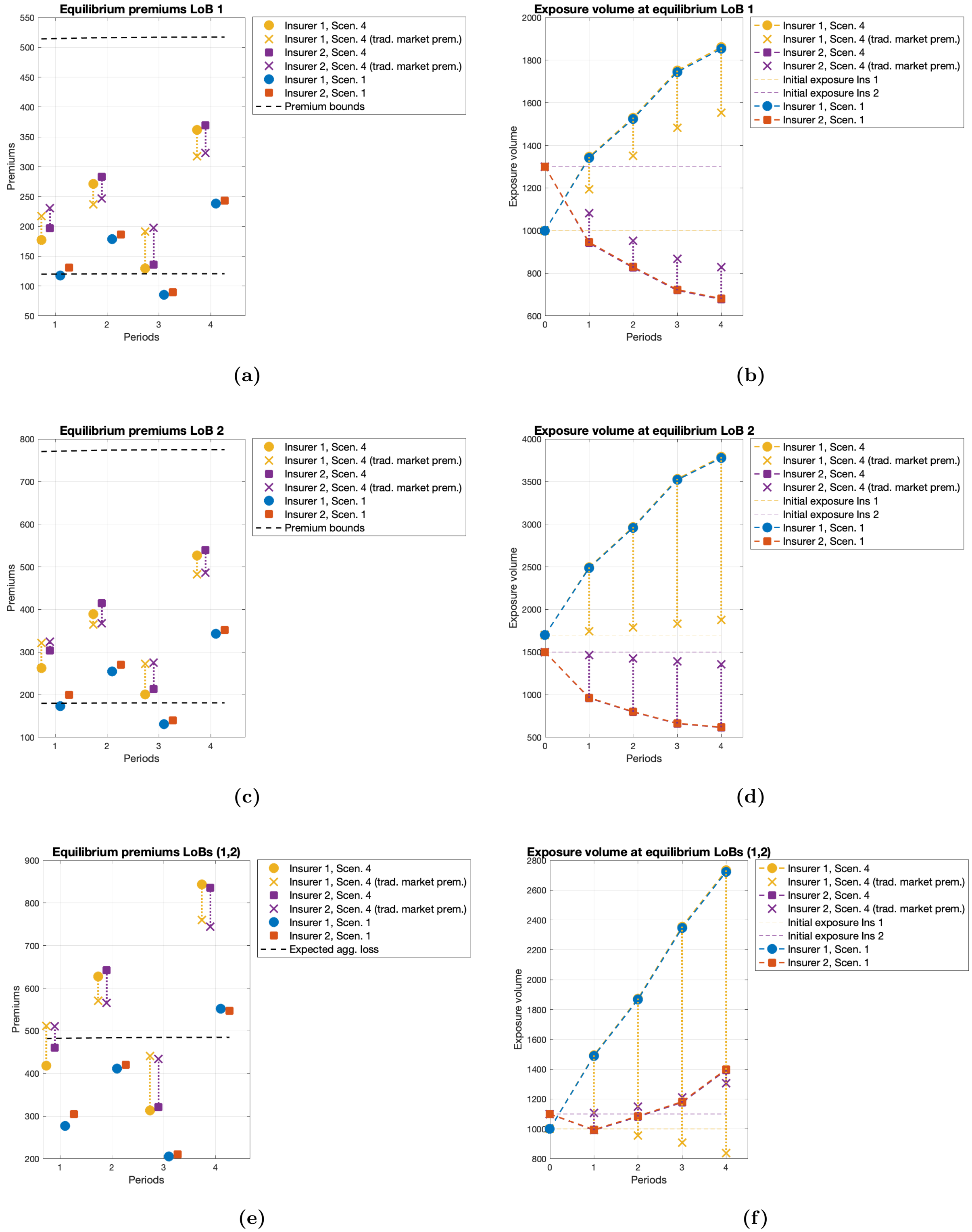


Figure 3: Scenario 4: Figs. (a), (c) and (e) show the PSNE premium profiles per contracts (1), (2) and $(1, 2)$, respectively, in a two-insurer two-LoB four-period insurance market, and Figs. (b), (d) and (f) show the associated exposure volumes, for the environment's modified transition probabilities as in Scenario 4. The solid markers represent the results obtained using the combinatorial market premium, and the x markers represent the results obtained using the traditional market premium. The PSNE results from Scenario 1 are also displayed. The dashed lines in Figs. (a) and (c) delimit the feasible premium region of the single-policy contracts, and the dashed line in Fig. (e) illustrates the total expected loss per contract $(1, 2)$. The horizontal dashed lines in Figs. (b), (d) and (f) illustrate the initial exposure volumes.

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	177.22	196.92	262.89	303.65	418.11	460.52
2	271.13	283.37	389.09	414.41	627.20	641.95
3	129.62	135.78	200.48	213.46	313.60	321.30
4	361.64	369.16	526.18	539.49	843.43	835.96

(a)

	LoB 1		LoB 2		LoB (1,2)	
Period	Ins 1	Ins 2	Ins 1	Ins 2	Ins 1	Ins 2
1	217.12	230.86	321.24	324.08	511.44	510.54
2	236.57	246.94	364.29	367.48	570.82	565.26
3	191.23	197.39	272.58	274.96	440.62	434.56
4	317.79	322.86	482.21	486.37	760.00	744.49

(b)

Table 3: Scenario 4: PSNE premium profiles in a two-insurer two-LoB four-period insurance market, when the initial/transition probabilities of the environment process are modified. Table (a) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the combinatorial market premium. Table (b) provides the equilibrium premium rates when insurers' competing premium rates are aggregated by the traditional market premium.

C.4 Percentage differences between the combinatorial- and traditional-market-premium PSNE outcomes

In all four scenarios, we evaluate the rate of change in the PSNE premiums and exposure volumes for the single- and double-policy contracts in every period, when they are evaluated using the combinatorial market premium instead of the traditional one. The subsequent figures present the ratios such that

$$\text{PSNE}_{\text{comb}} = (1 + \text{ratio})\text{PSNE}_{\text{trad}},$$

where PSNE stands for either the premiums or the exposure volumes per LoB.

Across all LoBs and scenarios, we consistently find that the PSNE premium values, when assessed using the combinatorial market premium, are lower than the traditionally evaluated counterparts during the odd periods of the time horizon and higher during the even times. This fact suggests that the differences in the premium values obtained by insurers in successive periods are greater in the case of combinatorial-market premiums compared to the traditional case.

For the single-policy contracts, in Scenarios 1, 2, and 4, Insurer 1 experiences a higher level of exposure volume with the combinatorial market premium compared to the traditional premium. On the other hand, Insurer 2's exposure volume with the combinatorial market premium is lower than its traditionally acquired level. Regarding the double-policy contracts in these scenarios, both insurers' strategies under the combinatorial market premium try to maximize their exposure volume. This is evident from the substantial greater levels of risk exposure for Insurer 1 when compared to the case of the traditional market premiums. Insurer 2's exposure volume of $(1, 2)$ under the combinatorial market premium is relatively close to its exposure levels under the traditional market premium.

In scenario 3, the reduction in Insurer 2's discount on the total premium for (1,2) has a contrasting impact on the exposure volumes compared to the other scenarios. Under the combinatorial market premium, Insurer 2 now has a higher exposure volume for single-policy contracts compared to its exposure volume under the traditional market premium. On the other hand, Insurer 1 has a lower exposure volume. Nevertheless, the insurers' levels of risk for the multi-policy contract $(1,2)$ are higher when using the combinatorial market premium compared to the traditional market price.

In the market with two LoBs and two insurers, our overall observation on the results obtained from the combined market premium is that the optimization process produces premium strategies for the insurers in order both to gain market share in their competition for the double-policy contracts.. This is In contrary to the traditional market premium; if one insurer loses market share in contract $(1,2)$, the competitor must gain market share.

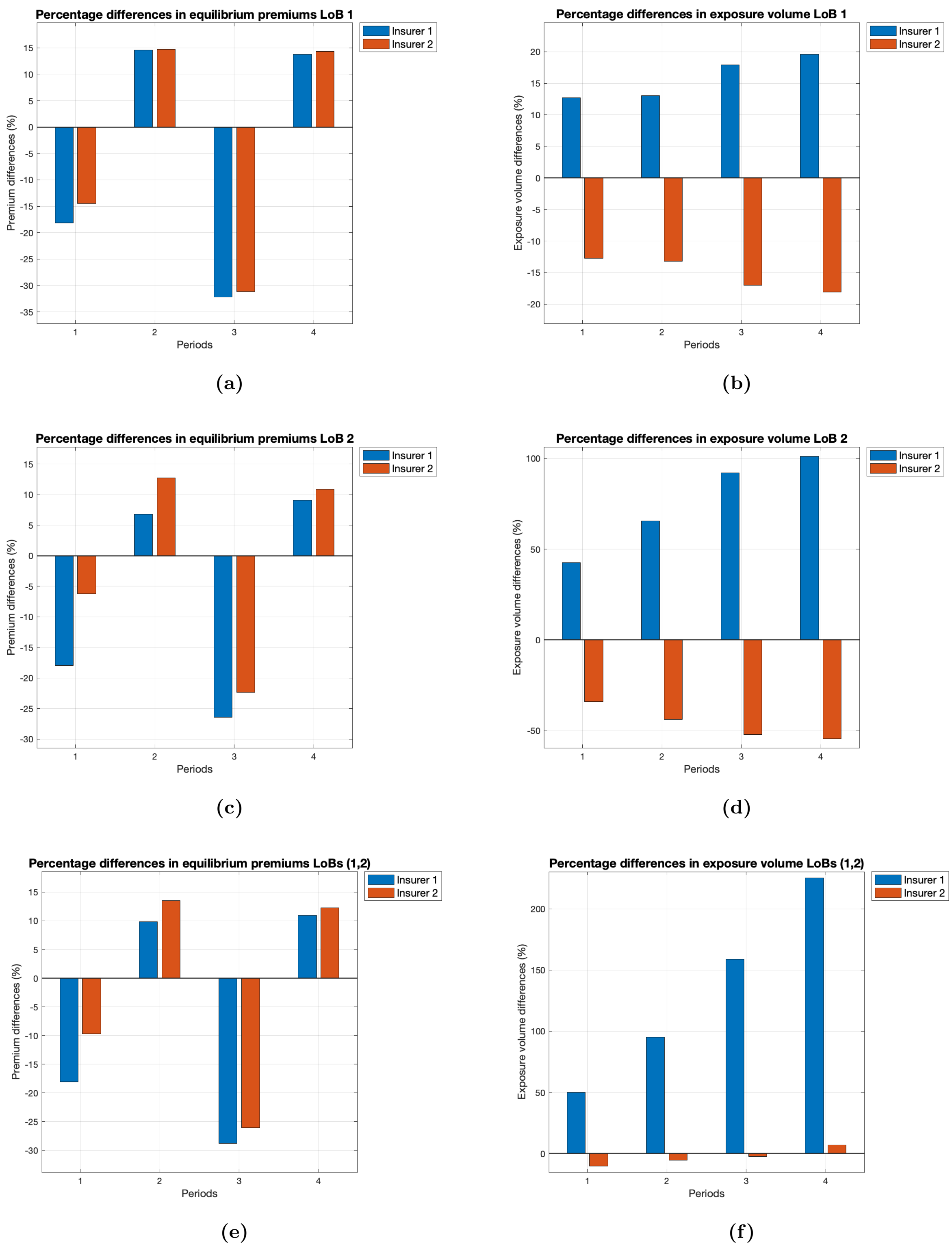
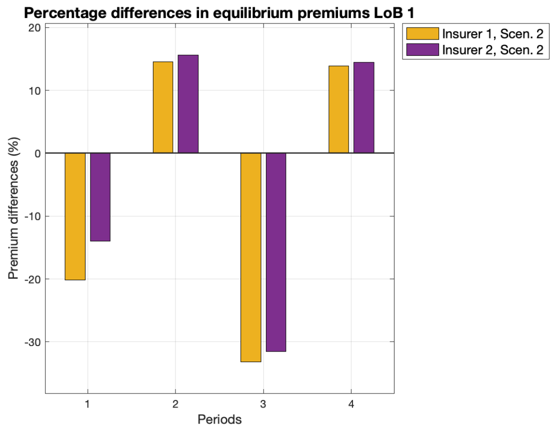
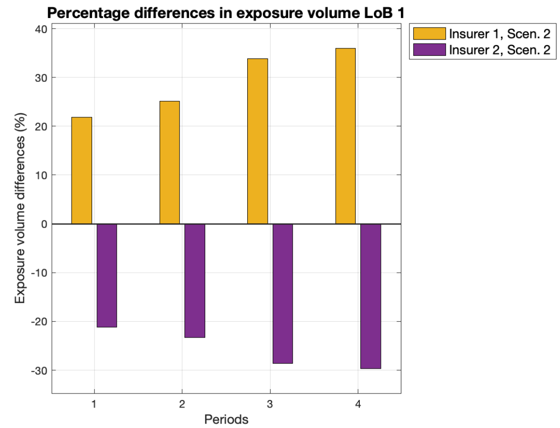


Figure 4: Scenario 1: The left-hand side displays the percentage difference in the PSNE premiums evaluated using the combinatorial market premium instead of the traditional one, namely, $(p^{*(m)}_{i,\text{comb}}(t)-p^{*(m)}_{i,\text{trad}}(t))/p^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\dots,4$. The right-hand side displays the percentage difference in the PSNE exposure volumes evaluated using the combinatorial market premium instead of the traditional one, namely, $(q^{*(m)}_{i,\text{comb}}(t) - q^{*(m)}_{i,\text{trad}}(t))/q^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\dots,4$.



(a)



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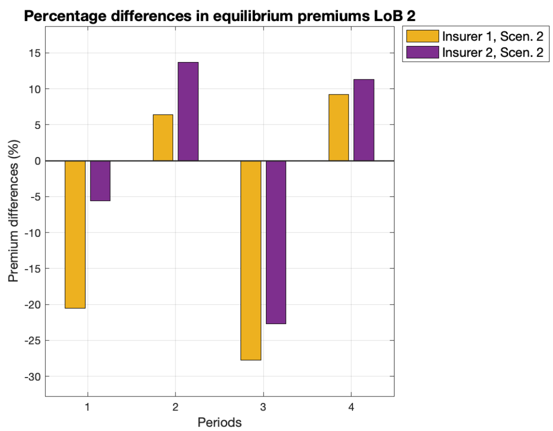

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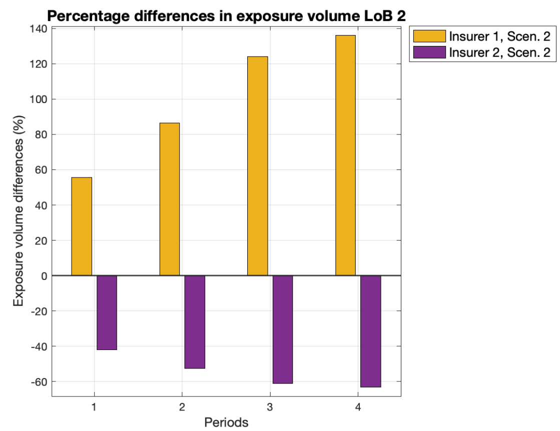
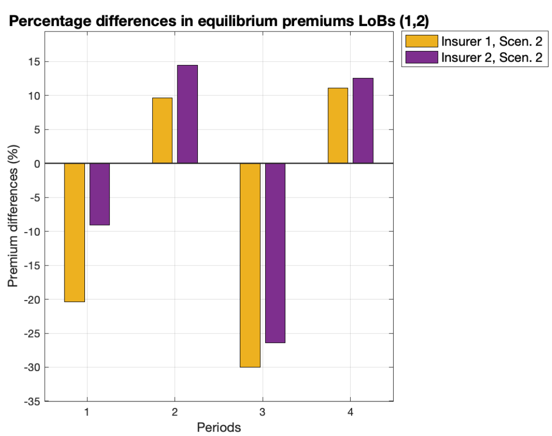
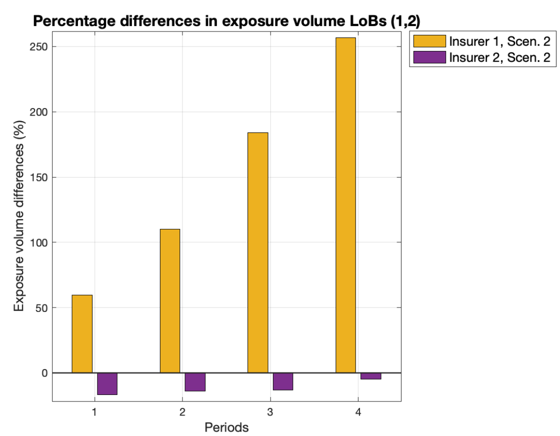


(d)



(e)



(f)

Figure 5: Scenario 2: The left-hand side displays the percentage difference in the PSNE premiums evaluated using the combinatorial market premium instead of the traditional one, namely, $(p^{*(m)}_{i,\text{comb}}(t)-p^{*(m)}_{i,\text{trad}}(t))/p^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\ldots,4$. The right-hand side displays the percentage difference in the PSNE exposure volumes evaluated using the combinatorial market premium instead of the traditional one, namely, $(q^{*(m)}_{i,\text{comb}}(t) - q^{*(m)}_{i,\text{trad}}(t))/q^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\ldots,4$.

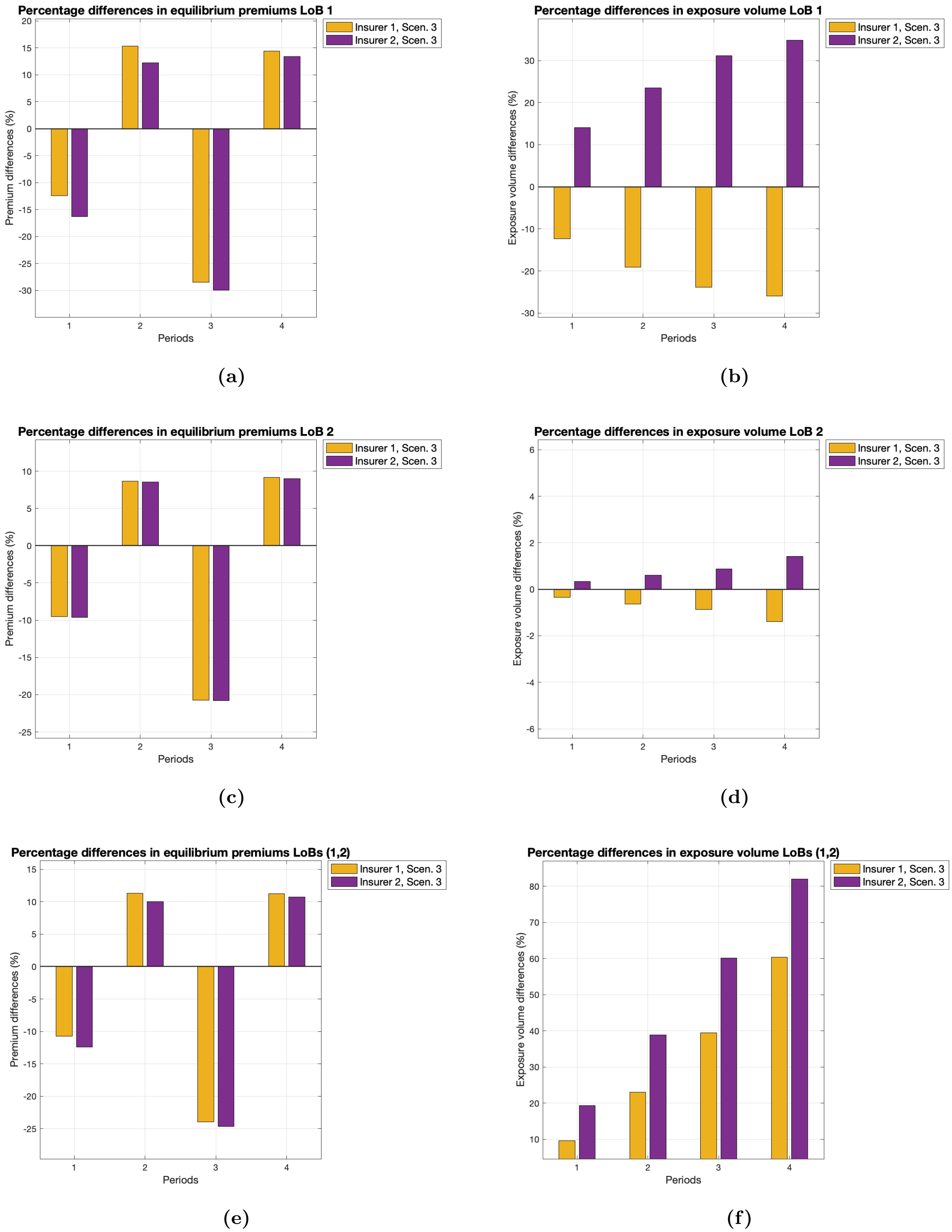


Figure 6: Scenario 3: The left-hand side displays the percentage difference in the PSNE premiums evaluated using the combinatorial market premium instead of the traditional one, namely, $(p^{*(m)}_{i,\text{comb}}(t) - p^{*(m)}_{i,\text{trad}}(t))/p^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1), (2), (1,2)\}$ and $t = 1, \ldots, 4$. The right-hand side displays the percentage difference in the PSNE exposure volumes evaluated using the combinatorial market premium instead of the traditional one, namely, $(q^{*(m)}_{i,\text{comb}}(t) - q^{*(m)}_{i,\text{trad}}(t))/q^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1), (2), (1,2)\}$ and $t = 1, \ldots, 4$.

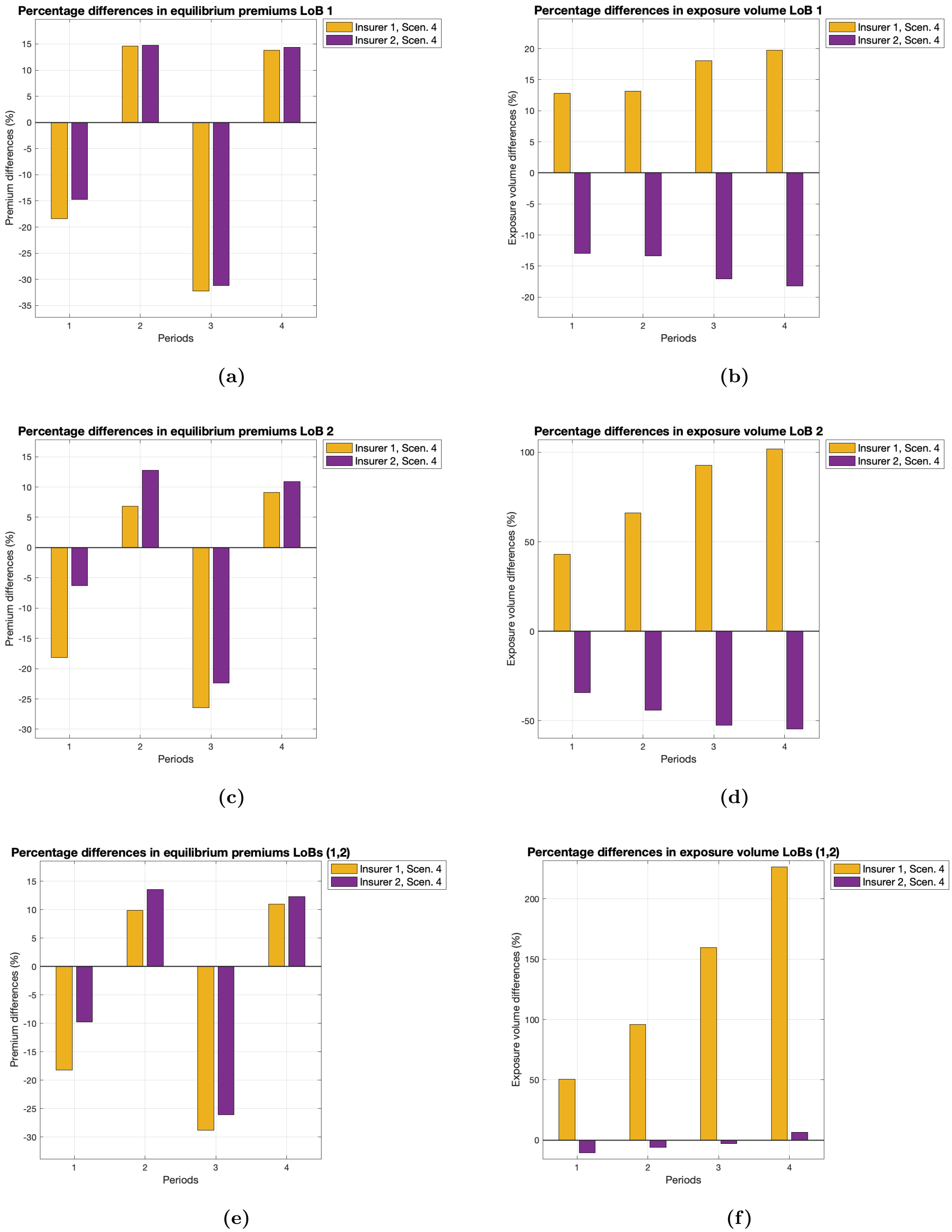


Figure 7: Scenario 4: The left-hand side displays the percentage difference in the PSNE premiums evaluated using the combinatorial market premium instead of the traditional one, namely, $(p^{*(m)}_{i,\text{comb}}(t) - p^{*(m)}_{i,\text{trad}}(t))/p^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\dots,4$. The right-hand side displays the percentage difference in the PSNE exposure volumes evaluated using the combinatorial market premium instead of the traditional one, namely, $(q^{*(m)}_{i,\text{comb}}(t) - q^{*(m)}_{i,\text{trad}}(t))/q^{*(m)}_{i,\text{trad}}(t)$, $i \in N$, $m \in \{(1),(2),(1,2)\}$ and $t = 1,\dots,4$.