Optimal Premium Pricing in a Competitive Stochastic Insurance Market with Incomplete Information: A Bayesian Game-theoretic Approach

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Abstract

This paper examines a stochastic one-period insurance market with incomplete information. The aggregate amount of claims follows a compound Poisson distribution. Insurers are assumed to be exponential utility maximizers, with their degree of risk aversion forming their private information. A premium strategy is defined as a mapping between risk-aversion types and premium rates. The optimal premium strategies are denoted by the pure-strategy Bayesian Nash equilibrium, whose existence and uniqueness are demonstrated under specific conditions on the insurer-specific demand functions. Boundary and monotonicity properties for equilibrium premium strategies are derived.

Keywords: Competitive Insurance Markets; Incomplete Information; Bayesian Nash Equilibrium; Combined Ratio.

JEL classification: G22; C72; C73; C11.

1 Introduction

1.1 Motivation

The insurance underwriting process, by its nature, is subjective and affected by various factors within the insurance market itself or the financial market in general. In addition to underwriting and operational costs, an insurer's premium must be sufficient to cover potential losses. In order to accomplish this, the actuarial (or technical) premium is computed using traditional premium principles, which rely heavily on various order moments and quantiles of the claim distribution (see, for example, Kaas et al. (2008)). However, a premium price must also be competitive. This means that insurers typically deviate from the actuarial premium in order to offer a more attractive premium and gain a competitive advantage. This deviation is an extremely challenging decision for an insurer since an aggressive underwriting strategy, i.e., extremely low prices, could compromise net profits when adverse claim experience arises, whereas a conservative insurance pricing would be outperformed by competitors resulting in a loss of market share.

An objective analysis of how insurers can optimally respond to competitors' actions dates back to Taylor (1986, 1987). Since then, there have been two mainstream optimization models: single-objective optimization problems and multi-person optimization problems. In the former, an insurer optimizes its pricing policy for some given exogenous beliefs about the competitors' prices, while the latter involves game-theoretic approaches. In a game-theoretic framework, each insurer tries to predict competitors' prices from some knowledge of the insurance market, while competitors choose their own prices based on their own predictions of the insurance market, including the insurer's price.

The majority of game-theoretic models in the insurance literature assume complete information. That is, all insurers are fully aware of the basic components of the insurance market, i.e., the set of insurers, the set of strategies available to each insurer and the payoffs of all insurers. This assumption may be hard to justify in practice, and this paper relaxes it. In particular, we question the complete information structure by assuming that insurers' payoffs depend on their risk appetite, which should not be disclosed to competitors.

1.2 Non-game-theoretic approaches

Observing a peculiar cyclical behavior of premium rates from insurers in the Liability section of the Australian insurance market within the decade 1973-1982, Taylor (1986) investigates the competitive nature of insurance underwriting in a deterministic discrete-time framework. He defines distinct forms of demand functions to capture the connection between premium rates and volume of exposure. Given the market average competitor premium rates, an insurer maximizes the expected discounted profit over a finite time horizon to derive the optimal premium rates. Taylor (1987) extends his initial model by considering the effect of expenses on optimal underwriting strategies and shows that optimal premium strategies might differ in the occurrence of non-constant expenses. Emms and Haberman (2005), Emms et al. (2007) and Emms (2007) further extend Taylor's deterministic model. Specifically, they apply stochastic continuous-time optimal control theory to characterize the optimal premium strategy that maximizes an insurer's expected terminal wealth. Pantelous and Passalidou (2013, 2015, 2017) develop a stochastic discrete-time model to capture the effect of an insurer's reputation on its premium strategy. Optimal premiums of the insurer are given as solutions to polynomial equations.

1.3 Game-theoretic approaches with complete information

The models mentioned in Section 1.2 involve single-objective optimisation problems and the key assumption in these models is that an insurer's premium strategy does not affect competitors' premiums. Abandoning this assumption leads to multi-person decision problems that focus on capturing how competitive pressures determine the pricing strategy of all insurers in the market using non-cooperative game theoretical procedures.[1] Emms (2012) considers insurers who are expected utility maximizers and characterizes the Nash equilibrium of an n-player, non-cooperative, deterministic or stochastic differential game, depending on whether the break-even premium is considered uncertain. Dutang et al. (2013) prove the existence and uniqueness of the Nash equilibrium as well as the existence of the Stackelberg equilibrium of a static (i.e., single-period) non-cooperative game. In their model, insurers' optimal premiums satisfy a solvency constraint and are the solutions to a maximization problem with a deterministic quadratic objective function. Furthermore, the transition probabilities from one insurer to another are modelled by a multinomial logit model and the aggregate claim amount follows a compound Poisson or negative binomial distribution. In Wu and Pantelous (2017), optimal premium strategies are determined by Nash equilibria in an n-player potential game in which the market average competitor premium is assessed by aggregating all of the market's paired competitions. Boonen et al. (2018) determine the open-loop Nash equilibrium premium strategies in an n-player differential game utilizing optimal control theory procedures. Asmussen et al. (2019) consider the customer's problem with market frictions and develop a stochastic differential game between two insurance companies of varying sizes. Mourdoukoutas et al. (2021) study a competitive stochastic non-life insurance market with exponential utility maximizing insurers. The total loss amount of insurers is described by the collective risk model, where the number of policies follows either a Poisson or negative binomial distribution. The connection between insurers' premiums and volume of exposure is captured by two distinct exponential demand functions. When the demand function is concave, the existence and uniqueness of a pure strategy Nash equilibrium premium profile in a single-period stochastic game are demonstrated.

In all the models mentioned above, insurers have the same set of information about

[1] Also in reinsurance markets, the premium competition can be modelled via non-cooperative game theory. In reinsurance, the pricing is often represented by a pricing kernel, and insurers can respond by selecting very general reinsurance indemnities. Selecting the premium kernel in a competitive market with multiple reinsurers has been studied recently by Zhu et al. (2023), who consider sub-game perfect Nash equilibria: a well-known dynamic refinement of Nash equilibria.

the insurance market. The departure from this information structure in the insurance market is the starting point for our current paper.

However, economic equilibrium concepts under asymmetric information have a long history in the insurance economics literature. There, the asymmetric information appears within the context of moral hazard (i.e., insureds partly control the contract outcome and the insurers cannot monitor to which extent a reported loss is attributable to insureds' behavior) and adverse selection (i.e., the risks associated with insureds are heterogeneous and cannot be predetermined by insurers so as all insureds to be charged the same premium rate). For the origins and first developments of asymmetric information, we refer the reader to Dionne (2000).[2]

1.4 Our model: Game-theoretic approach with incomplete information

This paper develops a model to analyse not only how insurers interact with one another, but also how this interaction affects the volume of their exposure. In particular, we employ a game-theoretic approach to model insurance market competition. Instead of analyzing an insurer's optimal response to a projected market price unaffected by the insurer's premium, we calculate the equilibrium premiums of all insurers in the market. In a state of equilibrium, all insurers select the optimal response to their competitors' actions, and no insurer has an incentive to deviate from this state.

Our game design assumes that insurers are risk averse and use utility functions to capture their risk aversion. In addition, we introduce uncertainty into our game by

[2] Arrow (1963) introduces asymmetric information attributable to moral hazard and adverse selection. Early developments on moral hazard can be found at Pauly (1968), Spence and Zeckhauser (1971), Marshall (1976), Shavell (1979, 1982, 1986), Dionne (1982), Stiglitz (1983), Arnott and Stiglitz (1991) and Arnott (1992). A development of great importance on adverse selection is the paper of Rothschild and Stiglitz (1976). They analyse a single-period competitive insurance market in which insurers provide a menu of contracts defined in terms of price and quantity. Insureds are classified into good risks and bad risks, and each group has private information on its probability of accident. Under adverse selection, an equilibrium exists if the proportion of good risks is not large enough. In equilibrium, the individuals with bad risks accept full insurance at a high price, whereas partial coverage at a low price is provided to the individuals with good risks. Extensions to multi-period insurance contracts can be found at Wilson (1977), Spence (1978) and Riley (1979).

assuming that an insurer's risk aversion is private information, i.e., it is known only to that insurer and not to its competitors. In this incomplete-information insurance market, insurers' returns depend not only on their premium selections but also on their risk profiles. Therefore, an insurer's payoff function is not observed by its competitors. It is common knowledge that there is a prior joint distribution over all insurers' risk-aversion types. Each insurer observes only its own risk-aversion type and, based on the prior joint distribution, maximizes its expected conditional payoff given its risk-aversion type. As a result, the equilibrium premium strategy maps each insurer's risk-aversion type to optimal premium options based on that insurer's beliefs about its competitors' risk-aversion types.[3]

The equilibrium concept invoked in our paper is the one defined by Harsanyi. Particularly, Harsanyi (1967, 1968) introduces a prior move by nature that determines players' types and transforms the incomplete information about players' types into imperfect information about nature's moves. Harsanyi defines the Bayesian Nash equilibrium by assuming that all players have the same prior beliefs about the probability distribution on nature's moves.

The paper is organized as follows. In Section 2, we define the model setup. In particular, we define the utility function of insurers, the characteristics of the demand function, the model for the aggregate claim amount, the objective of an insurer and the feasible region for the insurer's premiums. Section 3 proves the existence and uniqueness of a pure-strategy Bayesian equilibrium under two conditions on the insurer-specific demand function. Boundary and monotonicity properties for the equilibrium

[3]The concept of an information structure commonly appears in anonymous games as well. In such games, it is also assumed that the distribution of players is common knowledge while no one can see each other individually. However, the main feature in anonymous games is that a player's utility depends only on the player's strategy and the total number/quantity of players choosing the same strategy. In this sense, the only thing that matters is how the strategies are distributed over the set of players rather than the strategy profiles per se specifying the strategy of each individual. In other words, the players' identities do not affect the game. These games have numerous economic applications and some examples are the cases of network or congestion externalities. We refer interested readers to Milchtaich (1996) and Blonski (1999) for more information about anonymous games.

In this paper, the payoff function of an insurer depends on the premium strategy profile of all insurers in the market, since each individual competitor's strategy does affect the insurer's payoff. The identity of insurers cannot be disregarded as it is in anonymous games.

premium strategies are provided. Section 4 illustrates the theoretical results, and Section 5 concludes. The proofs are delegated to Appendix A.

2 Structure of insurance market

We consider an insurance market with n insurers who offer single-period policies in a non-life line of business. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ denote the set of insurers. At the beginning of the period, the insurers select the premium rates, which affect the expected number of policies that will be gained or lost over the one-period horizon. Let $p = (p_i)_{i \in \mathcal{N}} \in \mathbb{R}^n_+$ denote a non-negative premium vector, where p_i is the premium per policy set by insurer i.

It is common knowledge to all insurers that each one sets premium rates so as to maximize the expected utility of their terminal net wealth. However, we assume that an insurer's expected utility is not observed by competitors and depends on the insurer's sensitivity to risk. In particular, we assume that insurers have various private risk-aversion types that determine their premium decisions. Only a prior joint probability distribution over the risk-aversion types is common information to all insurers.

In the following subsections, we define the insurers' risk preferences, loss model and disutility function. Then, we present the incomplete-information insurance market and provide conditions for the existence and uniqueness of a *pure-strategy Bayesian Nash equilibrium*. Boundary and monotonicity properties for the equilibrium premium strategies follow.

2.1 Risk preferences of insurers

Similar to Emms (2012) and Mourdoukoutas et al. (2021), we assume that insurers are endowed with a utility function with constant absolute risk aversion. To be precise, insurers are exponential utility maximizers, and the utility function of insurer i is given by

$$u_i(x) = -\exp\{-\lambda_i x\}, \tag{1}$$

where $\lambda_i > 0$ denotes the risk-aversion parameter of insurer i.

In this paper, we mainly deviate from the corresponding literature by assuming that the parameter λ_i is private information to insurer i. In other words, when insurers set their premium rates concurrently at the beginning of the period, they are unable to observe their competitors' risk-aversion parameters. They only know their own risk-aversion parameter, and based on this private knowledge, they form a belief about their competitors' risk aversion.[4]

This privacy of information naturally leads to the representation of an insurer's risk aversion by a random variable. Let the positive real-valued random variable Λ_i denote insurer i's risk aversion and $\mathcal{L}_i$ be the space of risk-aversion types, i.e., the range of Λ_i. Insurer i knows only its own risk-aversion type Λ_i, whereas it can only form beliefs on its competitors' risk-aversion types on the basis of a prior joint probability distribution of the vector $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$. The corresponding joint probability density function is denoted by $g(\lambda_1, \ldots, \lambda_n)$. We consider only finite risk-aversion type spaces $\mathcal{L}_i$ to avoid measurability technicalities. For each insurer i, we define $\underline{\lambda}_i = \min \mathcal{L}_i$ and $\bar{\lambda}_i = \max \mathcal{L}_i$. The following two main assumptions are imposed in our analysis:

(A1) All insurers know the joint probability density function $g(\lambda_1, \ldots, \lambda_n)$, i.e., g is common knowledge to all insurers.

(A2) For each insurer i and every $\lambda_i \in \mathcal{L}_i$, the probability density function g_i is strictly positive. That is, $g_i(\lambda_i) := P(\Lambda_i = \lambda_i) > 0$ for all $\lambda_i \in \mathcal{L}_i$ and all $i \in \mathcal{N}$.

At the beginning of the period and before setting the premium values, insurer i knows only its own risk-aversion parameter λ_i. Conditioning on that knowledge, insurer i updates the belief about competitors' risk behavior by using Bayes' rule:

$$g_i(\lambda_{-i}|\lambda_i) := P(\{\Lambda_j = \lambda_j\}_{j \neq i} | \Lambda_i = \lambda_i) = \frac{g(\lambda_1, \ldots, \lambda_n)}{g_i(\lambda_i)},$$

[4]Modelling reinsurance markets with unobserved risk preferences was studied by Anthropelos and Kardaras (2017) and Anthropelos and Boonen (2020). While they focus on a mechanism in which agents strategically disclose their risk preferences, our focus is on the beliefs of the competitors' risk preferences in a pure-strategy Bayesian Nash equilibrium.

where $g_i(\lambda_{-i}|\lambda_i)$ is the conditional density function of $(\Lambda_j)_{j\neq i}$ conditional of $\Lambda_i = \lambda_i$. Here, $\lambda_{-i} = (\lambda_j)_{j\neq i}$ is the risk-aversion parameter vector of all insurers in $\mathcal{N}$ except insurer i. Notice that when Λ_i, $i \in \mathcal{N}$, are independent, there is no need for insurers to update their belief about competitors' risk-aversion parameter, and then $g_i(\lambda_{-i}|\lambda_i) = \prod_{j\neq i} g_j(\lambda_j)$.

2.2 Competition model

Let $N_i(p)$ denote the number of policyholders of insurer i realized at the end of the period. It is a random variable whose expected value, also referred to as exposure volume, is influenced by the premium profiles of all market insurers. Next, we present the demand function, which links insurers' premium rates with exposure volumes, and mention its basic characteristics.

In reality and in the existing literature, policyholders' choices for an insurer are related to their choices over successive time periods. As in Taylor (1986), we assume that the expected number of policyholders at the end of the period is proportional to the number of policyholders at the beginning of the period. Let $p_{-i} = (p_j)_{j\neq i}$ be a premium vector of all insurers' premium rates except insurer i. Intending to focus on insurer i in a game-theoretical model, it is useful to decompose the premium profile $p = (p_j)_{j\in\mathcal{N}}$ as $p = (p_i, p_{-i})$. Now, $q_i(p_i, p_{-i})$ denotes the expected number of policyholders of insurer i at the end of the period, and it is assumed that

$$q_i(p_i, p_{-i}) = f_i(p_i, \bar{p}_{-i})q_{i,0}, \tag{2}$$

where $q_{i,0}$ is the current number of policyholders, i.e., at the beginning of the period. We consider already existing insurers who offer the insurance product under question and currently possess a positive share of the market, i.e., $q_{i,0} > 0$ for all $i \in \mathcal{N}$.

Here, $f_i(p_i, \bar{p}_{-i})$ is a price-sensitivity function that denotes the relative change in the expected number of policyholders of insurer i. In line with Taylor (1986, 1987), it depends on the comparison of p_i with the aggregate competitor premium $\bar{p}_{-i}$ defined as the average premium of insurer i's competitors, i.e.,

$$\bar{p}_{-i} = \frac{1}{n-1} \sum_{j \neq i} p_j.$$

It is assumed that the insurance product in question displays positive price elasticity of demand. This means that any attempt by insurer i to gain by increasing its premium value while competitors underwrite at a loss will result in a reduction of its exposure volume. Therefore, $f_i(p_i, \bar{p}_{-i})$ is considered a decreasing function of p_i. Assuming that $f_i(p_i, \bar{p}_{-i})$ is infinitely differentiable with respect to p, given p_{-i}, for all p_i it holds

$$\frac{df_i(p_i, \bar{p}_{-i})}{dp_i} < 0. \tag{3}$$

As in Mourdoukoutas et al. (2021), a Poisson distribution with intensity equal to $q_i(p_i, p_{-i})$ is used to model the actual number of policyholders of insurer i, i.e.,

$$N_i(p) \sim \text{Poisson}(q_i(p_i, p_{-i})). \tag{4}$$

2.3 Loss model

Let $X_{i,k}$ be a random variable with non-negative values that denotes the total claim amount associated with policy k of insurer i during the coverage period. It is assumed no adverse selection among policyholders, i.e., the underlying risks faced by the policyholders of every insurer are homogeneous. Thus, $(X_{i,k})_k$ and $N_i(p)$ are independent, and $((X_{i,k})_k)_{i \in \mathcal{N}}$ are independent and identically distributed (i.i.d.). Let $M_X(t)$ denote the moment generating function of an individual claim amount X that is identically distributed as $X_{i,k}$. We impose the following assumption:

(A3) $M_X(\bar{\lambda}_i) < \infty$ for all $i \in \mathcal{N}$.

Using a frequency-average severity loss model, the aggregate claim amount for insurer i follows a compound Poisson distribution given by

$$S_i(N_i(p)) = \sum_{k=1}^{N_i(p)} X_{i,k},$$

with $N_i(p)$ as defined in (4).

2.4 Disutility function

In this section, we present the wealth of the insurers and how it generates the insurers' disutility functions. Given an insurer's risk-aversion type, the insurer's strategy for determining premium values is to minimize the conditional expected disutility. Given the premium vector $p = (p_i, p_{-i})$, insurer i's wealth at the end of the period is a random variable defined as

$$W_i(p_i, p_{-i}) = -a_i \pi_i + (1 - a_i) \left[p_i N_i(p_i, p_{-i}) - S_i(N_i(p_i, p_{-i})) \right], \tag{5}$$

where π_i denotes the deterministic initial reserve, and $a_i \in (0, 1)$ represents the expense rate of insurer i holding wealth. As we can see, insurers' capital at the end of the period is equal to the premium income collected by all policies minus the total policy claims and cost of holding capital.

We have assumed that insurers display risk aversion towards the uncertainty of the underwriting strategy and are exponential utility maximizers. That is, insurer i of risk-aversion type λ_i chooses the premium value p_i so as to maximize the expected utility of the wealth. Therefore, the objective function that insurer i of risk-aversion type λ_i maximizes is equal to

$$\begin{aligned} o_i(p_i, p_{-i}; \lambda_i) &= \mathbb{E}\left[u_i \left(W_i(p) \right) \right] \\ &= \mathbb{E}\left[\mathbb{E}\left[-\exp\left\{ -\lambda_i \left[-a_i \pi_i + (1 - a_i) \left(p_i N_i(p) - S_i(N_i(p)) \right) \right] \right\} | N_i(p) \right] \right]. \end{aligned}$$

From the independence of $(X_{i,k})_k$ and $N_i(p)$, and the fact that $((X_{i,k})_k)_{i \in \mathcal{N}}$ are i.i.d.

as X, we readily obtain

$$o_i(p_i, p_{-i}; \lambda_i) = -\exp\{\lambda_i a_i \pi_i\}\mathbb{E}\left[\left(e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i))\right)^{N_i(p)}\right].$$

From the distribution of $N_i(p)$ given in (4) and following Proposition 4 in Mourdoukoutas et al. (2021), the objective function of insurer i of risk-aversion type λ_i can be written as

$$o_i(p_i, p_{-i}; \lambda_i) = -\exp\left\{\lambda_i a_i \pi_i + q_i(p_i, p_{-i})\left[e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1\right]\right\},$$

for all p_i that satisfy

$$e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1 \leq 0. \tag{6}$$

Here, (6) is an individual rationality constraint, and implies that the insurer is better off by providing insurance than by withdrawing from the insurance market.

Maximizing $o_i(p_i, p_{-i}; \lambda_i)$ is equivalent to minimizing $\log\left(-o_i(p_i, p_{-i}; \lambda_i)\right)$, which results in the following disutility (cost) function:

$$c_i(p_i, p_{-i}; \lambda_i) = \lambda_i a_i \pi_i + q_i(p_i, p_{-i})\left[e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1\right]. \tag{7}$$

The definition of the disutility function in (7) assumes knowledge of competitors' premium profiles p_{-i}. However, as we will demonstrate analytically in Section 3, insurers' premium decisions are contingent on their risk-aversion parameters. To illustrate this relationship, we define premium strategies as mappings between risk-aversion types and premium choices. In an abuse of notation, $p_j(\Lambda_j)$ represents the premium rate of an insurer j whose risk-aversion type is Λ_j. Now, insurer i does not observe competitors' risk preferences but instead forms a belief about them based on its own risk-aversion parameter λ_i. Therefore, if insurer i of risk-aversion type λ_i knows the competitors' premium strategies $p_{-i}(\cdot) = (p_j(\cdot))_{j\neq i}$, then insurer i minimizes the conditional expected disutility function given λ_i, i.e., $\mathbb{E}\left[c_i(p_i, p_{-i}(\Lambda_{-i}); \lambda_i)|\Lambda_i = \lambda_i\right]$, with respect to the conditional probability law $g_i(\lambda_{-i}|\lambda_i)$.

2.5 Feasible premium region

The insurance market is tightly regulated, with the purpose of this oversight being to protect policyholders from insurer insolvency. The combined ratio is an important factor for insurers, regulators, shareholders, and policyholders. It serves as an indicator of insurers' financial health and provides insight into their underwriting policies. Further, the combined ratio reveals the percentage of earnings that is used for claims and underwriting costs like salaries, commissions, etc. The requirement that the combined ratio of an insurer i for each policy be at most equal to a given percentage, cr_i, imposes an inequality constraint. Thus, the premium rate of insurer i should satisfy

$$r_i^S(p_i) = \frac{\mu_X + e_i}{p_i} - cr_i \leq 0, \tag{8}$$

where μ_X and e_i are the expected claim amount and underwriting expenses per policy, respectively. Let us mention that the combined ratio might exceed 1 since excess losses and expenses can be offset by investment income, which is not accounted for in the calculation of the combined ratio. So, $cr_i \in [CR_i^L, CR_i^U]$, with $0 < CR_i^L < 1 < CR_i^U$.

Next, we aim to define a lower and upper bound for insurers' premium rates. Initially, these bounds can be interpreted as the minimum and maximum premium values controlled by external regulatory bodies. However, another interpretation is based on the behavioral aspect of both insurers and insureds whose risk preferences are characterized by the exponential utility function given in (1). For the latter interpretation, we refer to Mourdoukoutas et al. (2021).

Regarding the lower bound, we saw that the premium rate of insurer i of risk-aversion type λ_i should satisfy inequality (6), i.e.,

$$r_i^L(p_i) = e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1 \leq 0, \tag{9}$$

which is equivalent to

$$p_i \geq p_i^L(\lambda_i) := \frac{1}{\lambda_i(1-a_i)} \log M_X(\lambda_i(1-a_i)). \tag{10}$$

Therefore, $p_i^L(\lambda_i)$ is the minimum premium value that makes insurer i, with risk-aversion parameter λ_i, indifferent to underwriting insurance. Note that the lower bound is based on insurer i's risk-aversion type. However, as we shall see later, every risk-aversion type of insurer i is considered a distinct player who engages in competition with every risk-aversion type of insurer i's competitors.

For the upper premium bound, consider the most risk-averse individual in the market with risk-aversion parameter h_i contingent on insurer i, and let p_i^U denote the maximum premium value that the individual is willing to purchase. In contrast to Mourdoukoutas et al. (2021), the dependence of an individual's risk aversion on the insurer suggests that the individual might have different premium tolerances for insurers' underwriting strategies. That is, factors such as an insurer's market power and reputation can influence a person's willingness to purchase. Now, let $h_i > \max_{j \in \mathcal{N}} \bar{\lambda}_j > 0$, and assume that the moment generating function of X evaluated at h_i exists for all $i \in \mathcal{N}$. Then, each insurer i's premium rate satisfies the inequality constraint

$$r_i^U(p_i) = e^{h_i p_i} M_X^{-1}(h_i) - 1 \leq 0, \tag{11}$$

which is equivalent to

$$p_i \leq p_i^U := \frac{1}{h_i} \log M_X(h_i). \tag{12}$$

We have that $b(t) = t^{-1} \log M_X(t)$, $0 < t \leq h_i$, is an increasing function of t, see Appendix A.1. Given that $a_i \in (0,1)$ and $h_i > \max_{j \in \mathcal{N}} \bar{\lambda}_j > 0$, we get $p_i^L(\lambda_i) < p_i^U$ for all $i \in \mathcal{N}$ and every $\lambda_i \in \mathcal{L}_i$. Therefore, the premium range for insurer i of risk-aversion type λ_i is a non-empty closed interval denoted by $\mathcal{P}_i(\lambda_i) = [p_i^L(\lambda_i), p_i^U]$. Thus, we arrive at the following assumption, which states the insurers' premium range:

(A4) The feasible premium region of insurer i of risk-aversion type λ_i is defined by

$$\mathcal{R}_i(\lambda_i) = \left\{ p_i \in \mathcal{P}_i(\lambda_i) | r_i^S(p_i) \leq 0 \right\}, \tag{13}$$

and $\mathcal{R}_i(\lambda_i) \neq \{\emptyset\}$ for all $\lambda_i \in \mathcal{L}_i$.

The implication of Assumption (A4) is that the premium range of each insurer is a compact set. Remark that $\mathcal{R}_i(\lambda_i) \neq \{\emptyset\}$ holds true if and only if $p_i^U \geq \frac{\mu_X + e_i}{cr_i}$.

3 Equilibrium concept and results

We investigate an insurance market with incomplete information in which each insurer knows its own private risk-aversion level that is not observable by competitors. All insurers select their premium rates simultaneously at the beginning of the period in order to respond optimally to competitors' premium strategies, whereas the premiums selected by insurers are dependent on their risk-aversion parameters. With an abuse of notation, let $p_i : \mathcal{L}_i \to \mathcal{R}_i$ denote the premium strategy of insurer i, which is actually a map from risk-aversion types $\mathcal{L}_i$ to insurer i's premium choices $\mathcal{R}_i = \mathcal{R}_i(\underline{\lambda}_i)$.[5] Therefore, given λ_i, insurer i is not able to just minimize the disutility function $c_i(p_i, p_{-i}; \lambda_i)$, given in (7), with respect to p_i. The reason is that competitors' premium decisions are based on their risk-aversion parameters, which insurer i cannot observe. A reasonable approach for insurers is to evaluate the conditional expected disutility, based on their risk-aversion type and the premium strategy profile of their competitors. This is stated formally in the definition that follows.

In the Bayesian insurance market considered here, any insurer with a specific risk-aversion parameter may be viewed as a single player competing against the other insurers without knowing their exact type. Therefore, given the premium strategy profile, $p_{-i}(\cdot)$, of all insurers other than i, the payoff function of insurer i for risk-

[5]Notice that $\mathcal{R}_i = \bigcup_{\lambda_i \in \mathcal{L}_i} \mathcal{R}_i(\lambda_i) = \mathcal{R}_i(\underline{\lambda}_i)$. This is due to the fact that $p_i^L(\underline{\lambda}_i) \leq p_i^L(\lambda_i)$ for all $\lambda_i \in \mathcal{L}_i$. The last inequality holds because $b(t) = t^{-1} \log M_X(t)$ is increasing as shown in Appendix A.1 and hence, we derive from (10) that $p_i^L(\lambda_i) = b(\lambda_i(1 - a_i))$ is increasing in λ_i since $a_i \in (0, 1)$.

aversion type λ_i is equal to its conditional expected disutility function, constrained by λ_i. Equation (7) yields

$$\begin{aligned} C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) &= \mathbb{E}\left[c_i(p_i, p_{-i}(\Lambda_{-i}); \lambda_i)|\Lambda_i = \lambda_i\right] \qquad (14) \\ &= \lambda_i a_i \pi_i + \left[e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1\right] \sum_{\lambda_{-i}\in\mathcal{L}_{-i}} q_i(p_i, p_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i). \end{aligned}$$

Now, given λ_i and competitors' premium strategies, $p_{-i}(\cdot)$, insurer i can evaluate the payoff function $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)$, with respect to the probability law $g_i(\lambda_{-i}|\lambda_i)$. Then, the optimal premium choice for insurer i of risk-aversion type λ_i is the one that minimizes the conditional expected disutility function with respect to p_i. Next, we present the notion of pure-strategy Bayesian Nash equilibrium in a game with incomplete information (see, e.g., Fudenberg and Tirole, 1991).

Definition 1 *Let $G^{\mathcal{B}} = \left\langle \mathcal{N}, (\mathcal{L}_i)_{i\in\mathcal{N}}, g, ((\mathcal{R}_i(\lambda_i))_{\lambda_i\in\mathcal{L}_i})_{i\in\mathcal{N}}, (c_i)_{i\in\mathcal{N}} \right\rangle$ such that assumptions (A1)-(A4) hold, and $G^{\mathcal{B}}$ is referred to as the incomplete-information (Bayesian) insurance market. Moreover, let $\mathcal{R}_i^{\mathcal{L}_i}$ denote insurer i's space of premium strategies, i.e., $\mathcal{R}_i^{\mathcal{L}_i}$ is the set of mappings from $\mathcal{L}_i$ to $\mathcal{R}_i$. The premium strategy profile $p^*(\cdot) = (p_i^*(\cdot), p_{-i}^*(\cdot))$ is a pure-strategy Bayesian Nash equilibrium (PSBNE) if for each $i \in \mathcal{N}$ and every $\lambda_i \in \mathcal{L}_i$, $p_i^*(\lambda_i)$ is a solution to the following minimization problem:*

$$\min_{p_i\in\mathcal{R}_i(\lambda_i)} \mathbb{E}\left[c_i(p_i, p_{-i}^*(\Lambda_{-i}); \lambda_i)|\Lambda_i = \lambda_i\right], \qquad (15)$$

with

$$\mathbb{E}\left[c_i(p_i, p_{-i}^*(\Lambda_{-i}); \lambda_i)|\Lambda_i = \lambda_i\right] = \sum_{\lambda_{-i}\in\mathcal{L}_{-i}} c_i(p_i, p_{-i}^*(\lambda_{-i}); \lambda_i)g_i(\lambda_{-i}|\lambda_i).$$

In a PSBNE, every risk-aversion type λ_i of insurer i optimally responds to competitors' optimal premium strategies. When there is no confusion, we write $p_i^* := p_i^*(\lambda_i)$ as the equilibrium price strategy of insurer i. Next, we provide necessary conditions for the existence and uniqueness of a PSBNE.

Theorem 1 *Consider the incomplete-information insurance market* $G^{\mathcal{B}}$ *as defined in Definition 1. A PSBNE premium profile* $\mathrm{p}^*(\cdot)$*, where* $p_i^*(\cdot) \in \mathcal{R}_i^{\mathcal{L}_i}$ *for all* $i \in \mathcal{N}$*, exists and is unique in any of the two following cases for the price-sensitivity function* f_i*:*

Case 1. For any feasible premium profile $\mathrm{p} = (p_i, p_{-i})$*,*

$$\frac{\partial^2 f_i(p_i, \bar{p}_{-i})}{\partial p_i^2} < 0; \tag{16}$$

Case 2. For any feasible premium profile $\mathrm{p} = (p_i, p_{-i})$*, assume that* $f_i(p_i, \bar{p}_{-i}) = \exp\{h_i(p_i, \bar{p}_{-i})\}$*, where* $h_i(\cdot)$ *is a linear decreasing function of* p_i *with constant rate of decrease denoted by* $b_i := \dfrac{\partial h_i(p_i, \bar{p}_{-i})}{\partial p_i}$*. Then, it holds that*

$$\max \mathcal{R}_i(\lambda_i) < -\frac{1}{\lambda_i(1-a_i)} \log\left[\frac{1}{M_X(\lambda_i(1-a_i))} b_i^2 \left(\lambda_i(1-a_i) - b_i\right)^{-2}\right], \tag{17}$$

for all $\lambda_i \in \mathcal{L}_i$*.*

Proof. See Appendix A.2. ∎

Remark 1 *If each insurer i's risk-aversion random variable* Λ_i *follows a degenerate probability distribution with support a singleton, i.e., there exists a* $\lambda_i > 0$ *such that* $\Pr[\Lambda_i = \lambda_i] = 1$*, then there is no uncertainty about insurers' risk-aversion types and we have a complete-information insurance market. This case is studied in Mourdoukoutas et al. (2021), who show the existence and uniqueness of a Nash equilibrium.*

Our intuition suggests that an insurer would neither employ a very aggressive nor a very conservative premium pricing strategy. From a utility perspective, an insurer would be unable to generate profits if premiums were either extremely low or extremely high. A high premium value would make the insurer unattractive and uncompetitive, resulting in an exposure volume so low as to threaten the insurer's continued existence. When the dynamics of the insurance market are characterized by a demand function with a cut-off point, i.e., a premium value above that point causes the exposure volume to be zero, it is never optimal for an insurer to set a premium equal to the cut-off point.

On the other hand, a premium value equal to the lower premium bound could make the insurer the most desirable in the insurance market, but the premium loading would disappear, rendering the insurer indifferent to its large market share. The following proposition shows the aforementioned arguments and provides the characterization for the PSBNE.

Proposition 1 *Let $p^*(\cdot)$ be a PSBNE of the incomplete-information insurance market $G^{\mathcal{B}}$ as defined in Definition 1. Then, for each insurer i and every λ_i, $p_i^*(\lambda_i)$ is not equal to the lower premium bound $p_i^L(\lambda_i)$. Further assume that for all $i \in \mathcal{N}$, $f_i(p_i^U, \bar{p}_{-i}) = 0$ for all feasible premium profiles p_{-i}. Then, for each insurer i and every λ_i, $p_i^*(\lambda_i)$ is not equal to the upper premium bound p_i^U. As interior points of the premium range, the components of the PSBNE are characterized by the first-order conditions (FOCs):*

$$
\begin{aligned}
&\frac{\partial C_i(p_i, p_{-i}^*, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)}{\partial p_i} = \\
&-\lambda_i(1-a_i)e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) q_{i,0} \sum_{\lambda_{-i} \in \mathcal{L}_{-i}} f_i(p_i, \bar{p}_{-i}^*(\lambda_{-i})) g_i(\lambda_{-i}|\lambda_i) \\
&+ \left[e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1\right] q_{i,0} \sum_{\lambda_{-i} \in \mathcal{L}_{-i}} \frac{\partial f_i(p_i, \bar{p}_{-i}^*(\lambda_{-i}))}{\partial p_i} g_i(\lambda_{-i}|\lambda_i) = 0,
\end{aligned}
\tag{18}
$$

for all $\lambda_i \in \mathcal{L}_i$ and all $i \in \mathcal{N}$.

Proof. See Appendix A.3. ■

Next, we can observe some straightforward monotonicity properties of the PSBNE when the combined ratio constraint is binding. Specifically, let $p^*(\cdot)$ be a PSBNE of the incomplete-information insurance market $G^{\mathcal{B}}$ as defined in Definition 1. If the constraint $r_i^S(p_i^*) \leq 0$ is binding for insurer i with risk-aversion type λ_i, then the equilibrium premium $p_i^* \equiv p_i^*(\lambda_i)$ increases with the expected claim amount μ_X and the underwriting expenses e_i, and decreases with the combined ratio cr_i. Indeed, from (8), we obtain $p_i^* = cr_i^{-1}(\mu_X + e_i)$. Recalling that the quantities cr_i, μ_X, and e_i are all positive, it follows that p_i^* increases with μ_X and e_i, and decreases with cr_i.

According to the exponential utility function in (1), the greater the risk-aversion parameter, the more risk-averse the insurer. In terms of premium pricing, an insurer's risk aversion is reflected in the premium loading it seeks. Typically, insurers with a greater risk aversion charge such high premiums in order to generate a substantial amount of premium loading. Moreover, we can assume that in a purely competitive insurance market, there is no insurer with sufficient market power to influence the risk attitudes of its competitors. In such markets, it makes sense to consider the risk-aversion random variables independent. In the following statement, the monotonicity of the PSBNE with respect to the risk-aversion parameter is demonstrated.

Proposition 2 *Let $p^*(\cdot)$ be a PSBNE of the incomplete-information insurance market $G^{\mathcal{B}}$ as defined in Definition 1. If the risk-aversion random variables Λ_i, $i \in \mathcal{N}$, are independent, then the equilibrium premium strategy $p_i^*(\cdot)$ is non-decreasing in the risk-aversion type λ_i for all $i \in \mathcal{N}$.*

Proof. See Appendix A.4. ■

4 Numerical application

In the previous sections, the price-sensitivity function $f_i(p_i, \bar{p}_{-i})$ was fairly general. The application of any demand function that satisfies the conditions in Section 2.2 and belongs into any of the two cases of Theorem 1 guarantees the existence and uniqueness of the PSBNE. To illustrate our new approach to premium strategies in an incomplete information structure, we employ the same two exponential demand functions as in Mourdoukoutas et al. (2021) and examine a similar hypothetical insurance market containing five insurers. Recall that in Mourdoukoutas et al. (2021), the pure-strategy Nash equilibrium (NE) is evaluated in a market with complete information, whereas our focus in this paper is on random unobserved risk-aversion types for the competitors.

Through Sections 4.1 to 4.4, we investigate four scenarios of incomplete information for the hypothetical insurance market by considering different ranges of risk-aversion types $\mathcal{L}_i$ and corresponding probability density functions $g_i(\lambda_i)$. In all four scenarios, the insurers determine their risk-aversion types independently of the competitors and therefore, Λ_i, $i \in \mathcal{N}$, are independent random variables. We choose independent Λ_i, $i \in \mathcal{N}$, for simplicity, since the main result about existence and uniqueness of PS-BNE is unaffected by the dependence structure of Λ_i, $i \in \mathcal{N}$. Furthermore, we study independent Λ_i, $i \in \mathcal{N}$, and therefore, Proposition 2 applies.

In the hypothetical insurance market, there are five insurers that compete in a particular line of business: $\mathcal{N} = \{1, \ldots, 5\}$. It is believed that Insurer 3 is the market leader, that Insurers 2 and 4 have comparable market power, followed by Insurer 1, and Insurer 5 is the insurer with the lowest market power.

The two exponential demand functions in Mourdoukoutas et al. (2021) are defined by

$$\tilde{q}_i(p_i, p_{-i}) = b\left[1 - \exp\left\{-\tilde{\alpha}_i \frac{p^U - p_i}{p^U - \bar{p}_{-i}}\right\}\right] q_{i,0}, \tag{19}$$

for $p_i \in [p_i^L(\lambda_i), p^U]$, and

$$\hat{q}_i(p_i, p_{-i}) = \exp\left\{-\hat{\alpha}_i \frac{p_i - \bar{p}_{-i}}{\bar{p}_{-i}}\right\} q_{i,0}, \tag{20}$$

for $p_i \in [p_i^L(\lambda_i), p_i^U]$. Here, p^U is the cut-off point for insurers' demand function $\tilde{f}$, whereas no such restriction takes place in demand function $\hat{f}$. Moreover, $b > 1$ is a market scale parameter that characterizes the size of inflows and outflows of individuals from the insurance market. A value greater than one indicates a market expansion with considerable potential. In both demand functions, α_i is interpreted as the price-sensitivity parameter associated with the market power of insurer i. Specifically, when an insurer charges a premium rate that is 20% higher than the average of its competitors' premiums, the proportion of the current exposure volume that the insurer maintains increases proportionally to the insurer's market power. Considering the price-sensitivity function $\tilde{f}_i$, when the market on average charges 80% of the premium bound p^U, then the price-sensitivity parameters $\tilde{\alpha}_i$ are evaluated by $\tilde{f}_i(1.2(0.8p^U), 0.8p^U) = \tilde{R}_i$. Regarding $\hat{f}_i$, the price-sensitivity parameters are evaluated by $\hat{f}_i(1.2\bar{p}_{-i}, \bar{p}_{-i}) = \hat{R}_i$. The relative changes R and the associated values of α are given in Table 1. Therefore, considering the demand function $\tilde{q}$, larger values of $\tilde{\alpha}$ characterize insurers with greater market power, whereas less market power is associated with a greater price-sensitivity parameter $\hat{\alpha}$ for the demand function $\hat{q}$.

As far as the upper premium bound in the price-sensitivity function $\tilde{f}$ is concerned, in all four scenarios we assume a universal upper premium bound that applies to all insurers, i.e., $p_i^U = p^U$ for all $i \in \mathcal{N}$, and any premium rate above it makes an insurer's exposure volume vanish. As explained in Section 2.5, this single upper premium bound might be interpreted either as a premium value set by external regulators or as the maximum premium value that the most risk-averse individual in the market, with risk-aversion parameter $\tilde{h}$, is willing to purchase. When the dynamics in the insurance market are described by the price-sensitivity function $\hat{f}$, we assume that the most risk-averse individual's risk aversion is higher when they consider to purchase insurance coverage from an insurer with greater market power. That is, the individual is willing to pay a higher premium to be insured by an insurer who might provide more benefits, be more reliable, have better reputation, etc. Therefore, in each of the four scenarios of this section it always holds that $\hat{h}_5^{(l)} < \hat{h}_1^{(l)} < \hat{h}_2^{(l)} = \hat{h}_4^{(l)} < \hat{h}_3^{(l)}$, for $l = 1, 2, 3$ and 4 (the superscript (l) indicates the index number of the scenario whereas the subscript indicates the insurer).

For all insurers, the individual claim sizes follow an exponential distribution with mean $\mu_X = 100$. Thus, the moment generating function of X is equal to $M_X(t) = (1 - \mu_X t)^{-1}$ for $t < 0.01$. For the existence of M_X, we will check that all the risk-aversion parameters h and λ in the the four scenarios are less than $1/\mu_X = 0.01$. Moreover, we consider the same underwriting expenses per policy, and the same range for the combined ratio for all insurers. In all scenarios, the feasible premium region $\mathcal{R}_i(\lambda_i)$ is non-empty for all $\lambda_i \in \mathcal{L}_i$ and every $i \in \mathcal{N}$ because of the following two conditions. First, $\max_{j \in \mathcal{N}} \bar{\lambda}_j$ is less than $\tilde{h}$ and $\hat{h}_i^{(l)}$ and hence, the premium range $\mathcal{P}_i(\lambda_i)$ is a non-empty set for all risk-aversion types λ_i and every insurer i. Also, $(\mu_X + e_i)/cr_i \in [80.8, 150]$ for $cr_i \in [0.7, 1.3]$, and remark that for all i and all cr_i, p^U and p_i^U are greater than $(\mu_X + e_i)/cr_i$.

The set of all model parameters is summarized in Table 1. Since insurers' risk preferences are characterized by exponential utilities, it is well known that the initial reserve π_i does not affect insurer i's risk attitude, and thus, it does not affect the equilibrium prices.

Parameters	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
$q_{i,0}$	1,000	2,000	3,000	2,000	500
$\tilde{R}_i$	0.35	0.38	0.40	0.38	0.32
$\tilde{\alpha}_i$	1.7242	1.9039	2.0273	1.9039	1.5508
$\hat{R}_i$	0.57	0.58	0.60	0.58	0.55
$\hat{\alpha}_i$	2.8106	2.7236	2.5541	2.7236	2.9892
$\hat{h}_i^{(l)},\ l = 1, 2$	0.007574	0.007855	0.008173	0.007855	0.007247
$\hat{h}_i^{(3)}$	0.008124	0.008149	0.008197	0.008149	0.008048
$\hat{h}_i^{(4)}$	0.007607	0.007884	0.008244	0.007884	0.007286

$\tilde{h} = 0.007$	in all four scenarios
$a_i = 0.05$	
$\mu_X = 100$	
$e_i = 5$	
$cr_i \in [0.7, 1.3]$	and thus, $(\mu_X + e_i)/cr_i \in [80.8, 150]$
$b = 1.2$	

Table 1: Basic model parameters in Section 4.

The optimal premiums in the PSBNE premium strategy profile are given as solutions to minimisation problem (15) and characterized by Proposition 1. That is, for all $i \in \mathcal{N}$ and every $\lambda_i \in \mathcal{L}_i$, we solve the first-order conditions $\partial \mathbb{E}\left[c_i(p_i, p_{-i}(\Lambda_{-i}); \lambda_i) | \Lambda_i = \lambda_i\right] / \partial p_i = 0$ with respect to p_i.

The PSBNE premium strategy profile associated with demand function $\tilde{q}$ is denoted by $\tilde{p}^*(\cdot)$. Notice that the price-sensitivity function $\tilde{f}$ satisfies inequality (16) of Theorem 1 and hence, a PSBNE premium strategy profile $\tilde{p}^*(\cdot)$ always exists and is unique.

The PSBNE premium strategy profile associated with demand function $\hat{q}$ is denoted by $\hat{p}^*(\cdot)$. In our four scenarios, the sum in (A.4) is always positive for all feasible premium profiles.[6] Therefore, the payoff $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)$ is a strictly convex function of p_i for each insurer i and every λ_i, and it follows from Theorem 1 that there

[6]For each insurer i, every $\lambda_i \in \mathcal{L}_i$ and all $p_i \in \mathcal{P}_i(\lambda_i) = [p_i^L(\lambda_i), p_i^U]$ it holds that $\partial^2 C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)/\partial p_i^2 > 0$, for all insurers $j \neq i$, every $\lambda_j \in \mathcal{L}_j$ and all $p_j \in \mathcal{P}_j(\lambda_j)$.

exists a unique PSBNE premium strategy profile. Moreover, Condition (17) is always satisfied.[7]

Considering the PSBNE premium strategy profiles in all four scenarios, both $\tilde{p}^*$ and $\hat{p}^*$ satisfy the property in Proposition 1. Moreover, the optimal premiums in both equilibrium strategy profiles $\tilde{p}^*$ and $\hat{p}^*$ are interior points of their premium range and the monotonicity property in Proposition 2 holds true, i.e., $\tilde{p}_i^*$ and $\hat{p}_i^*$ are non-decreasing functions of the risk aversion λ_i for all $i \in \mathcal{N}$.

Briefly, the findings in our four scenarios are as follows. Scenarios 1 and 3 show that the leaders in the insurance market are more likely to set higher premium values, as a consequence of the common belief that their higher risk-aversion values are associated with greater probabilities, resulting in a reduction in their expected exposure volume at the end of the period compared to their current exposure volume. However, Scenario 3 further shows that the competition can be tighter, by means of the difference between the lowest and the highest equilibrium premium value in the market, when insurers share the same range of risk-aversion values (not necessary the same probabilities). In Scenario 2, we see that the PSBNE premium strategy profiles consist of lower premium values than in Scenario 1 as a result of the common belief that every risk-aversion type of each insurer in the market is equally likely. Finally, Scenario 4 presents the PSBNE premium strategy profiles in an incomplete-information instance of the insurance market in parallel with the NE premium profiles for two particular cases of a complete-information variation for the insurance market.

Finally, let us comment on the distinct characteristics of the two demand functions employed in this paper and their effect on our results in the four scenarios that follow. The gains or losses in the end-of-period expected exposure volume (compared to the current exposure volume) appear more severe in the insurance market whose dynamics are described by $\hat{q}$ than in the insurance market whose dynamics are described by $\tilde{q}$. It is an anticipated phenomenon considering the fact that the competition under $\tilde{f}$ is stricter and more controllable due to the following reasons. First, under $\tilde{f}$ the deviation

[7] For verification in the algorithm, we refer to Appendix A.5.

of an insurer's premium from the cut-off bound is competing with the deviation of the market average premium from the cut-off bound, whereas under $\hat{f}$ the competition is between the insurer's premium and the market average premium. Second, the size of gains and losses in exposure volume at the end of the period is further constrained by the market scale parameter b under $\tilde{f}$. Third, the two demand functions display opposite curvature.

Before proceeding, let us mention that the expected exposure volume of insurer i at the end of the period, in all scenarios, is calculated by

$$\mathbb{E}_\Lambda \left[q_i(p_i^*(\Lambda_i), p_{-i}^*(\Lambda_{-i}))\right] = \sum_{\lambda \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_n} \left(\prod_{j \in \mathcal{N}} g_j(\lambda_j)\right) q_i(p_i^*(\lambda_i), p_{-i}^*(\lambda_{-i})),$$

where $\lambda = (\lambda_i, \lambda_{-i}) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$.

4.1 Scenario 1

Insurers' ranges of risk-aversion types $\mathcal{L}_i$ and the associated probability density functions $g_i(\lambda_i)$ are presented in Table 2. This scenario implies that insurers with greater market power are commonly believed to be more sensitive to the large volume of risk undertaken and assign higher probabilities to greater values of risk aversion in order to generate sufficient premium loadings and protect themselves against potential losses; see in Table 3 that the difference $p^L(\lambda) - \mu_X > 0$ increases in λ.

The PSBNE premium strategy profiles $\tilde{p}^*$ and $\hat{p}^*$ associated with the demand functions $\tilde{f}$ and $\hat{f}$, respectively, are given in Table 3 and illustrated in Figure 1, as well as the same figure depicts that the combined ratio constraint $r^S(p^*) \leq 0$ is slack (i.e., $r^S(p^*) < 0$) over the equilibrium premium range for ratios $cr \in [0.7, 1.3]$.

In probabilistic terms, Insurers 3 and 4 are more likely to select higher premium values than the other insurers, because their higher risk-aversion types are associated with greater probabilities. As a result, it is expected for Insurers 3 and 4 to lose some amount from their current market share at the end of the period, with the highest losses occurring under the market dynamics described by $\hat{q}$. On the contrary, lower premium values are more likely for the remaining insurers, resulting in gaining expected exposure volume at the end of the period compared to their initial exposure volume.

Because of the distinct characteristics of the two demand functions, when insurers' risk-aversion types coincide, insurers' optimal premiums are very close under the demand function $\tilde{f}$. This is not the case with the demand function $\hat{f}$. These instances occur infrequently and yield the deviation of some insurers' expected exposure volume at the end of the period from their initial exposure volume to be higher for $\hat{q}$ than $\tilde{q}$.

Insurer 1

λ_1	0.002	0.003	0.004
$g_1(\lambda_1)$	0.15	0.70	0.15

Insurer 2

λ_2	0.003	0.004	0.005
$g_2(\lambda_2)$	0.20	0.70	0.10

Insurer 3

λ_3	0.004	0.005	0.006
$g_3(\lambda_3)$	0.10	0.10	0.80

Insurer 4

λ_4	0.003	0.004	0.005
$g_4(\lambda_4)$	0.10	0.10	0.80

Insurer 5

λ_5	0.001	0.002	0.003
$g_5(\lambda_5)$	0.60	0.30	0.10

Table 2: Scenario 1: The probability density functions $g_i(\lambda_i)$ for all $\lambda_i \in \mathcal{L}_i$ and all i.

4.2 Scenario 2

In this scenario, we keep the ranges of risk-aversion types $\mathcal{L}_i$ as defined in Scenario 1, but the insurers use a discrete uniform distribution over the risk-aversion types, i.e., we consider $g_i(\lambda_i) = 0.33$ for all $\lambda_i \in \mathcal{L}_i$ and every $i \in \mathcal{N}$. This scenario investigates which are the optimal responses of the insurers when it is commonly believed that they equally weigh their risk-aversion preferences independently of their market power and the size of risks undertaken.

The PSBNE premium strategy profiles $\tilde{p}^*$ and $\hat{p}^*$ are given in Table 4 and illustrated in Figure 2, as well as the same figure depicts that the combined ratio constraint, $r^S(p^*) \leq 0$, is slack over the equilibrium premium range for ratios $cr \in [0.7, 1.3]$.

Under this scenario, Insurers 2 and 4 are characterized by the same set of parameters and hence, their equilibrium premium strategies are identical, leading to equal expected exposure volumes at the end of the period. Similar to Scenario 1, in the PSBNE it holds that the follower insurers are more likely to set lower premium values than the

λ_i	$p^L(\cdot)$	$\tilde{p}_1^*(\cdot)$	$\tilde{p}_2^*(\cdot)$	$\tilde{p}_3^*(\cdot)$	$\tilde{p}_4^*(\cdot)$	$\tilde{p}_5^*(\cdot)$	p^U
0.001	105.07					156.54	172
0.002	110.91	157.64				157.11	172
0.003	117.71	158.38	158.85		158.71	157.90	172
0.004	125.80	159.45	159.86	159.83	159.73		172
0.005	135.65		161.37	161.35	161.27		172
0.006	148.06			163.83			172
$\mathbb{E}_\Lambda[\tilde{q}_i(\tilde{p}_i^*(\Lambda_i), \tilde{p}_{-i}^*(\Lambda_{-i}))]$		1034	2042	2671	1961	523	

λ_i	$p^L(\cdot)$	$\hat{p}_1^*(\cdot)$	$\hat{p}_2^*(\cdot)$	$\hat{p}_3^*(\cdot)$	$\hat{p}_4^*(\cdot)$	$\hat{p}_5^*(\cdot)$
0.001	105.07					166.99
0.002	110.91	173.91				171.10
0.003	117.71	178.97	180.22		179.76	176.31
0.004	125.80	185.44	186.62	188.77	186.19	
0.005	135.65		194.91	196.94	194.49	
0.006	148.06			207.79		
	p_i^U	187	196	208	196	178
$\mathbb{E}_\Lambda[\hat{q}_i(\hat{p}_i^*(\Lambda_i), \hat{p}_{-i}^*(\Lambda_{-i}))]$		1144	2008	2176	1799	701

Table 3: Scenario 1: Bayesian equilibrium premiums, premium bounds and expected exposure volumes for demand functions $\tilde{f}$ and $\hat{f}$.

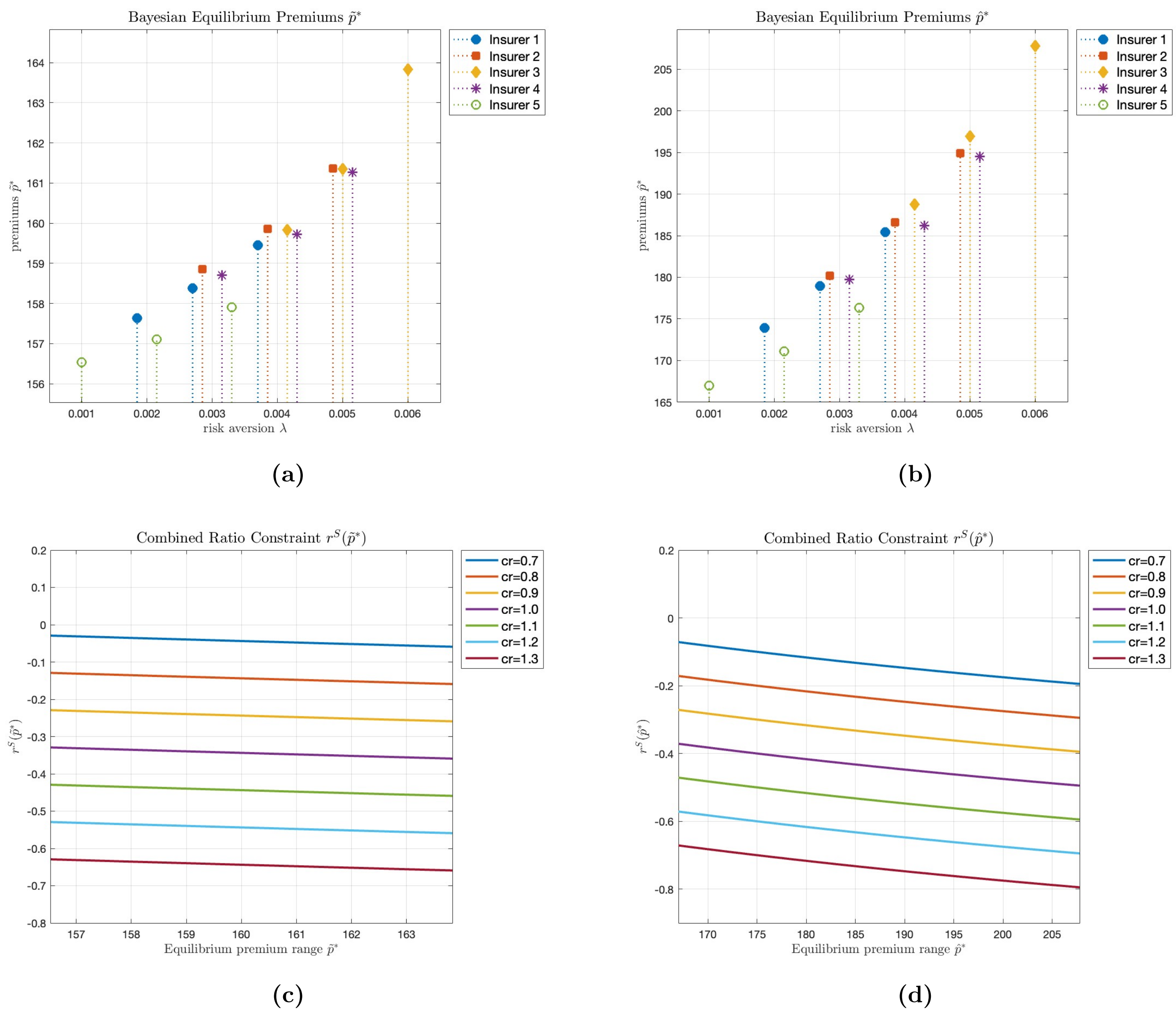

Figure 1: Scenario 1: Pure-strategy Bayesian equilibrium premium profiles (first row) and combined ratio constraint over the equilibrium premium range (second row). The left-hand side is for $\tilde{f}$ and the right-hand side is for $\hat{f}$.

leader insurers and consequently, they expect increases in their end-of-period exposure volume relative to their current exposure volume. For the same levels of risk aversion, the associated equilibrium premium values of insurers are closer to each other under $\tilde{f}$ than under $\hat{f}$, in which it still follows that the leader insurers select higher premium values than the followers. This leads to larger gains in the end-of-period expected exposure volume for followers and larger losses in the end-of-period expected exposure volume for the leaders under $\hat{f}$ than under $\tilde{f}$, as compared with their current market shares.

The discrete uniform distribution of the risk-aversion levels leads to slightly lower equilibrium premium strategies for all insurers compared to those in Scenario 1. In contrast to Scenario 1, higher probabilities are now attached to the lowest risk-aversion values for all insurers except Insurer 5. The leaders (Insurers 3 and 4) randomize between their middle and smallest risk-aversion level with total probability of 0.66, which is higher than the total probability of 0.2 under Scenario 1, whereas Insurer 5 randomizes between the middle and biggest risk-aversion level with total probability of 0.66, which is higher than the total probability of 0.4 under Scenario 1. These factors contribute to PSBNE premium strategy profiles in this scenario being lower than in Scenario 1, leading market leaders to maintain higher end-of-period expected exposure volume and Insurer 5 to maintain less end-of-period expected exposure volume compared to Scenario 1.

4.3 Scenario 3

In this scenario, we investigate the optimal premium strategies when all insurers have exactly the same set of possible risk-aversion values. However, it is commonly believed that insurers with greater current market power put more weight on high risk-aversion values in order to generate sufficient premium loadings, as also assumed in Scenario 1. Table 5 shows the set of risk-aversion types $\mathcal{L}_i = \mathcal{L}$ and the probability density functions $g_i(\lambda_i)$.

The PSBNE premium strategy profiles $\tilde{p}^*$ and $\hat{p}^*$ are given in Table 6 and illustrated

λ_i	$p^L(\cdot)$	$\tilde{p}_1^*(\cdot)$	$\tilde{p}_2^*(\cdot)$	$\tilde{p}_3^*(\cdot)$	$\tilde{p}_4^*(\cdot)$	$\tilde{p}_5^*(\cdot)$	p^U
0.001	105.07					155.97	172
0.002	110.91	157.15				156.58	172
0.003	117.71	157.95	158.40		158.40	157.42	172
0.004	125.80	159.08	159.47	159.62	159.47		172
0.005	135.65		161.07	161.19	161.07		172
0.006	148.06			163.73			172
$\mathbb{E}_\Lambda[\tilde{q}_i(\tilde{p}_i^*(\Lambda_i), \tilde{p}_{-i}^*(\Lambda_{-i}))]$		1020	2000	2830	2000	513	

λ_i	$p^L(\cdot)$	$\hat{p}_1^*(\cdot)$	$\hat{p}_2^*(\cdot)$	$\hat{p}_3^*(\cdot)$	$\hat{p}_4^*(\cdot)$	$\hat{p}_5^*(\cdot)$
0.001	105.07					165.93
0.002	110.91	173.00				170.10
0.003	117.71	178.11	179.28		179.28	175.36
0.004	125.80	184.62	185.73	188.40	185.73	
0.005	135.65		194.05	196.59	194.05	
0.006	148.06			207.45		
	p_i^U	187	196	208	196	178
$\mathbb{E}_\Lambda[\hat{q}_i(\hat{p}_i^*(\Lambda_i), \hat{p}_{-i}^*(\Lambda_{-i}))]$		1108	1917	2372	1917	655

Table 4: Scenario 2: Bayesian equilibrium premiums, premium bounds and expected exposure volumes for $\tilde{f}$ and $\hat{f}$ demand functions.

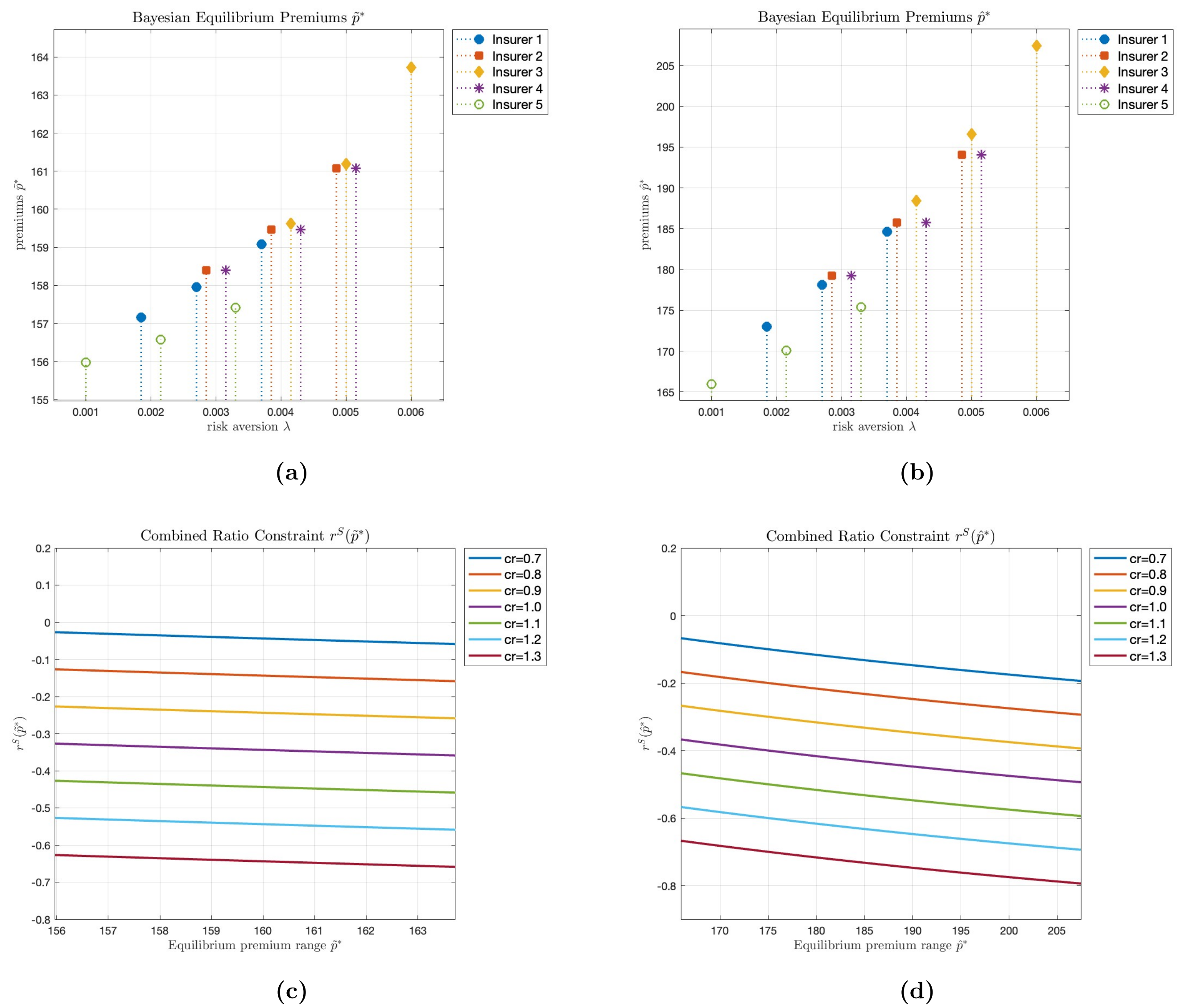

Figure 2: Scenario 2: Pure-strategy Bayesian equilibrium premium profiles (first row) and combined ratio constraint over the equilibrium premium range (second row). The left-hand side is regarding $\tilde{f}$ and the right-hand side is regarding $\hat{f}$.

in Figure 3, and the figure also depicts that the combined ratio constraint, $r^S(p^*) \leq 0$, is slack over the equilibrium premium range for ratios $cr \in [0.7, 1.3]$.

However, these strategies are on the basis of the probability distribution g. Thus, Insurer 2 is more likely to be less risk averse than Insurers 3 and 4 resulting in an increase in Insurer 2's expected exposure volume at the end of the period (compared to Insurer 2's current exposure volume). Similarly, Insurers 1 and 5 put more weight on the lower risk-aversion values, which leads to an increase in their end-of-period expected exposure volumes compared to their initial ones.

At the PSBNE, insurers' premium values associated with the same risk aversion display a higher deviation from one another in the equilibrium strategy profile $\hat{p}^*$ than in $\tilde{p}^*$, but they still follow the pattern of higher values being set by insurers with higher (current) market power. Thus, similar to the previous two scenarios, the losses in the expected exposure volume at the end of the period for Insurers 3 and 4 and the gains in the expected exposure volume at the end of the period for the rest of the insurers are higher under $\hat{f}$ than under $\tilde{f}$ when compared to their current market shares.

The belief in a common set of risk-aversion levels for all insurers results in more symmetric competition. Compared to the previous two scenarios where insurers are believed to have distinct ranges of risk-aversion values, the difference between the smallest premium value in the market (associated with the lowest risk-aversion value of the weakest insurer) and the largest one (associated with the highest risk-aversion value of the leader insurer) is slightly narrower under this scenario for both demand functions.

$\mathcal{L}$	0.002	0.004	0.006
$g_1(\lambda_1)$	0.45	0.45	0.10
$g_2(\lambda_2)$	0.25	0.70	0.05
$g_3(\lambda_3)$	0.05	0.25	0.70
$g_4(\lambda_4)$	0.05	0.55	0.40
$g_5(\lambda_5)$	0.80	0.15	0.05

Table 5: Scenario 3: Range of risk-aversion values $\mathcal{L}_i = \mathcal{L}$ and probability density functions $g_i(\lambda_i)$ for all i.

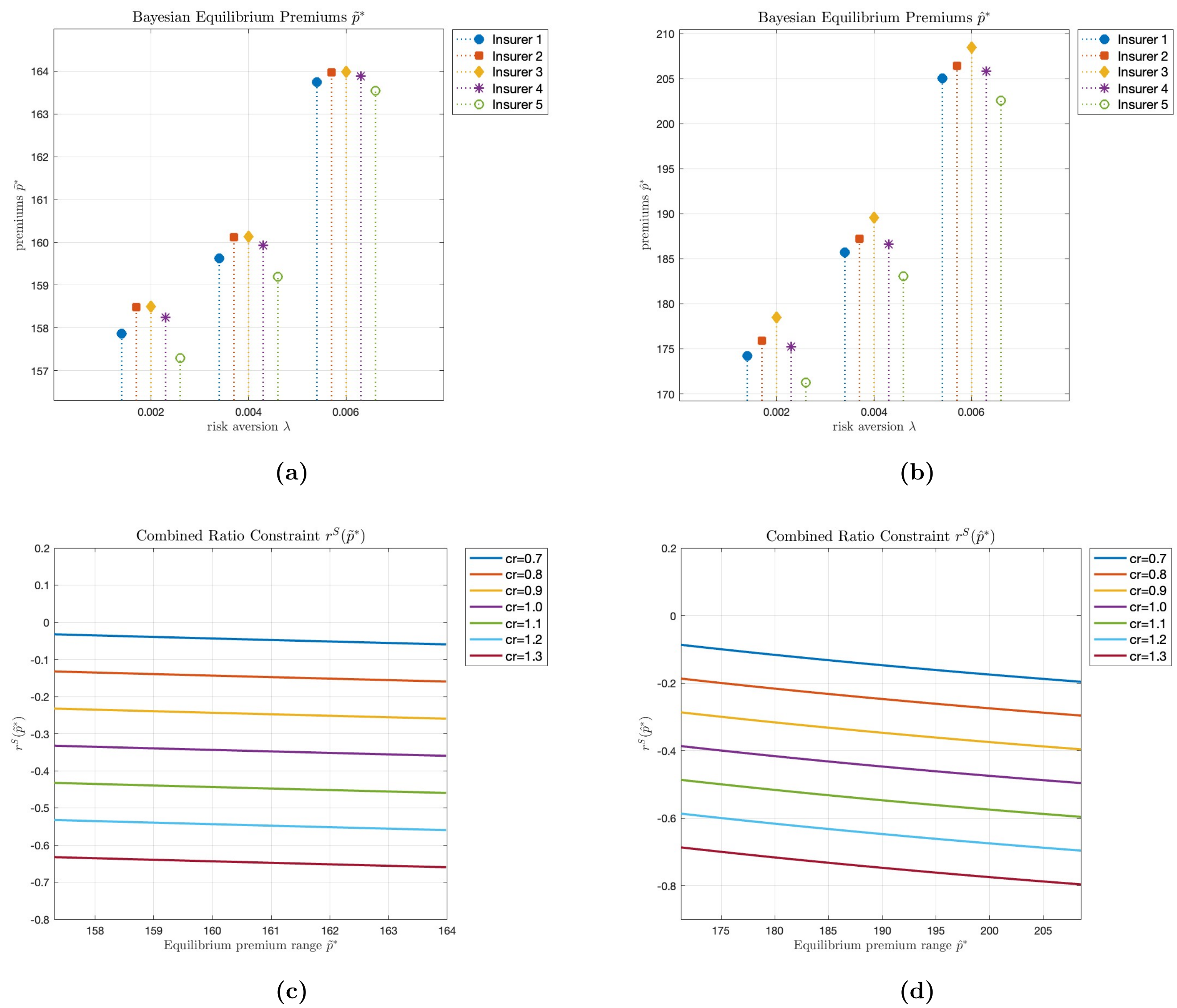

Figure 3: Scenario 3: Pure-strategy Bayesian equilibrium premium profiles (first row) and combined ratio constraint over the equilibrium premium range (second row). The left-hand side is regarding $\tilde{f}$ and the right-hand side is regarding $\hat{f}$.

λ_i	$p^L(\cdot)$	$\tilde{p}_1^*(\cdot)$	$\tilde{p}_2^*(\cdot)$	$\tilde{p}_3^*(\cdot)$	$\tilde{p}_4^*(\cdot)$	$\tilde{p}_5^*(\cdot)$	p^U
0.002	110.91	157.86	158.48	158.50	158.24	157.30	172
0.004	125.80	159.63	160.12	160.13	159.93	159.20	172
0.006	148.06	163.74	163.97	163.98	163.88	163.54	172
$\mathbb{E}_\Lambda[\tilde{q}_i(\tilde{p}_i^*(\Lambda_i), \tilde{p}_{-i}^*(\Lambda_{-i}))]$		1016	2060	2775	1924	513	

λ_i	$p^L(\cdot)$	$\hat{p}_1^*(\cdot)$	$\hat{p}_2^*(\cdot)$	$\hat{p}_3^*(\cdot)$	$\hat{p}_4^*(\cdot)$	$\hat{p}_5^*(\cdot)$
0.002	110.91	174.23	175.91	178.49	175.22	171.25
0.004	125.80	185.73	187.23	189.54	186.61	183.05
0.006	148.06	205.04	206.40	208.49	205.84	202.60
	p_i^U	206	207	209	207	203
$\mathbb{E}_\Lambda[\hat{q}_i(\hat{p}_i^*(\Lambda_i), \hat{p}_{-i}^*(\Lambda_{-i}))]$		1112	2097	2354	1811	651

Table 6: Scenario 3: Bayesian equilibrium premiums, premium bounds and expected exposure volumes for $\tilde{f}$ and $\hat{f}$ demand functions.

4.4 Scenario 4

In this scenario, it is assumed that the insurance market presented in Table 1 consists of two types of risk-averse insurers who randomize between the risk preferences with equal probabilities as in Scenario 2. Additionally, we consider two specific complete-information variations of the insurance market under question, in which insurers' risk aversion is not random and characterized by a single-valued parameter (with probability 1). We study the two possible extreme insurance markets in which insurers possess full knowledge about competitors' risk aversion. Specifically, we investigate the following two cases of complete information:

$$\textit{Case 1}:\ \lambda_i = \underline{\lambda}_i \text{ for all } i = 1, \ldots, 5$$

$$\textit{Case 2}:\ \lambda_i = \bar{\lambda}_i \text{ for all } i = 1, \ldots, 5,$$

where $\underline{\lambda}_i$ and $\bar{\lambda}_i$ are the minimum and maximum element of the risk-aversion space $\mathcal{L}_i$ of insurer i in the incomplete-information market, respectively. Case 1 represents a market in which all insurers display the least possible risk aversion, whereas Case 2 asserts that all insurers display the highest possible risk aversion. Note that the NE is a special case of PSBNE as indicated in Remark 1. The various risk-aversion types per insurer, the marginal distributions and the equilibrium results (PSBNE denoted by p^* in the incomplete-information market and NE denoted by p^{*m} in the two complete-information markets) are presented in Table 7.

As expected, the PSBNE premium strategy profiles lie in between these two extreme cases of complete information. Specifically, we observe for the NE that $\tilde{p}_i^{*m} < \tilde{p}_i^*(\underline{\lambda}_i)$ and $\hat{p}_i^{*m} < \hat{p}_i^*(\underline{\lambda}_i)$ in Case 1, whereas $\tilde{p}_i^{*m} > \tilde{p}_i^*(\bar{\lambda}_i)$ and $\hat{p}_i^{*m} > \hat{p}_i^*(\bar{\lambda}_i)$ in Case 2, for $i = 1, \ldots, 5$. That is, when all insurers are assumed to be little risk averse (Case 1) and set low NE premium values, the lack of such information leads insurers to underwrite moderately higher in the corresponding PSBNE premium component, and vice-versa for Case 2 with its corresponding PSBNE premium component. This relationship is in line with our intuition, and it is an immediate consequence of the fact that the belief in competitors' risk aversion is spread between the types $\underline{\lambda}_i$ and $\bar{\lambda}_i$ in the incomplete-information market.

Insurer 1

λ_1	0.003	0.004
$g_1(\lambda_1)$	0.50	0.50
incomplete-information		
p^U	172	
$\tilde{p}_1^*(\lambda_1)$	158.54	159.59
$\mathbb{E}_\Lambda\left[\tilde{q}_1\left(\tilde{p}^*(\Lambda)\right)\right]$	1026	
complete-information	Case 1	Case 2
$\tilde{p}_1^{*m}$	157.75	160.27
$\tilde{q}_1\left(\tilde{p}^{*m}\right)$	1019	1033
incomplete-information		
p_1^U	188	
$\hat{p}_1^*(\lambda_1)$	179.53	185.96
$\mathbb{E}_\Lambda\left[\hat{q}_1\left(\hat{p}^*(\Lambda)\right)\right]$	1114	
complete-information	Case 1	Case 2
$\hat{p}_1^{*m}$	177.81	187.51
$\hat{q}_1\left(\hat{p}^{*m}\right)$	1103	1121
$p_1^L(\lambda_1)$	117.71	125.80

Insurer i, for $i = 2, 4$

λ_i	0.004	0.005
$g_i(\lambda_i)$	0.50	0.50
incomplete-information		
p^U	172	
$\tilde{p}_i^*(\lambda_i)$	159.96	161.45
$\mathbb{E}_\Lambda\left[\tilde{q}_i\left(\tilde{p}^*(\Lambda)\right)\right]$	2000	
complete-information	Case 1	Case 2
$\tilde{p}_i^{*m}$	159.31	161.96
$\tilde{q}_i\left(\tilde{p}^{*m}\right)$	2006	2000
incomplete-information		
p_i^U	197	
$\hat{p}_i^*(\lambda_i)$	187.06	195.32
$\mathbb{E}_\Lambda\left[\hat{q}_i\left(\hat{p}^*(\Lambda)\right)\right]$	1909	
complete-information	Case 1	Case 2
$\hat{p}_i^{*m}$	185.45	196.78
$\hat{q}_i\left(\hat{p}^{*m}\right)$	1909	1901
$p_i^L(\lambda_i)$	125.80	135.65

Insurer 3

λ_3	0.005	0.006
$g_3(\lambda_3)$	0.50	0.50
incomplete-information		
p^U	172	
$\tilde{p}_3^*(\lambda_3)$	161.54	163.95
$\mathbb{E}_\Lambda\left[\tilde{q}_3\left(\tilde{p}^*(\Lambda)\right)\right]$	2785	
complete-information	Case 1	Case 2
$\tilde{p}_3^{*m}$	161.07	164.24
$\tilde{q}_3\left(\tilde{p}^{*m}\right)$	2876	2712
incomplete-information		
p_3^U	211	
$\hat{p}_3^*(\lambda_3)$	197.86	208.66
$\mathbb{E}_\Lambda\left[\hat{q}_3\left(\hat{p}^*(\Lambda)\right)\right]$	2332	
complete-information	Case 1	Case 2
$\hat{p}_3^{*m}$	196.33	210.04
$\hat{q}_3\left(\hat{p}^{*m}\right)$	2366	2286
$p_3^L(\lambda_3)$	135.65	148.06

Insurer 5

λ_5	0.002	0.003
$g_5(\lambda_5)$	0.50	0.50
incomplete-information		
p^U	172	
$\tilde{p}_5^*(\lambda_5)$	157.26	158.04
$\mathbb{E}_\Lambda\left[\tilde{q}_5\left(\tilde{p}^*(\Lambda)\right)\right]$	518	
complete-information	Case 1	Case 2
$\tilde{p}_5^{*m}$	156.35	158.86
$\tilde{q}_5\left(\tilde{p}^{*m}\right)$	512	524
incomplete-information		
p_5^U	179	
$\hat{p}_5^*(\lambda_5)$	171.55	176.73
$\mathbb{E}_\Lambda\left[\hat{q}_5\left(\hat{p}^*(\Lambda)\right)\right]$	662	
complete-information	Case 1	Case 2
$\hat{p}_5^{*m}$	169.80	178.32
$\hat{q}_5\left(\hat{p}^{*m}\right)$	651	671
$p_5^L(\lambda_5)$	110.91	117.71

Table 7: Scenario 4: PSBNE in an incomplete-information market consisting of risk-averse insurers with two possible types, and NE in two extreme cases of complete-information variations for the market; namely, all insurers are associated either with their smallest possible risk aversion (Case 1) or with their highest (Case 2). For each insurer, we discuss demand function $\tilde{f}$ in the top panel and $\hat{f}$ in the bottom panel.

5 Conclusions

This paper presents a model of a stochastic insurance market with incomplete information over a single period. We challenge the complete information structure in existing literature by assuming that insurers possess knowledge which is unavailable to their competitors. In our analysis, insurers' risk aversion is regarded as private information and is modeled using a prior joint probability law. This private information influences the insurers' decision making, resulting in the definition of premium strategies as mappings from risk-aversion types to premium rates. Therefore, insurers are uncertain about whether their competitors are aggressive or passive underwriters, and optimal premium strategies are derived from the prior joint probability distribution over insurers' risk-aversion types. Instead of a single equilibrium premium value for each insurer, we can determine optimal premium strategies across all risk-aversion types. Under certain conditions on the demand function, the existence and uniqueness of a PSBNE (pure-strategy Bayesian Nash Equilibrium) is guaranteed, and the equilibrium premium of an insurer is a non-decreasing function of the insurer's own risk-aversion type. This is consistent with our intuition, as greater risk aversion in insurers leads to higher premium loadings.

In our first foray into Bayesian games, we only considered finite risk-aversion type spaces to avoid measurability issues. For future research, we plan to demonstrate the existence of a pure-strategy Bayesian equilibrium under a continuum of insurer risk-aversion types. Additionally, we are concerned about the limitations and anomalies of the negative exponential demand function, as well as the lack of control over the market's total number of insured individuals with this method. In this regard, we plan to examine a closed market in which migrations between insurers occur according to a Markov chain, and the intensity of migration from one insurer to another is directly proportional to the price difference between them.

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A Proofs

A.1 Monotonicity of indifference premium

By assumption, we have that X is a non-negative random variable whose moment generating function, $M_X(t)$, exists for all $0 \leq t \leq h$. Now, we want to show that the function $b(t) = t^{-1} \log M_X(t)$, $0 < t \leq h$, is an increasing function of t. Firstly, notice that $M_X(t)$ is a log-convex function and $M_X(t) \geq 1$ for all $0 \leq t \leq h$. Therefore,

$$\log M_X(0) \geq \log M_X(t) + \frac{d \log M_X(t)}{dt}(0 - t),$$

or

$$\frac{M_X^{'}(t)}{M_X(t)} \geq t^{-1} \log M_X(t). \tag{A.1}$$

So, we have

$$b^{'}(t) = -t^{-2} \log M_X(t) + t^{-1} \frac{M_X^{'}(t)}{M_X(t)} = t^{-1} \left[\frac{M_X^{'}(t)}{M_X(t)} - t^{-1} \log M_X(t) \right] \geq 0,$$

for $0 < t \leq h$. The last inequality comes from (A.1).

A.2 Proof of Theorem 1

For all $i \in \mathcal{N}$, $\mathcal{L}_i$ is a finite subset of positive real numbers. Moreover, for all $i \in \mathcal{N}$ and every $\lambda_i \in \mathcal{L}_i$, the premium range, $\mathcal{R}_i(\lambda_i)$, is a nonempty compact convex set, and the inequality constraint, $r_i^S(p_i)$, defined in (8) is a convex function. Notice that the payoff function, given in (14), is infinitely differentiable with respect to p, due to our assumption that $f_i(p_i, \bar{p}_{-i})$ is infinitely differentiable with respect to p too. Next, we evaluate the first and second partial derivatives of the payoff function of Insurer i of risk-aversion type λ_i, and we prove that under the two cases the payoff is a strictly convex function of $p_i \equiv p_i(\lambda_i)$. Then, the existence of a PSBNE is guaranteed by Debreu (1952), Glicksberg (1952) and Fan (1952). Thereafter, we show the uniqueness of PSBNE using a similar argument as in Theorem 2 of Rosen (1965).

Existence. Differentiating the payoff function $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)$ with respect to p_i and substituting q_i from (2), we obtain

$$\frac{\partial C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)}{\partial p_i} = \tag{A.2}$$
$$-\lambda_i(1-a_i)e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i))q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i)$$
$$+\left[e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i)) - 1\right]q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} \frac{\partial f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))}{\partial p_i}g_i(\lambda_{-i}|\lambda_i),$$

where $\bar{p}_{-i}(\lambda_{-i}) = [1/(n-1)]\sum_{j\neq i} p_j(\lambda_j)$. Recall that insurers currently possess a positive share of the market, i.e., $q_{i,0} > 0$.

Now, differentiating $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)$ twice with respect to p_i, we obtain

$$\frac{\partial^2 C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)}{\partial p_i^2} = \tag{A.3}$$
$$[\lambda_i(1-a_i)]^2 e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i))q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i)$$
$$-2\lambda_i(1-a_i)e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i))q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} \frac{\partial f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))}{\partial p_i}g_i(\lambda_{-i}|\lambda_i)$$
$$+\left[e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i)) - 1\right]q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} \frac{\partial^2 f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))}{\partial p_i^2}g_i(\lambda_{-i}|\lambda_i).$$

Considering Case 1 in (16) and recalling the inequalities in (3) and (9), we easily derive from (A.3) that $\partial^2 C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)/\partial p_i^2 > 0$ for all $p_i \in \mathcal{R}_i(\lambda_i)$ and all feasible premium profiles p_{-i}. Hence, the payoff of Insurer i of risk-aversion type λ_i is a strictly convex function of p_i in $\mathcal{R}_i(\lambda_i)$.

In Case 2, the second partial derivative of $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)$ with respect to p_i is given by

$$\frac{\partial^2 C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)}{\partial p_i^2} =$$
$$[\lambda_i(1-a_i)]^2 e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i))q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i)$$
$$-2\lambda_i(1-a_i)e^{-\lambda_i(1-a_i)p_i}M_X(\lambda_i(1-a_i))q_{i,0}\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} \frac{\partial h_i(p_i, \bar{p}_{-i}(\lambda_{-i}))}{\partial p_i}f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i)$$

$$+\left[e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - 1\right] q_{i,0} \sum_{\lambda_{-i}\in\mathcal{L}_{-i}} \left(\frac{\partial h_i(p_i,\bar{p}_{-i}(\lambda_{-i}))}{\partial p_i}\right)^2 f_i(p_i,\bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i).$$

We simplify the above expression as

$$\begin{aligned}
&\frac{\partial^2 C_i(p_i,p_{-i},\Lambda_i,\Lambda_{-i}|\Lambda_i=\lambda_i)}{\partial p_i^2} = \\
&\sum_{\lambda_{-i}\in\mathcal{L}_{-i}} [\lambda_i(1-a_i)-b_i]^2 e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) q_{i,0} f_i(p_i,\bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i) \\
&- \sum_{\lambda_{-i}\in\mathcal{L}_{-i}} b_i^2 q_{i,0} f_i(p_i,\bar{p}_{-i}(\lambda_{-i}))g_i(\lambda_{-i}|\lambda_i). \qquad \text{(A.4)}
\end{aligned}$$

Given the feasible premium profile p_{-i} and requiring

$$[\lambda_i(1-a_i)-b_i]^2 e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) > b_i^2, \text{ for all } p_i \in \mathcal{R}_i(\lambda_i),$$

leads to Condition (17). Therefore, all the summands in (A.4) are positive and hence, Condition (17) guarantees that for all feasible premium profiles p the payoff $C_i(p_i,p_{-i},\Lambda_i,\Lambda_{-i}|\Lambda_i=\lambda_i)$ is a strictly convex function of p_i in $\mathcal{R}_i(\lambda_i)$. Notice that the premium range is not possible to be an empty set since it is always true that [8]

$$\begin{aligned}
-\frac{1}{\lambda_i(1-a_i)}\log\left[\frac{1}{M_X(\lambda_i(1-a_i))} b_i^2 \left(\lambda_i(1-a_i)-b_i\right)^{-2}\right] &> \frac{1}{\lambda_i(1-a_i)}\log\left[M_X(\lambda_i(1-a_i))\right] \\
&= p_i^L(\lambda_i).
\end{aligned}$$

Uniqueness. Following a proof by contradiction as in Theorem 2 of Rosen (1965), we will show the uniqueness of the PSBNE. Let $p^{(1)}(\cdot)$ and $p^{(2)}(\cdot)$ be two distinct PSBNE premium strategy profiles with values in the space of the feasible premium region in which the payoff function $C_i(p_i,p_{-i},\Lambda_i,\Lambda_{-i}|\Lambda_i=\lambda_i)$ is a strictly convex function of p_i for all $\lambda_i \in \mathcal{L}_i$ and all $i \in \mathcal{N}$. Then, by the necessity of the Karash-Kuhn-Tucker

[8]Condition (17) implies something stronger. Namely, for all feasible premium profiles p_{-i} it holds

$$p_i^U < -\frac{1}{\lambda_i(1-a_i)}\log\left[\frac{1}{M_X(\lambda_i(1-a_i))} b_i^2 \left(\lambda_i(1-a_i)-b_i\right)^{-2}\right].$$

conditions and Definition 1, for each $i \in \mathcal{N}$ and every $\lambda_i \in \mathcal{L}_i$, there exist non-negative multipliers $\varrho_{i,\lambda_i}^{(k),l}$, $k = 1, 2$ and $l \in \{S, L, U\}$, such that

$$r_i^l(p_i^{(k)}(\lambda_i)) \leq 0, \tag{A.5}$$

$$\varrho_{i,\lambda_i}^{(k),l} r_i^l(p_i^{(k)}(\lambda_i)) = 0, \ \varrho_{i,\lambda_i}^{(k),l} \geq 0, \tag{A.6}$$

and

$$\frac{\partial C_i(p_i^{(k)}, p_{-i}^{(k)}, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)}{\partial p_i} + \sum_{l \in \{S,L,U\}} \varrho_{i,\lambda_i}^{(k),l} \frac{dr_i^l(p_i^{(k)}(\lambda_i))}{dp_i} = 0. \tag{A.7}$$

We multiply (A.7) by $(p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i))$ when $k = 1$ and by $(p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i))$ when $k = 2$. Summing on all insurers and all risk-aversion types yields

$$A + B = 0, \tag{A.8}$$

where

$$\begin{aligned} A \ &= \ \sum_{i \in \mathcal{N}} \sum_{\lambda_i \in \mathcal{L}_i} (p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i)) \frac{\partial C_i(p_i^{(1)}, p_{-i}^{(1)}, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)}{\partial p_i} \\ &\quad + \sum_{i \in \mathcal{N}} \sum_{\lambda_i \in \mathcal{L}_i} (p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i)) \frac{\partial C_i(p_i^{(2)}, p_{-i}^{(2)}, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i)}{\partial p_i}, \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} B \ &= \ \sum_{i \in \mathcal{N}} \sum_{\lambda_i \in \mathcal{L}_i} \sum_{l \in \{S,L,U\}} \Bigg[\varrho_{i,\lambda_i}^{(1),l} (p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i)) \frac{dr_i^l(p_i^{(1)}(\lambda_i))}{dp_i} \\ &\quad + \varrho_{i,\lambda_i}^{(2),l} (p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i)) \frac{dr_i^l(p_i^{(2)}(\lambda_i))}{dp_i} \Bigg]. \end{aligned} \tag{A.10}$$

For all $l \in \{S, L, U\}$, $r_i^l(p_i)$ is a convex function of p_i. Therefore,

$$r_i^l(p_i^{(2)}(\lambda_i)) \geq r_i^l(p_i^{(1)}(\lambda_i)) + (p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i)) dr_i^l(p_i^{(1)}(\lambda_i))/dp_i$$

and

$$r_i^l(p_i^{(1)}(\lambda_i)) \geq r_i^l(p_i^{(2)}(\lambda_i)) + (p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i))dr_i^l(p_i^{(2)}(\lambda_i))/dp_i.$$

Now, (A.10) yields

$$\begin{aligned} B &\leq \sum_{i\in\mathcal{N}}\sum_{\lambda_i\in\mathcal{L}_i}\sum_{l\in\{S,L,U\}}\left[\varrho_{i,\lambda_i}^{(1),l}\left(r_i^l(p_i^{(2)}(\lambda_i)) - r_i^l(p_i^{(1)}(\lambda_i))\right) + \varrho_{i,\lambda_i}^{(2),l}\left(r_i^l(p_i^{(1)}(\lambda_i)) - r_i^l(p_i^{(2)}(\lambda_i))\right)\right] \\ &= \sum_{i\in\mathcal{N}}\sum_{\lambda_i\in\mathcal{L}_i}\sum_{l\in\{S,L,U\}}\left[\varrho_{i,\lambda_i}^{(1),l} r_i^l(p_i^{(2)}(\lambda_i)) + \varrho_{i,\lambda_i}^{(2),l} r_i^l(p_i^{(1)}(\lambda_i))\right] \\ &\leq 0, \end{aligned}$$

where the last two relations are derived by the conditions (A.5) and (A.6). Similarly, the payoff function $C_i(p_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)$ is a strictly convex function of p_i and hence, we obtain the following two inequalities:

$$\begin{aligned} C_i(p_i^{(2)}, p_{-i}^{(2)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) \quad &- \quad C_i(p_i^{(1)}, p_{-i}^{(1)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) \\ &> \quad (p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i))\partial C_i(p_i^{(1)}, p_{-i}^{(1)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)/\partial p_i \end{aligned}$$

and

$$\begin{aligned} C_i(p_i^{(1)}, p_{-i}^{(1)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) \quad &- \quad C_i(p_i^{(2)}, p_{-i}^{(2)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) \\ &> \quad (p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i))\partial C_i(p_i^{(2)}, p_{-i}^{(2)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)/\partial p_i. \end{aligned}$$

Now, combining these two inequalities we have

$$(p_i^{(2)}(\lambda_i) - p_i^{(1)}(\lambda_i))\frac{\partial C_i(p_i^{(1)}, p_{-i}^{(1)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)}{\partial p_i} + (p_i^{(1)}(\lambda_i) - p_i^{(2)}(\lambda_i))\frac{\partial C_i(p_i^{(2)}, p_{-i}^{(2)}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i)}{\partial p_i} < 0,$$

and (A.9) gives $A < 0$. Finally, the fact that $A < 0$ and $B \leq 0$ contradicts the parity in (A.8). Thus, the PSBNE is unique.

A.3 Proof of Proposition 1

If $p_i^* = p_i^L(\lambda_i)$, the non-positivity constraint of r_i^L evaluated at p_i^* is binding (i.e., $r_i^L(p_i^*) = 0$) and hence, $\exp\{-\lambda_i(1-a_i)p_i^*\}M_X(\lambda_i(1-a_i)) - 1 = 0$. Then, the payoff function of Insurer i of type λ_i, given in (14), satisfies

$$\begin{aligned} C_i(p_i^L(\lambda_i), p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) &= \lambda_i a_i \pi_i \\ &> C_i(\tilde{p}_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i), \end{aligned}$$

for all $\tilde{p}_i \in (p_i^L(\lambda_i), p_i^U)$ and all feasible premium profiles p_{-i}.

In the case of the cut-off point p_i^U for Insurer i's demand function, if $p_i^* = p_i^U$, by assumption $q_i(p_i^U, p_{-i}) = 0$ for all feasible p_{-i}. Thus, for all $\tilde{p}_i \in (p_i^L(\lambda_i), p_i^U)$, it holds that $\exp\{-\lambda_i(1-a_i)\tilde{p}_i\}M_X(\lambda_i(1-a_i)) - 1 < 0$ and hence,

$$\begin{aligned} C_i(p_i^U, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i) &= \lambda_i a_i \pi_i \\ &> C_i(\tilde{p}_i, p_{-i}, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i). \end{aligned}$$

A.4 Proof of Proposition 2

Given that Λ_i, $i \in \mathcal{N}$, are independent, we have $g_i(\lambda_{-i}|\lambda_i) = g_i(\lambda_{-i})$. To simplify our notation, we write

$$\mathbb{E}\left[f_i(p_i, \bar{p}_{-i})\right] = \sum_{\lambda_{-i} \in \mathcal{L}_{-i}} f_i(p_i, \bar{p}_{-i}(\lambda_{-i}))q_{i,0}g_i(\lambda_{-i}).$$

Let $p_i^{'} = p_i^*(\lambda_i^{'})$ and $p_i^{''} = p_i^*(\lambda_i^{''})$. Considering that $p^*(\cdot)$ is a PSBNE premium strategy profile, Insurer i of risk-aversion type $\lambda_i^{'}$ prefers $p_i^{'}$ to $p_i^{''}$ and of risk-aversion type $\lambda_i^{''}$ prefers $p_i^{''}$ to $p_i^{'}$. In terms of preferences, we have the inequalities

$$C_i(p_i^{'}, p_{-i}^*, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i^{'}) \leq C_i(p_i^{''}, p_{-i}^*, \Lambda_i, \Lambda_{-i}|\Lambda_i = \lambda_i^{'})$$

and

$$C_i(p_i^{''}, p_{-i}^*, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i^{''}) \leq C_i(p_i^{'}, p_{-i}^*, \Lambda_i, \Lambda_{-i} | \Lambda_i = \lambda_i^{''}).$$

Utilising the definition of the payoff function in (14), and subtracting the right-hand side of the latter inequality from the left-hand side of the former and the left-hand side of the latter from the right-hand side of the former, we obtain

$$\begin{aligned} &\left[e^{-\lambda_i^{'}(1-a_i)p_i^{'}} M_X(\lambda_i^{'}(1-a_i)) - e^{-\lambda_i^{''}(1-a_i)p_i^{'}} M_X(\lambda_i^{''}(1-a_i))\right] \mathbb{E}\left[f_i(p_i^{'}, \bar{p}_{-i}^*)\right] \\ &\leq \qquad \qquad \text{(A.11)} \\ &\left[e^{-\lambda_i^{'}(1-a_i)p_i^{''}} M_X(\lambda_i^{'}(1-a_i)) - e^{-\lambda_i^{''}(1-a_i)p_i^{''}} M_X(\lambda_i^{''}(1-a_i))\right] \mathbb{E}\left[f_i(p_i^{''}, \bar{p}_{-i}^*)\right]. \end{aligned}$$

Now, we define the function

$$T_i(p) = \left[e^{-\lambda_i^{'}(1-a_i)p} M_X(\lambda_i^{'}(1-a_i)) - e^{-\lambda_i^{''}(1-a_i)p} M_X(\lambda_i^{''}(1-a_i))\right] \mathbb{E}\left[f_i(p, \bar{p}_{-i}^*)\right],$$

where p is a premium rate. The first derivative of T_i with respect to p is equal to

$$\begin{aligned} \frac{dT_i(p)}{dp} &= e^{-\lambda_i^{'}(1-a_i)p} M_X(\lambda_i^{'}(1-a_i)) \left[\mathbb{E}\left[\frac{\partial f_i(p, \bar{p}_{-i}^*)}{\partial p}\right] - \lambda_i^{'}(1-a_i)\mathbb{E}\left[f_i(p, \bar{p}_{-i}^*)\right]\right] \\ &\quad - e^{-\lambda_i^{''}(1-a_i)p} M_X(\lambda_i^{''}(1-a_i)) \left[\mathbb{E}\left[\frac{\partial f_i(p, \bar{p}_{-i}^*)}{\partial p}\right] - \lambda_i^{''}(1-a_i)\mathbb{E}\left[f_i(p, \bar{p}_{-i}^*)\right]\right], \end{aligned}$$

where for simplification purposes we denote

$$\mathbb{E}\left[\frac{\partial f_i(p, \bar{p}_{-i}^*)}{\partial p}\right] = \sum_{\lambda_{-i} \in \mathcal{L}_{-i}} \frac{\partial f_i(p, \bar{p}_{-i}^*)}{\partial p} q_{i,0} g_i(\lambda_{-i}).$$

If $\lambda_i^{'} < \lambda_i^{''}$, we derive $dT_i(p)/dp \geq 0$, i.e., $T_i(p)$ is a non-decreasing function of p. Then, Inequality (A.11) is rewritten as $T_i(p_i^{'}) \leq T_i(p_i^{''})$, and the monotonicity of T_i yields $p_i^{'} \leq p_i^{''}$ for $\lambda_i^{'} < \lambda_i^{''}$. Thus, p_i^* is a non-decreasing function of λ_i.

A.5 Numerical verification of Condition (17)

Firstly, we discretize the compact premium range $\mathcal{P}_i(\lambda_i) = [p_i^L(\lambda_i), p_i^U]$ into k equidistant points. Let $\mathcal{P}_i^D(\lambda_i) = \left\{p_i^L(\lambda_i) = p_{i,1}, p_{i,2}, \ldots, p_{i,k} = p_i^U\right\}$ denote the discretized premium range of Insurer i. The discretized set of all premium profiles of the insurers is given by the Cartesian product

$$\mathcal{P}^D = \prod_{i \in \mathcal{N}} \prod_{\lambda_i \in \mathcal{L}_i} \mathcal{P}_i^D(\lambda_i).$$

Now, we verify that the sum in (A.4) is positive over all possible combinations of the discretized feasible premium profiles $p \in \mathcal{P}^D$, and this is shown by Algorithm 1 below.

Algorithm 1

Result: Give the first point $p \in \mathcal{P}^D$ that sum (A.4) is **non-positive**;

Repeat for all i in $\mathcal{N}$;

for $\lambda_i \in \mathcal{L}_i$ **do**

  for $p_i \in \mathcal{P}_i^D(\lambda_i)$ **do**

    for $\lambda_{-i} \in \mathcal{L}_{-i}$ **do**

      for $p_{-i} \in \mathcal{P}_{-i}^D(\lambda_{-i})$ **do**

        Set $b_i = \dfrac{\partial h_i(p_i, \bar{p}_{-i})}{\partial p_i}$;

        if $\sum_{\lambda_{-i} \in \mathcal{L}_{-i}} \left\{ [\lambda_i(1-a_i) - b_i]^2 \, e^{-\lambda_i(1-a_i)p_i} M_X(\lambda_i(1-a_i)) - b_i^2 \right\} q_{i,0} f_i(p_i, \bar{p}_{-i}) g_i(\lambda_{-i}|\lambda_i) \leq 0$

        then

          print $\{(p_i, p_{-i})$ and $(\lambda_i, \lambda_{-i})\}$;

          break;

        end

      end

    end

  end

end

Outcome: the empty set.

Next, we verify that Condition (17) also holds. This is shown by Algorithm 2 below.

Algorithm 2

Result: Give the first point p at which Condition (17) is **<u>not</u>** valid;

Repeat for all i in $\mathcal{N}$;

for $\lambda \in \mathcal{L}$ **do**

  Set $p_i^{\text{Ref}} = p_i^U$;

  for $p_{-i} \in \mathcal{P}_{-i}^D(\lambda_{-i})$ **do**

    Set $b_i = \dfrac{\partial h_i(p_i^{\text{Ref}}, \bar{p}_{-i})}{\partial p_i}$;

    if $p_i^{Ref} \geq -\dfrac{1}{\lambda_i(1-a_i)} \log\left[\dfrac{1}{M_X(\lambda_i(1-a_i))} b_i^2 \left(\lambda_i(1-a_i) - b_i\right)^{-2}\right]$ **then**

      print $\left\{(p_i^{\text{Ref}}, p_{-i}) \text{ and } (\lambda_i, \lambda_{-i})\right\}$;

      break;

    end

  end

end

Outcome: the empty set.