Redistribution of Longevity Risk: The effect of heterogeneous mortality beliefs*

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Abstract

Existing literature regarding the natural hedge potential that arises from combining different longevity-linked liabilities typically does not address the question how changes in the liability mix can be obtained. We consider firms who aim to exploit the benefits of natural hedge potential by redistributing their risks, and characterize the risk redistributions that will arise when the parties bargain for a redistribution of risk that weakly benefits them all. We analyze the effects of heterogeneity in the beliefs regarding the probability distribution of future mortality rates on the properties of these risk redistributions, and provide a numerical illustration for a case where an insurer with a portfolio of term assurance contracts and a pension fund with a portfolio of life annuities redistribute their risks.

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1 Introduction

Well-functioning markets can serve as mechanisms to reallocate risk among firms. However, for some classes of risk, markets are non-existent or there are significant obstacles to trade. In such cases, firms could approach each other to negotiate a redistribution of their risks. In this paper, we investigate the extent to which firms can benefit from such risk redistributions. Our main focus is on how heterogeneity in the beliefs regarding the underlying probability distribution of the risks affects the characteristics of the risk redistributions.

Although the model that we introduce allows for any type of risk, our focus in this paper is on redistribution of longevity risk. Longevity risk is the systematic risk in life-contingent liabilities due to uncertain upward or downward deviations of future survival rates from their best-estimate values.[1] Existing literature shows that longevity risk can have a significant impact on the liabilities of pension funds and insurers (see, e.g., Coughlan et al. 2007; Hári et al. 2008; Pitacco et al., 2009). Pension funds and insurers could reduce the impact of longevity risk on their liabilities via reinsurance or longevity-linked derivatives, such as, longevity swaps, longevity bonds, or q-forwards (see, e.g., Blake et al., 2006; Ngai and Sherris, 2011). However, the capacity of reinsurance is limited (see, e.g., Basel Committee on Banking Supervision, 2013), and lack of consensus regarding the price of longevity risk hampers trade of longevity-linked derivatives.[2] In this paper, we focus on how firms with longevity-linked liabilities can mitigate the impact of longevity risk on their liabilities by redistributing their risks. While the existing literature on the natural hedge potential that arises from combining different liabilities (see, e.g., Cox and Lin, 2007; Tsai et al., 2010; Wang et al., 2010; Zhu and Bauer, 2014; Li and Haberman, 2015) focuses on quantifying the corresponding risk reduction, it does not address the question whether and how an improved product mix can be obtained. This is the question that we address in this paper.

We consider a small number of firms with longevity-linked liabilities (e.g., insurers and pension sponsors) who wish to reduce the impact of longevity risk on their liabilities by redistributing their risks. The main focus of our analysis is on how heterogeneity in the beliefs regarding the probability distribution of future survival rates affects the characteristics of the risk redistribution. Disagreement regarding the underlying probability distribution is potentially an important concern when parties need to agree on a redistribution of longevity-linked risks. Starting with the seminal contribution of Lee and Carter (1992), a relatively large variety of stochastic mortality forecast models has been developed (see, e.g., Brouhns et al., 2002; Cairns et al., 2006; Cossette et al., 2007; Plat, 2009; Dowd et al., 2010; Haberman and Renshaw, 2012; Niu and Melenberg, 2014; Börger et al., 2014). However, no single model outperforms the other models in terms of in-sample fit, out-of-sample forecast accuracy, biological reasonableness, etc.

[1] While annuity providers are mainly concerned about underestimation of survival rates, insurers who offer (term) assurance (i.e. a payment in the event of death) are mainly concerned about overestimation of survival rates. In each case, we refer to the corresponding risks as longevity risk.

[2] Blake et al. (2006) address the main obstacles for trading longevity linked products in the market.

As a consequence, there is generally no consensus regarding which model is "best"(see, e.g., Cairns et al., 2007, 2008, 2011; Haberman and Renshaw, 2011). Moreover, several studies have shown that the estimated probability distribution of future mortality rates can differ significantly when different models are used, or when model parameters are estimated on different datasets (see, e.g., Dowd et al., 2008; Li et al., 2015). Hence, heterogeneous beliefs regarding the "true" model can have a significant impact on the preferred risk redistributions of pension funds and insurers. In our model, we allow the parties who wish to redistribute risk to "agree to disagree" on the probability distribution of future mortality rates, and we investigate the consequences of such disagreements on the resulting risk redistributions.

We first show that there always exists at least one redistribution that is "*stable*" in the sense that no subset of firms can be better off by not participating in the redistribution, or by excluding some other firms from the redistribution. Then, we use the constrained Nash bargaining solution to characterize the stable risk redistribution that results from a bargaining process in which each involved party bargains for the risk redistribution that it prefers. Several studies have used Nash bargaining solutions to characterize risk redistributions (Kihlstrom and Roth, 1982; Schlesinger, 1984; Aase, 2009; Quiggin and Chambers, 2009; Zhou et al., 2015; Boonen, 2016). Our paper focuses on how heterogeneity in the beliefs regarding the probability distribution of mortality rates affects the characteristics of the Nash bargaining solution when firms bargain for a redistribution of longevity risk. With heterogeneous beliefs, the firms can all benefit from shifting risk in a scenario to a firm that assigns the lowest probability to the scenario. This implies that the effects of heterogeneity on the redistribution are larger when the firms are less risk averse. Regardless of the degrees of risk aversion, we find that it is *more* likely that parties will achieve strict Pareto improvement via this risk redistribution when they disagree on the underlying probability distribution of future mortality rates.

In our numerical illustration, we consider redistribution of longevity risk between a pension fund with a portfolio of life annuities and an insurer with a portfolio of term assurance contracts.[3] We quantify the benefits from the risk redistribution by determining the maximum premium that the pension fund (the insurer) would have been willing to pay for a reinsurance contract that yields the same degree of risk reduction. We consider degrees of risk aversion varying from close to risk neutral to very risk averse, and measure the benefits from a redistribution by determining the premium that the pension fund (insurer) would be willing to pay for a reinsurance contract that yields the same degree of risk reduction as the risk redistribution, expressed as percentage of the best-estimate value of the liabilities. Our results suggest that the benefits from risk redistribution are significant, even when the insurer is small relative to the pension fund and risk is redistributed over a short (one-year) horizon. Depending on the firms' degrees of risk aversion and their beliefs regarding the probability distribution of survival rates, the relative zero utility premium corresponding to the distribution varies from 0.1% to

[3]Term assurance provides coverage (a death benefit) in the event of decease of the insured life during a prespecified period.

18.4% for the pension fund, and from 0.7% to 37.5% for the insurer. While the perceived benefits from the redistribution are increasing in the degree of risk aversion if the two firms have homogeneous beliefs regarding the probability distribution of future mortality rates, this is no longer the case when they have heterogeneous beliefs; the perceived benefits can then be high both when risk aversion is very low and when it is very high. This occurs because when firms are close to risk neutral, they can benefit significantly from shifting risk in a specific scenario to firms that assign a lower probability to this scenario. On the other hand, when firms are highly risk averse, the (perceived) utility gains from risk reduction via pooling risks are large. Moreover, heterogeneity in the beliefs may, but need not, be beneficial to the firms. The relative zero utility premium can be both higher and lower than when the firms have homogeneous beliefs.

The remainder of this paper is organized as follows. We introduce the model for redistribution of longevity risk in Section 2. In Section 3, we characterize properties that the risk redistribution will satisfy if the involved parties act rationally, and in Section 4 we model the choice of a particular redistribution as the outcome of a bargaining process that weighs the utility gains of all involved parties. In Section 5, we use the model to numerically illustrate the extent to which a pension fund and an insurer can benefit from redistributing longevity risk. Section 6 concludes.

2 Redistributing longevity risk

In this section we present the basic model for risk redistribution between firms with longevity-linked liabilities. We consider the case where on date 0, firms with longevity-linked liabilities (e.g., pension funds and insurers) redistribute risk in order to reduce the impact of longevity risk on their net asset value on a prespecified date T. Henceforth, we refer to this net asset value as the date-T NAV.[4] Throughout the paper, we let $T \in \{1, 2, \dots\}$ be given, and we consider a given set of firms, indexed by N. Unless mentioned otherwise, we will without loss of generality let $N = \{1, \dots, n\}$. Moreover, for dates $t \in \{1, 2, \dots\}$, we refer to the time period $(t-1, t]$ as year t.

In Section 2.1, we model the date-T NAV of firms with longevity linked liabilities. In Section 2.2, we discuss the assumptions regarding the firms' risk preferences and their (subjective) beliefs regarding the underlying probability distribution of future mortality rates.

2.1 The risk profiles

The date-T NAV of firm $i \in N$ is given by

$$X_i = A_i(T) - L_i(T), \tag{1}$$

[4]The focus on Net Asset Value is in line with current regulation. Under Solvency II, for example, the regulator requires that the current level of assets is sufficient to reduce the probability of a negative Net Asset Value on a one-year horizon to a sufficiently low level.

where $A_i(T)$ denotes the (market) value of the assets of firm i on date T, and $L_i(T)$ denotes the date-T value of the liabilities of firm i.

For firms with life-contingent liabilities, the date-T NAV is affected by uncertainty in future mortality rates in two ways:

1. Uncertainty in future mortality rates affects the level of payments made during the period, which in turn affects the asset value $A_i(T)$ at the end of the period.

2. Forecasts for mortality rates in years $T+1, T+2, \ldots,$ depend on realized mortality rates in years $1, 2, \ldots, T$. Therefore, uncertainty in mortality rates in years $1, 2, \ldots, T$ affects the way in which the liabilities are valued on date T, i.e., it affects $L_i(T)$. The effect depends on how $L_i(T)$ is determined.

In order to focus on longevity risk, we assume a deterministic annual return on assets, which we denote by r. Moreover, we assume that all liability payments occur at the end of a year. Then, the asset value for firm $i \in N$ on date T equals

$$A_i(T) = (1+r)^T \cdot A_i - \sum_{\tau=1}^{T} \widetilde{L}_{i,\tau} \cdot (1+r)^{T-\tau}, \tag{2}$$

where $A_i = A_i(0)$ denotes the asset value of firm i on date $t = 0$, and $\widetilde{L}_{i,\tau}$ denotes the (stochastic) liability payment of firm i in year τ.
Ideally, $L_i(T)$ would represent the market value on date T of the future liabilities (i.e., the value at which the liabilities can be sold to a third party). However, because there is (not yet) a liquid market for longevity-linked derivatives, there is not yet a market price, and pension funds and insurance companies have to valuate their liabilities using mark-to-model valuation.[5] For example, if firms valuate their liabilities at the *best-estimate value* with respect to their own (subjective) beliefs regarding the probability distribution of future mortality rates, then

$$L_i(T) = \sum_{\tau=1}^{T^{\max}-T} \frac{\widetilde{L}_{i,T+\tau}^{(\mathrm{BE(T)})}}{(1+r)^{\tau}}, \tag{3}$$

where $\widetilde{L}_{i,T+\tau}^{(\mathrm{BE(T)})}$ denotes the expected value of the liability payment in year $T+\tau$, based on the date-T best-estimate scenario for survival probabilities in years $T+1, T+2, \ldots, T+\tau$. We describe how we determine the date-T best-estimate scenario in Appendix D. We emphasize that none of our analytical results depend on how exactly valuation is done on date T. For example, an alternative that fits into our model would be the case where $L_i(T)$ includes a risk premium on top of the best-estimate value. While our analytical

[5]Due to market incompleteness, mark-to-market valuation (valuation based on observed market prices) is typically not possible for longevity linked liabilities. Therefore, in practice, one typically resorts to mark-to-model based valuation, which means that a model is used to determine the value of the liabilities. A mark-to-model approach that is used often in practice is to determine the expected present value of future payments, using "best-estimate" predictions of future mortality rates.

model and results in Sections 3 and 4 allow for a risk premium to be included in $L_i(T)$, we focus on valuation as in (3) in our numerical illustration.

Combining (1)-(3) yields that the date-T NAV for firm $i \in N$ is given by

$$X_i = [A_i - Y_i]\,(1+r)^T, \tag{4}$$

where

$$Y_i = \sum_{\tau=1}^{T} \frac{\widetilde{L}_{i,\tau}}{(1+r)^\tau} + \frac{L_i(T)}{(1+r)^T} = \sum_{\tau=1}^{T} \frac{\widetilde{L}_{i,\tau}}{(1+r)^\tau} + \sum_{\tau=T+1}^{T^{\max}} \frac{\widetilde{L}_{i,\tau}^{(\mathrm{BE(T)})}}{(1+r)^\tau}. \tag{5}$$

Hence, the date-T NAV equals the initial asset value, A_i, increased by the return on assets and reduced by $Y_i \cdot (1+r)^T$. The random variable Y_i represents the sum of the present value of *the actual liability payments* up to year T, and the present value of *the date-T best-estimate expected payments* in years beyond date T. We refer to Y_i as the *prior risk profile* of firm i. The prior risk profiles are affected by longevity risk due to uncertainty in the population survival rates in years $1 \leq \tau \leq T$. This uncertainty affects Y_i in two ways:

(i) it induces uncertainty in the actual liability payments $\widetilde{L}_{i,\tau}$ for years $1 \leq \tau \leq T$,

(ii) it induces uncertainty in the date-T best-estimate value of the liabilities $\widetilde{L}_{i,\tau}^{(\mathrm{BE(T)})}$ for years $\tau > T$; this uncertainty arises because uncertainty in the population survival rates in years $1 \leq \tau \leq T$:

  - induces uncertainty in survival/decease of participants in years $1, \ldots, T$, which in turn induces uncertainty in the composition of the fund on date T;
  - induces uncertainty in the *best-estimate survival rates for years $\tau > T$* that will be used on date T to determine the best-estimate value of the liabilities for the participants who are still alive. This uncertainty is referred to as "longevity trend risk".

Some studies on the natural hedge potential that arises from combining different longevity-linked liabilities focus on the effect of liability mix on the present value of the liability payments over complete run-off, i.e., the focus is on the case where $T = T^{\max}$, so that $Y_i = \sum_{\tau=1}^{T^{\max}} \widetilde{L}_{i,\tau}/(1+r)^\tau$ (see, e.g., Tsai et al., 2010; Wang et al, 2010). A potential drawback of that approach is that firms need to agree on a redistribution of risk over complete run-off. Moreover, firms may not benefit optimally in terms of reducing the impact of longevity trend risk. Several studies have shown that longevity trend risk can have a significant impact on the value of life-contingent liabilities (see, e.g., Plat, 2011; Richards et al., 2014). Considering instead the effect on the NAV over a shorter horizon, as in our study, allows firms to reduce their sensitivity to longevity trend risk. It also allows for more flexibility in renegotiating contracts when conditions change. For example, once a contract ends, firms can re-evaluate their liabilities according to new mortality data, changes in portfolio composition, or maybe even new regulations, and then negotiate a new contract.

2.2 Risk redistributions

Via redistribution of their risks, firms with longevity-linked liabilities may be able to mitigate the adverse effects of longevity risk on their date-T NAV. Because our focus is on how heterogeneous beliefs regarding the probability distribution of future survival rates affects the characteristics of the redistributions that firms will agree to, we do not impose any restrictions on the form of the redistribution; we allow for any redistribution of risk that leads to *posterior risk profile* $(Y_i^{\text{post}})_{i\in N}$ that satisfy $(Y_i^{\text{post}})_{i\in N} \in \mathcal{F}(N)$, where

$$\mathcal{F}(N) = \left\{ (Y_i^{\text{post}})_{i\in N} : \sum_{i\in N} Y_i^{\text{post}} = \sum_{i\in N} Y_i \right\}. \tag{6}$$

We refer to $\mathcal{F}(N)$ as the set of *feasible* posterior risk profiles.

For any given posterior risk profiles $(Y_i^{\text{post}})_{i\in N} \in \mathcal{F}(N)$, the date-$T$ NAV of firm i after risk redistribution, which we denote X_i^{post}, is given by:

$$X_i^{\text{post}} = (1+r)^T \cdot \left(A_i - Y_i^{\text{post}} \right), \qquad \text{for } i \in N. \tag{7}$$

The redistribution that the firms will agree on will depend on their risk preferences, and on their (subjective) beliefs regarding the joint probability distribution of the risk profiles of the firms involved in the redistribution. Because for most of the commonly used mortality forecast methods in practice (e.g., the Lee and Carter, 1992, model or one of its extensions), there is typically not a closed form expression for the probability distribution of the underlying risks, we assume that the firms approximate the joint probability distribution of the risks via simulations. Specifically, scenarios for the future development of mortality rates are generated using one or more stochastic forecast models. These scenarios give rise to a discretized probability distribution of $(Y_i)_{i\in N}$. We allow for the possibility that the firms use different mortality forecast models, which in turn implies that they may have different beliefs regarding the probability distribution of future mortality rates. We discuss the details of the discretization procedure in Appendix E. We use the following notation and assumptions:

- Ω denotes the set of scenarios for future mortality rates; Ω is finite.
- $\mathbb{P}_i : \Omega \to \mathbb{R}_{++}$ denotes firm i's subjective probability measure over scenarios.
- To evaluate the attractiveness of a risk profile, firm $i \in N$ uses von Neumann-Morgenstern expected utility with respect to its own subjective beliefs. Hence, the utility of firm i of a risk profile $Y : \Omega \to \mathbb{R}$, is given by

  $$U_i(Y) := \sum_{\omega\in\Omega} u_i(Y(\omega)) \cdot \mathbb{P}_i(\{\omega\}), \tag{8}$$

  where the utility function $u_i : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and strictly concave, with $\lim_{x\to-\infty} u_i'(x) = \infty$, and $\lim_{x\to+\infty} u_i'(x) = 0$.

3 Properties of rational risk redistributions

If firms behave myopically, i.e., if each firm bargains for a redistribution that maximizes its own utility, the firms will typically not reach an agreement even though each firm knows that if they behave cooperatively, they can all benefit.[6] We therefore consider a setting in which firms behave cooperatively, and strive to reach a redistribution of risk that benefits all. In Section 3.1, we characterize properties that the risk redistribution will satisfy if firms behave rationally and redistribution is not mandated. In Section 3.2, we show how heterogeneity in the beliefs regarding the probability distribution of future mortality rates affects characteristics of risk redistributions that satisfy these properties.

3.1 Individual rationality, Pareto optimality and Stability

For any given feasible posterior risk profiles $(Y_i^{\text{post}})_{i\in N} \in \mathcal{F}(N)$, we denote $\Delta U_i(Y_i^{\text{post}})$ for the difference in the expected utility of the date-T NAV of firm $i \in N$ before and after redistribution, i.e.,

$$\Delta U_i(Y_i^{\text{post}}) = U_i\left((1+r)^T \cdot \left(A_i - Y_i^{\text{post}}\right)\right) - U_i\left((1+r)^T \cdot (A_i - Y_i)\right). \tag{9}$$

Moreover, for any (sub)set of firms $S \subseteq N$, we let $\mathcal{NI}(S)$ be the set of risk profiles for the firms in S for which no Pareto improvement is possible, i.e., there do not exist other posterior risk profiles for the firms in S that yield weakly higher utility for all firms in S, and strictly higher utility for at least one firm in S, i.e.,

$$\mathcal{NI}(S) = \left\{ (Y_i^{\text{post}})_{i\in S} : \nexists (\widetilde{Y}_i^{\text{post}})_{i\in S} \text{ s.t. } \begin{array}{l} \sum_{i\in S} \widetilde{Y}_i^{\text{post}} = \sum_{i\in S} Y_i, \\ (\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in S} \gneqq (\Delta U_i(Y_i^{\text{post}}))_{i\in S} \end{array} \right\}, \tag{10}$$

where for any two vectors x and y with the same dimension, $x \gneqq y$ if and only if $x \geq y$ component-wise and $x \neq y$.

If firms act rationally, they will only agree to a certain redistribution if it does not make them worse off in expected utility terms. Hence, a necessary condition for all firms to be willing to participate in the redistribution is that the redistribution is individually rational. We denote $\mathcal{IR}$ as the set of feasible posterior risk profiles that satisfy *Individual Rationality*, i.e.,

$$\mathcal{IR} = \left\{ (Y_i^{\text{post}})_{i\in N} \in \mathcal{F}(N) : (\Delta U_i(Y_i^{\text{post}}))_{i\in N} \geq 0 \right\}. \tag{11}$$

Moreover, all firms have incentives to not engage in a particular redistribution if it is not Pareto optimal. The set of feasible posterior risk profiles that satisfy *Pareto Optimality*, which we denote $\mathcal{PO}$, is given by

$$\mathcal{PO} = \left\{ (Y_i^{\text{post}})_{i\in N} \in \mathcal{F}(N) : (Y_i^{\text{post}})_{i\in N} \in \mathcal{NI}(N) \right\}. \tag{12}$$

[6] In game theory, this phenomenon is known as the Prisoner's dilemma. See Boonen (2016) for an illustration in bargaining for Over-The-Counter (OTC) insurance risk redistributions.

If a redistribution is Pareto optimal, there does not exist another feasible redistribution that makes each firm weakly better off, and at least one firm strictly better off. *Pareto Optimality*, however, does not rule out the possibility that a subset of firms could be better off if they decide to redistribute risk amongst each other, excluding the other firms from the negotiation. Allowing more firms to cooperate in the redistribution has the potential advantage that the set of posterior risk profiles that a firm can reach increases. However, it may have the drawback that each firm negotiates with a larger number of other firms who each want to benefit from the redistribution. It is therefore not a priori clear that firms cannot do better by excluding some firms from the negotiation. We therefore consider the subset of risk redistributions that satisfy a stronger condition, referred to as *Stability*. The set of feasible posterior risk profiles that satisfy *Stability*, which we denote by $\mathcal{S}$, is given by

$$\mathcal{S} = \left\{ (Y_i^{\text{post}})_{i \in N} \in \mathcal{F}(N) : (Y_i^{\text{post}})_{i \in S} \ \in \mathcal{NI}(S) \text{ for all } S \subseteq N \right\}. \tag{13}$$

If a redistribution leads to posterior risk profiles that satisfy *Stability,* no set of firms $S \subseteq N$ can be better off when they exclude the other firms from the negotiation and redistribute their risk amongst each other.[7] Because this holds true in particular for the set of all firms, i.e., for $S = N$, *Stability* implies that all firms together cannot achieve a redistribution from which they all weakly benefit and one firm strictly benefits. Hence, *Stability* implies *Pareto optimality.* Moreover, because the condition in (13) also needs to hold for any individual firm, i.e., for $S = \{i\}$ for all $i \in N$, no firm should be better off if it does not participate in the redistribution. This implies that *Stability* also implies *Individual Rationality.* Moreover, if risk is redistributed between two firms, *Stability* is satisfied if and only if both *Individual Rationality* and *Pareto Optimality* are satisfied.[8] Hence,

$$\begin{aligned} & \mathcal{S} \subseteq \mathcal{PO} \cap \mathcal{IR}, \\ \text{and} \qquad & \mathcal{S} = \mathcal{PO} \cap \mathcal{IR}, \quad \text{if } |N| = 2. \end{aligned}$$

Using game-theoretic methods, we show in the following proposition that there exists at least one redistribution that satisfies *Stability.*

Proposition 1 *There exist feasible posterior risk profiles that satisfy Stability, i.e.,* $\mathcal{S}$ *is non-empty.*

3.2 The effect of heterogeneous beliefs

A key focus of our analysis is on how heterogeneity in the firms' beliefs regarding the probability distribution of future mortality rates affects the risk redistribution. We start

[7] In game-theoretic terms, this condition implies that the redistribution is an element of the core of the corresponding game (Gillies, 1953; Scarf, 1967).

[8] This follows immediately from (12) and (13), and the fact that $(Y_i^{\text{post}})_{i \in N} \in \mathcal{IR}$ implies $Y_i^{\text{post}} \in \mathcal{NI}(\{i\})$ for all $i \in N$, and $(Y_i^{\text{post}})_{i \in N} \in \mathcal{PO}$ implies $(Y_i^{\text{post}})_{i \in N} \in \mathcal{NI}(N)$.

by analyzing how heterogeneity in the beliefs regarding the probability distribution of future mortality rates affects the structure of Pareto optimal redistributions.

It follows from Wilson (1968) that $(Y_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ if and only if there exists a $\overline{k} = (k_i)_{i\in N} \in \mathbb{R}_{++}^N$ such that

$$Y_j^{\text{post}}(\omega) = (1+r)^T A_j - \left(u_j'\right)^{-1}\left[\left(\frac{k_1\mathbb{P}_1(\omega)}{k_j\mathbb{P}_j(\omega)}\right)\cdot u_1'\left((1+r)^T\left(A_1 - Y_1^{\text{post}}(\omega)\right)\right)\right], \text{ for all } j \neq 1, \tag{14}$$

$$\sum_{j\in N} Y_j^{\text{post}}(\omega) = \sum_{j\in N} Y_j(\omega). \tag{15}$$

Hence, the set of (infinitely many) Pareto optimal risk redistributions can be found by solving the system of equations (14) and (15) for every $\overline{k} \in \mathbb{R}_{++}^N$, where without loss of generality one can impose as normalization that $k_1 = 1$. The fact that *Pareto Optimality* is a necessary condition for *Stability* implies that *Stable* risk redistributions need to satisfy (14) and (15).

We now use the characterization in (14) and (15) to analyze the effect of heterogeneity in the beliefs regarding the probability distribution of future mortality rates on the structure of Pareto optimal redistributions.

Proposition 2 *Let $\overline{k} \in \mathbb{R}_{++}^N$ be given, and let $(Y_i^{post})_{i\in N}$ be the corresponding Pareto optimal posterior risk profiles from (14) and (15). Then, for all firms $i \in N$, and for all states $\omega \in \Omega$, it holds that:*

- *$Y_i^{post}(\omega)$ is decreasing in $\mathbb{P}_i(\{\omega\})$;*
- *$Y_i^{post}(\omega)$ is increasing in $\mathbb{P}_j(\{\omega\})$ for $j \neq i$.*

Proposition 2 implies that when the probability that a firm assigns to a specific scenario $\omega \in \Omega$ for the development of future mortality rates increases, then, ceteris paribus, the risk assigned to this firm in scenario ω decreases, and the risk assigned to all other firms in scenario ω increases. This suggests that heterogeneity regarding the subjective probability distributions has non-trivial effects on the structure of Pareto optimal redistributions. It is well-known (see, e.g., Gerber and Pafumi, 1998) that when firms have homogeneous beliefs regarding the underlying probability distribution of the risks, Pareto optimal risk redistributions for a broad class of utility functions (including exponential, quadratic, and logarithmic) are of the form

$$Y_i^{\text{post}} = f_i\left(\sum_{j\in N} Y_j\right) - d_i, \text{ for } i \in N, \tag{16}$$

where $f_i(\cdot)$ are deterministic functions with $\sum_{i\in N} f_i(x) = x$ for all x, and $(d_i)_{i\in N}$ represent the deterministic (net) reinsurance premium received (or paid if negative) by firm i, with $\sum_{i \in N} d_i = 0$. The underlying intuition is that pooling the risk is optimal because (e.g., due to natural hedge potential between different types of longevity-linked liabilities) $\sum_{j \in N} Y_j$ is typically less sensitive to longevity risk than each of the individual risks separately. However, when firms use different mortality forecast models, and, hence, have heterogeneous beliefs regarding the probability distribution of the risks, redistributions as in (16) may no longer be optimal.

Consider, for example, the case where the firms use exponential utility functions.[9] Specifically, the utility function of firm $i \in N$ is given by

$$u_i(x) = -\frac{1}{\lambda_i} \exp(-\lambda_i x), \text{ for all } x \in \mathbb{R}, \tag{17}$$

where $\lambda_i > 0$ denotes the degree of risk aversion of firm i. Now let $\lambda = \left(\sum_{i \in N} \frac{1}{\lambda_i}\right)^{-1}$. Solving (14) and (15) yields that the Pareto optimal posterior risk profiles are given by[10]

$$Y_i^{\text{post}}(\omega) = \delta_i \cdot \left(\sum_{j \in N} Y_j(\omega)\right) + Z_i(\omega) - d_i, \tag{18}$$

where

$$\begin{aligned}
\delta_i &= \frac{\lambda}{\lambda_i}, &(19)\\
Z_i(\omega) &= -\frac{\log(\mathbb{P}_i(\{\omega\}))}{\lambda_i} + \frac{\lambda}{\lambda_i} \sum_{j \in N} \frac{\log(\mathbb{P}_j(\{\omega\}))}{\lambda_j}, \text{ for all } \omega \in \Omega, &(20)\\
d_i &= \frac{\log(k_i)}{\lambda_i} - \frac{\lambda}{\lambda_i} \sum_{j \in N} \frac{\log(k_j)}{\lambda_j}, &(21)
\end{aligned}$$

for $\overline{k} \in \mathbb{R}^N_{++}$. Note that $\sum_{j \in N} d_j = 0$, $\sum_{j \in N} \delta_j = 1$, and $\sum_{j \in N} Z_j(\omega) = 0$, for all $\omega \in \Omega$.

Equation (18) shows that the posterior risk profile of firm i consists of a proportional share of the aggregate risk, $\delta_i \cdot \left(\sum_{j \in N} Y_j\right)$, an additional risk Z_i, and a deterministic side-payment $-d_i$. When firms have homogeneous beliefs regarding the probability distribution of the scenarios, i.e., when $\mathbb{P}_1 = \mathbb{P}_2 = \cdots = \mathbb{P}_n$, it follows from (20) that $Z_i(\omega) = 0$ for all ω, and so the redistribution of risk is proportional: firm $i \in N$ is allocated a fraction δ_i of the aggregate risk $\sum_{j \in N} Y_j$. When firms have heterogeneous beliefs regarding the probability distribution, the redistribution is no longer proportional. In

[9] Boonen (2014) contains an earlier version of this paper in which also quadratic and logarithmic utility functions are considered. The qualitative results that we show here for the exponential distribution extend to these cases.

[10] With an exponential utility function, the certainty equivalent of a risky payoff is the negative of a cash invariant risk measure applied to the payoff. So, this result is a special case of Pareto optimal risk sharing with cash-invariant utilities with different reference probabilities. Existence of Pareto optimal risk redistributions is studied by Acciaio and Svindland (2009).

addition to the fraction δ_i of the aggregate risk, firm $i \in N$ is assigned the risk Z_i. The value $Z_i(\omega)$ represents the net risk shifted to firm i in state ω to benefit from differences in the probabilities $\mathbb{P}_j(\omega)$, $j \in N$. Ceteris paribus, $Z_i(\omega)$ is high when $\mathbb{P}_i(\omega)$ is low relative to $\mathbb{P}_j(\omega)$ for $j \neq i$ (see also Proposition 2).

These results illustrate several effects of heterogeneous beliefs on Pareto optimal risk redistributions. First, although pooling risks yields maximal risk reduction, the Pareto optimal risk redistributions are not functions of the aggregate risk. This occurs because firms who have different beliefs regarding the likelihood of the scenarios have an incentive to shift payments in a given scenario to those firms that assign the lowest probability to the scenario. Hence, as compared to the case of homogeneous beliefs, firms face an additional tradeoff between benefiting from risk reduction, and reducing expected payments by exploiting different probability beliefs. To illustrate this, consider the hypothetical case in which pooling liabilities eliminates all risk, i.e., $\sum_{i \in N} Y_i$ is risk-free. Then, while with homogeneous beliefs (i.e., when $\mathbb{P}_1 = \mathbb{P}_2 = \cdots = \mathbb{P}_n$), all Pareto optimal posterior risk profiles are risk-free for each firm, this is not the case when firms have heterogeneous beliefs.

Next, the example shows that the effect of heterogeneity on the set of Pareto optimal risk redistributions (as measured by the size of the additional risk transfers, $|Z_i(\omega)|$) is larger when the firms are less risk averse. To illustrate this, suppose that we replace each λ_i by $c \cdot \lambda_i$ for some $c > 0$. Then it follows from (19) and (20) that for all i, $Z_i(\omega)$ is replaced by $Z_i(\omega)/c$, while $\delta_i \cdot \left(\sum_{j \in N} Y_j(\omega)\right)$ is unaffected. Hence, the effect of heterogeneity becomes smaller when the degrees of risk aversion increase (c increases). The intuition is as follows. If the firms have different beliefs regarding the probability distribution (i.e., $\mathbb{P}_i(\omega) \neq \mathbb{P}_j(\omega)$ for at least some $i \neq j$, and $\omega \in \Omega$), the aggregate perceived expected liabilities decrease when payments in state ω are shifted to the party that assigns the lowest probability to this state. These shifts are represented by the values $Z_i(\omega)$. However, when $|Z_i(\omega)|$ becomes large, the benefits of lower aggregate expected liabilities are outweighed by the drawbacks of increased risk in Y_i^{post}. Hence, Pareto optimal shifts $|Z_i(\omega)|$ get smaller in magnitude when the parties are more risk averse.

We conclude this section by showing that heterogeneous beliefs may increase the likelihood that firms (believe that they) can strictly benefit from redistributing their risks. To illustrate this, again consider the case where the firms are (close to) risk averse, so that the objective is (close) to minimize $\mathbb{E}_{\mathbb{P}}[Y_i^{\text{post}}]$. Because $\sum_{i \in N} \mathbb{E}_{\mathbb{P}}[Y_i^{\text{post}}] = \sum_{i \in N} \mathbb{E}_{\mathbb{P}}[Y_i]$ for any probability measure $\mathbb{P}$ and any vector of feasible posterior risk profiles, Pareto improvement cannot be obtained when firms have homogeneous beliefs. However, when firms have heterogeneous beliefs, there exist feasible posterior risk profiles satisfying $\sum_{i \in N} \mathbb{E}_{\mathbb{P}_i}[Y_i^{\text{post}}] < \sum_{i \in N} \mathbb{E}_{\mathbb{P}_i}[Y_i]$. Therefore, heterogeneous beliefs may imply that all firms believe that they can gain in expectation from redistributing their risks. This suggests that heterogeneous beliefs might make redistribution of risk more attractive. To show that this holds true more generally, we define the condition *No Constant Ratio of Marginal Utilities,* referred to as *NCRMU.*

- Condition *NCRMU*: there exists a $j \in N$, such that $\frac{u_1'((1+r)^T(A_1-Y_1(\omega))\cdot\mathbb{P}_1(\{\omega\})}{u_j'((1+r)^T(A_j-Y_j(\omega))\cdot\mathbb{P}_j(\{\omega\})}$ is not constant in $\omega \in \Omega$.

The next proposition shows that *NCRMU* is a necessary condition for the existence of posterior risk redistributions that make each firm strictly better off.

Proposition 3 *It holds that:*

(i) *Condition NCRMU is necessary for the existence of posterior risk profiles* $(Y_i^{post})_{i\in N} \in \mathcal{S}$ *for which at least one firm strictly benefits, i.e.,* $\Delta U_i(Y_i^{post}) > 0$ *for at least one* $i \in N$.

(ii) *If* $|N| = 2$, *condition NCRMU is necessary and sufficient for the existence of posterior risk profiles* $(Y_i^{post})_{i\in N} \in \mathcal{S}$ *for which both firms strictly benefit, i.e.,* $\Delta U_i(Y_i^{post}) > 0$ *for all* $i \in N$.

Proposition 3 implies that heterogeneous beliefs regarding the underlying probability distribution may make it *more* likely that there exists a Pareto optimal risk redistribution that weakly benefits all firms and strictly benefits at least one firm. To illustrate this, consider the case where risk is redistributed between two firms. In that case, *NCRMU* is necessary and sufficient for the existence of a Pareto optimal risk redistribution that strictly benefits both firms. Now suppose that the two firms have the same prior risk profile and the same risk preferences. Then, *NCRMU* is satisfied if and only if $\frac{\mathbb{P}_1(\{\omega\})}{\mathbb{P}_2(\{\omega\})}$ depends on ω, which is satisfied if and only if the firms have heterogeneous beliefs. Hence, strict improvement can be achieved in case of heterogeneous beliefs regarding the underlying probability distribution, but not in case of homogeneous beliefs. We conclude that only in very special cases, there is no room for improvement.

4 The bargaining problem

The previous section shows that there exist risk redistributions that benefit all firms in the sense that each firm weakly gains from the redistribution in expected utility terms, and no subset of firms can be better off when they exclude the other firms from the negotiation and redistribute their risk amongst each other. In general, however, the set of redistributions that satisfy these criteria is not single-valued, and the issue arises which redistribution is selected. In each redistribution, all firms weakly benefit, but the extent to which a particular firm benefits depends on the particular redistribution that is chosen. In general, a redistribution that yields a high expected utility gain for a particular firm does not yield a high expected utility gain of another involved firm. This implies that the firms bargain over the redistribution that they choose, and so the selection of a particular redistribution reflects a bargaining process that can be modeled via a *bargaining rule* (Nash, 1950).[11] In order to reflect the fact that firms will only agree

[11] Nash (1950) and Kalai (1977) provide a cooperative game-theoretic characterization based on four properties. Rubinstein (1982) provides a bilateral non-cooperative game in which if players are perfectly

to a redistribution if it satisfies *Stability*, we consider the *constrained Nash bargaining solution*, which is given by:[12]

$$\mathcal{CNB} = \underset{(Y_i^{\text{post}})_{i\in N}\in\mathcal{S}}{\operatorname{argmax}} \left\{ \prod_{i\in N} \Delta U_i(Y_i^{\text{post}}) \right\}. \tag{22}$$

The objective function in (22) equally weighs the utility gains of a redistribution for all the involved parties. The following proposition shows that there exists a solution to the constrained Nash bargaining problem in (22).

Proposition 4 *There exists a solution to optimization problem (22), i.e.,* $\mathcal{CNB} \neq \emptyset$.

We conclude this section by characterizing the optimal risk redistribution when risk is distributed between two firms. Recall that with two firms, *Stability* is equivalent to *Pareto Optimality* and *Individual Rationality*. Hence, the set of potential risk redistributions that the firms bargain over is the set of redistributions that satisfy both *Pareto Optimality* and *Individual Rationality*, which implies that the constrained Nash bargaining solution coincides with the Nash bargaining solution from Nash (1950). This allows us to show that when risk is distributed between two firms, there is a unique redistribution that corresponds to the Nash bargaining solution. Moreover, the characterization in (14) and (15) combined with (11), allows to reduce the optimization problem in (22) to a one-dimensional optimization problem with a compact constraint set. These results are summarized in the next proposition.

Proposition 5 *If* $|N| = 2$*, it holds that:*

(i) *There exists a unique solution to optimization problem (22), i.e.,* $\mathcal{CNB}$ *is single-valued.*

(ii) *For any given* $k > 0$*, let* $(f_1(k), f_2(k))$ *be the unique solution of (14) and (15) for* $\overline{k} = (1, k)$*. The unique solution to optimization problem (22) is given by* $(Y_1^{post}, Y_2^{post}) = (f_1(k^*), f_2(k^*))$*, where*

$$k^* = \underset{k\in[k_{\min},k_{\max}]}{\operatorname{argmax}} \left\{ \prod_{i\in\{1,2\}} \Delta U_i(f_i(k)) \right\}, \tag{23}$$

and

- $k_{\max} > 0$ *is the unique solution of* $\Delta U_1(f_1(k_{\max})) = 0$,
- $k_{\min} > 0$ *is the unique solution of* $\Delta U_2(f_2(k_{\min})) = 0$.

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patient, the equilibrium division converges to the Nash bargaining solution. Moreover, Van Damme (1986) shows that the Nash bargaining solution constitutes the unique equilibrium if two firms have different opinions about what is the appropriate solution concept to use.

[12]This rule is called the *coalitional Nash bargaining solution* (Compte and Jehiel, 2010) in case of Transferable Utility games.

(iii) *If condition NCRMU is satisfied, the unique posterior risk profiles that yield the (constrained) Nash bargaining solution yield strict utility improvement for both firms, i.e.,* $\Delta U_i(f_i(k^*)) > 0$ *for* $i \in \{1, 2\}$.

5 Benefits from risk redistributions

In this section we use the model developed in the previous sections to investigate the extent to which pension funds and insurers can benefit from redistributing their risks. This section is organized as follows. We first specify the liabilities, risk preferences, and subjective probability distributions of the pension fund and the insurer. Then, we numerically illustrate the benefits from the redistribution of risk that corresponds to the Nash bargaining solution. In practice, these benefits may be affected by disagreement regarding the true distribution of future mortality rates, reluctance to engage in contracts with long horizons, and insufficient capacity in the life insurance market to yield significant risk reduction for pension funds. Moreover, we know from Section 3.2 that the effects of disagreement regarding the probability distribution depend non-trivially on the firms' risk preferences. We therefore consider a case where the liabilities of the insurer are small relative to those of the pension fund, risk is redistributed over a one-year horizon ($T = 1$), and analyze the effects on the utility gains from redistribution of heterogeneity in beliefs regarding the true probability distribution of future survival rates, for varying degrees of risk aversion.

5.1 Liabilities and subjective probabilities

Throughout this section, we let the set of firms participating in the risk redistribution problem be given by $N = \{PF, INS\}$, where PF is a pension fund and INS an insurer. The potential benefits from risk redistribution depend on the characteristics of their liabilities (i.e., the liability payments $\widetilde{L}_{i,\tau}$), their risk preferences, and their (subjective) beliefs regarding the probability distribution on the underlying state space.

We start by discussing the characteristics of the liabilities. These liabilities depend on future survival rates in the insured population. For any given age x on date $t = 0$, we let ${}_\tau p_{x,0}$ denote the *future* probability that an individual belonging to the cohort aged x in year $t = 0$ will survive at least τ more years.[13] Note that ${}_\tau p_{x,0}$ is a random variable at any time $s \leq \tau$, and is known on date $s = \tau + 1$. Then, the liabilities of the portfolios are as follows:[14]

[13] Following Cairns et al. (2006), we define $p_{x+s,s} = \mathbb{P}(T_x \geq s+1 | T_x \geq s, \mathcal{F}_\infty)$, where $\mathcal{F}_\infty$ denotes the set that contains of all information regarding mortality rates at all future dates, and where T_x denotes the random remaining lifetime of an individual aged x on date $t = 0$. Then, ${}_\tau p_{x,0} = \prod_{s=0}^{\tau-1} p_{x+s,s}$.

[14] We note that we have not explicitly modeled lapse. However, lapse does not affect the results that we present in this section, provided that lapse conditions are actuarially fair. The reason is that the random variable of interest for each party is the net asset value (NAV) at date $T = 1$. If lapse conditions are actuarially fair, both the value of the assets and the best-estimate value of the liabilities decrease with an amount equal to the best-estimate value of the liabilities for the policies that lapse. Hence, the net asset value is unaffected by lapse. This in turn implies that the risk redistributions that we find are unaffected by lapse.

1. The pension fund has a portfolio of 50,000 (deferred) single life annuities for male participants. The (deferred) single life annuity yields a fixed yearly payment. The first payment occurs at the beginning of the year in which the insured reaches age 65; the last payment at the beginning of the year in which the insured dies. For sufficiently large portfolios, the aggregate portfolio payment of the pension fund in year τ can be approximated by:[15]

$$\widetilde{L}_{PF,\tau} = \sum_{j=1}^{50{,}000} \delta_j \cdot {}_{\tau}p_{x_j,0} \cdot 1_{\{x_j+\tau \geq 65\}}, \tag{24}$$

   where $x_j \in \{20, 21, \ldots, 100\}$ denotes the age of participant j on date $t = 0$, and δ_j is the annual annuity payment of participant j. We consider the case where accrual of pension right starts at age 20, and increases linearly to a normalized value of 1 at age 65, i.e., $\delta_j = \min\left\{\frac{x_j-19}{46}, 1\right\}$. The age composition of the pension fund is based on the age composition of the Dutch population aged 20 and older, and is displayed in Figure 2 in Appendix B.

2. The insurer has a portfolio of $13,354$ term assurance contracts that pay a lump sum amount (i.e., the death benefit) of 10 (10 times the annual annuity payment of a 65 year old) at the end of the year in which the insured dies, in case of decease of the insured before age 65. Then

$$\widetilde{L}_{INS,\tau} = \sum_{j=1}^{13{,}354} 10 \cdot \left({}_{\tau-1}p_{x_j,0} - {}_{\tau}p_{x,_j,0} \right) \cdot 1_{\{x_j+\tau < 65\}}. \tag{25}$$

   The age composition of the pension fund is based on the age composition of the Dutch population aged 20 until 65, and is displayed in Figure 3 in Appendix B. The number of insureds $(13,354)$ is chosen such that, based on LC(1977-2009), the date-0 expected present value of the liabilities of the insurer is 20% of the date-0 best-estimate value of the liabilities of the pension fund.

Longevity risk arises from the fact that the future survival probabilities ${}_{\tau}p_{x,0}$ for $\tau \geq 0$ are unknown on date 0. We distinguish the case where the pension fund and the insurer have homogeneous beliefs regarding the underlying probability distribution of the future survival probabilities, and the case where they have heterogeneous beliefs. Specifically, we assume that both the pension fund and the insurer use the Lee and Carter (1992) approach to estimate the probability distribution of future mortality rates (see Appendix C), but they may disagree on the appropriate historical time period that is used to estimate the model parameters. For the case of homogeneous beliefs, they each estimate the model parameters based on data for Dutch males as reported in the Human Mortality

[15] The level of the payments is also affected by *individual mortality risk* which arises due to the fact that the survival fraction in the insured portfolio can deviate from the population fractions. Given the large portfolio sizes that we consider, we assume that individual mortality risk is negligible, and focus on the impact of systematic risk.

	LC(1977-2009)		LC(1987-2009)	
	PF	INS	PF	INS
$\mathbb{E}[Y_i]$	$2.82 * 10^5$	$5.65 * 10^4$	$2.89 * 10^5$	$5.66 * 10^4$
$\sigma(Y_i)/\mathbb{E}[Y_i]$	1.08%	4.63%	1.55%	5.43%
$(Q_{97.5}(Y_i) - \mathbb{E}[Y_i])/\mathbb{E}[Y_i]$	2.10%	9.42%	3.00%	11.10%
	LC(1977-2009)		LC(1987-2009)	
$\rho(Y_{PF}, Y_{INS})$	-0.90		-0.93	

Table 1: Top panel: summary statistics of the marginal probability distributions of Y_{PF} and Y_{INS}, for mortality model LC(1977-2009) (first two columns) and for mortality model LC(1987-2009) (last two columns). The first row displays the expected value; the second row displays the standard deviation expressed as percentage of the expected value; the last row displays the 97.5% quantile, as percentage deviation from the expectation. Bottom panel: correlation between Y_{PF} and Y_{INS} for mortality models LC(1977-2009) and LC(1987-2009). Summary statistics are determined using the simulation procedure described in Appendix D.

Database[16] for the period 1977 to 2009. For the case of heterogeneous beliefs, however, the insurer uses data from the shorter time period from 1987 until 2009. We refer to these models as LC(1977-2009) and LC(1987-2009), respectively.[17] We display the corresponding parameter estimates in Appendix C.

In Table 1, we present summary statistics of the probability distribution of (Y_{PF}, Y_{INS}) under LC(1977-2009) and under LC(1987-2009). The simulation approach used to determine these summary statistics is described in Appendix D. The return on assets is set equal to $r = 3\%$.

Table 1 shows that the liabilities of the insurer are more risky than the liabilities of the pension fund (higher $\sigma(Y_i)/\mathbb{E}[Y_i]$ and higher $(Q_{97.5}(Y_i) - \mathbb{E}[Y_i])/\mathbb{E}[Y_i]$).[18] Moreover, Table 1 shows that the liabilities of the pension fund and of the insurer are significantly more risky under LC(1987-2009) than under LC(1977-2009), and that the correlation between Y_{PF} and Y_{INS} is significantly more negative under LC(1987-2009) than under LC(1977-2009).

The negative correlation between the risk profiles of the pension fund and the insurer suggests strong potential for hedge benefit for both parties. Moreover, the fact that

[16] See http://www.mortality.org/.

[17] Zhu and Bauer (2014) show that the fact that the Lee-Carter model is a one factor model implies that the benefits from hedge potential may be overestimated. We use this model because in practice, it is still one of the most popular models.

[18] The mortality forecast model is "almost" symmetric, in the sense that scenarios in which mortality rates are higher than the best-estimates (positive shock) occur with almost the same probability as scenarios in which mortality rates are lower than the best-estimates (negative shock). Due to the nature of the liabilities, however, the impact of a positive shock on the value of term assurance is bigger that the impact of a negative shock of the same magnitude on the value of a pension annuity (see, e.g., Van Gulick et al., 2012).

the summary statistics depend on which model is used suggests that heterogeneity in beliefs regarding the mortality model could significantly affect the risk redistribution. In the next section, we numerically illustrate the extent to which the pension fund and the insurer can benefit from hedge potential by redistributing their risks amongst each other, distinguishing the case of homogeneous and heterogeneous beliefs. The extent to which the pension fund and the insurer can benefit from redistributing their risks depends on their risk preferences. We consider the case where the pension fund and the insurer each use an exponential utility function (see (17)) with risk aversion parameter $\lambda_{PF} = \lambda_{INS} = \lambda$.

5.2 Risk profiles after risk redistribution

To determine the Pareto optimal risk redistributions corresponding to the Nash bargaining solution, we first discretize the joint distribution of (Y_{PF}, Y_{INS}), using the procedure described in Appendix E. Then, we use Proposition 5(ii) to determine the constrained Nash bargaining solution from (22), which we denote $(Y_{PF}^{\text{post}}, Y_{INS}^{\text{post}})$. We note that when the pension fund and the insurer use an exponential utility function, the set of Pareto optimal posterior risk profiles is independent of the initial asset values (A_{PF}, A_{INS}). Hence, the posterior distributions $(Y_{PF}^{\text{post}}, Y_{INS}^{\text{post}})$ are also independent of (A_{PF}, A_{INS}).

Figure 1 displays the probability distributions of the risk profiles of the pension fund (left panel) and the insurer (right panel) before and after the redistribution of risk, for the case where $\lambda = 10^{-3}$.[19] The grey histograms in Figure 1 display the discretized probability distributions of Y_{PF} and Y_{INS}, expressed as percentage deviations from their expected values. The black histograms display the probability distribution of Y_{PF}^{post} and Y_{INS}^{post}, also expressed as percentage deviation from the expected value of Y_{PF} and Y_{INS}. The upper (lower) panels correspond to the case of homogeneous (heterogeneous) beliefs.

We first discuss the properties of the redistribution in case of homogeneous beliefs. The upper panels of Figure 1 show that in case of homogeneous beliefs, the risk redistribution implies that the date-1 NAV of both the pension fund and the insurer becomes significantly less dispersed. Because the pension fund and the insurer have the same utility function, it is optimal to pool the risk and each take an equal share. Hence, the posterior risk profiles are identical, up to a deterministic payment (see (18)-(21)). Because the prior liabilities of the insurer are smaller in expectation, the percentage deviations from the prior best estimate value are more dispersed for the insurer than for the pension fund.

To investigate the effects of heterogeneous beliefs on the shape of the risk redistribution, we compare the upper and the lower panels in Figure 1. Both for the insurer and the pension fund, the probability to face payments lower than the date-0 best-estimate value after redistribution (the black histograms), is higher with heterogeneous beliefs (bottom

[19]This level of risk aversion imply that the maximum premium that the pension fund would be willing to pay for a full buy-out of its liabilities equals 104.8% of the best-estimate value of the liabilities, if the beliefs are given by LC(1977-2009). For the insurer, the corresponding maximum premium equals 141.3% of the best-estimate value.

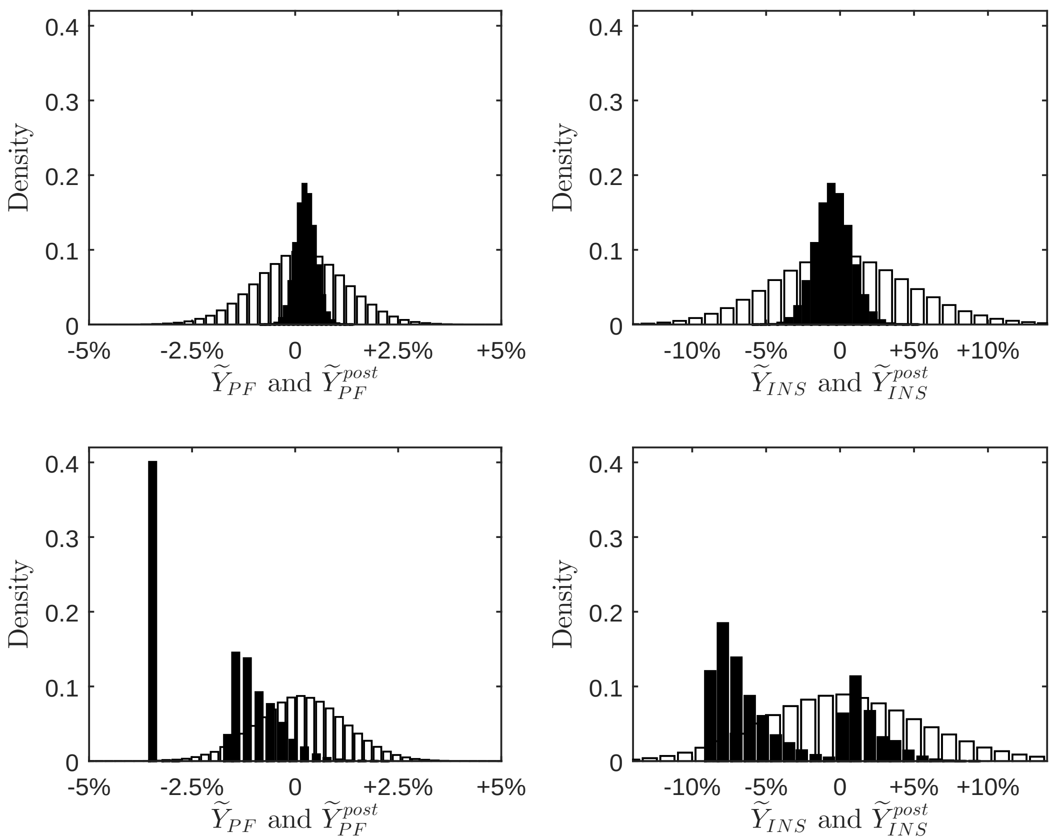

Figure 1: The grey histograms represent the probability distributions of $\widetilde{Y}_i = (Y_i - \mathbb{E}_{\mathbb{P}_i}[Y_i]) / \mathbb{E}_{\mathbb{P}_i}[Y_i] \cdot 100\%$. The black histograms represent the probability distributions of $\widetilde{Y}_i^{\text{post}} = \left(Y_i^{\text{post}} - \mathbb{E}_{\mathbb{P}_i}[Y_i]\right) / \mathbb{E}_{\mathbb{P}_i}[Y_i] \cdot 100\%$. The left (right) panel corresponds to the pension fund (insurer). The posterior distribution is given by (22). The upper (lower) panel corresponds to the case of homogeneous (heterogeneous) beliefs.

panels) than with homogeneous beliefs (top panels). Moreover, both for the insurer and for the pension fund, the probability distribution of the posterior risk profile in case of heterogeneous beliefs (lower panels; black histograms) is bimodal. This occurs because the pension fund and the insurer can both benefit from shifting risk in a state of the world to the party that assigns the lowest probability to that state (see Proposition 2). Therefore, in addition to pooling the risk and each taking an equal share (i.e., the optimal redistribution in case of homogeneous beliefs), the pension fund is assigned the additional risk Z_{PF}, while the insurer is assigned the opposite risk $Z_{INS} = -Z_{PF}$ (see (18)-(20)). The probability distribution of this zero-sum additional risk transfer depends on differences in beliefs regarding the likelihood of the different states of the world.[20] As compared to the pension fund, the insurer assigns higher probabilities to states of the world in the tails of the distribution and lower probabilities to states of the world closer to the mean (see also Table 1). Therefore, as compared to the case of homogeneous beliefs, some probability mass in the tails of the distribution of the posterior risk profile of the pension fund (left panels; black histograms) is shifted towards higher values (i.e., the pension fund bears more risk in these scenarios), while some probability mass around the mean of the distribution is shifted towards lower values (i.e., the pension fund bears less risk in these scenarios); for the insurer (right panels; black histograms), some probability mass in the tails of the distribution of the posterior risk profile is shifted towards lower values, while some probability mass around the mean is shifted towards higher values. These shifts imply that both for the insurer and for the pension fund, the probability distribution of the risk profile after risk redistribution is bimodal.

5.3 The benefits from the risk redistribution

In this section, we quantify the benefits from the Nash bargaining solution, as characterized in Proposition 5(ii). Recall that we denote Y_i^{post} for the corresponding posterior risk profiles. We focus on the effect of heterogeneous beliefs on these benefits. Moreover, to investigate the effect of the degree of risk aversion, we will consider three values of λ. To quantify the benefits from the redistribution, we consider the following two criteria:

(i) The *percentage decrease in the date-0 expected present value of the liabilities.* We determine the percentage reduction in the expected value of the liabilities as

$$\%RedEV_i = \frac{\mathbb{E}_{\mathbb{P}_i}\left[Y_i\right] - \mathbb{E}_{\mathbb{P}_i}\left[Y_i^{\text{post}}\right]}{\mathbb{E}_{\mathbb{P}_i}\left[Y_i\right]}, \text{ for } i \in \{PF, INS\}. \tag{26}$$

(ii) The *relative zero-utility premium.* The redistribution implies that firm $i \in \{PF, INS\}$ effectively receives a net payment equal to $(1+r)(Y_i - Y_i^{\text{post}})$ on date $T = 1$. The value to firm i of the risk redistribution can therefore be quantified by determining the maximum premium that firm i would have been willing to pay on date 0 for

[20] It follows from (20) that $Z_{PF}(\omega) = \frac{1}{\lambda_{PF}+\lambda_{INS}} \ln\left(\frac{\mathbb{P}_{INS}(\{\omega\})}{\mathbb{P}_{PF}(\{\omega\})}\right)$ and $Z_{INS}(\omega) = -Z_{PF}(\omega)$, for all $\omega \in \Omega$.

		homogeneous beliefs		heterogeneous beliefs	
		PF	INS	PF	INS
$\%RedEV_i$	$\lambda = 0.01$	-0.1%	0.6%	0.8%	-8.0%
	$\lambda = 0.001$	-0.1%	0.6%	2.1%	4.3%
	$\lambda = 0.0001$	-0.0%	0.1%	17.9%	111%
$\%ZU_i$	$\lambda = 0.01$	3.7%	18.4%	2.7%	13.4%
	$\lambda = 0.001$	1.5%	7.4%	2.2%	11.0%
	$\lambda = 0.0001$	0.1%	0.7%	7.5%	37.5%

Table 2: The simulated gains of risk redistribution, i.e., $\%RedEV_i$ as defined in (26) and $\%ZU_i$ as defined in (28), for the constrained Nash bargaining solution from (22), distinguishing the case of homogeneous beliefs and the case of heterogeneous beliefs.

a contract that yields this net payment on date $T = 1$. This maximum premium, which we denote $p_i \in \mathbb{R}$, is the premium at which firm $i \in \{PF, INS\}$ would be indifferent between buying the contract and not buying the contract, and is therefore referred to as the *zero-utility premium.* It is the unique solution of the following equation:

$$U_i((1 + r) \cdot (A_i - Y_i^{\text{post}} - p_i)) = U_i((1 + r) \cdot (A_i - Y_i)). \tag{27}$$

Solving (27) yields

$$p_i = \frac{1}{\widehat{\lambda}} \ln \left(\mathbb{E}_{\mathbb{P}_i} \left[e^{\widehat{\lambda} Y_i} \right] / \mathbb{E}_{\mathbb{P}_i} \left[e^{\widehat{\lambda} Y_i^{\text{post}}} \right] \right), \text{ for } i \in \{PF, INS\},$$

where $\widehat{\lambda} = (1 + r)\lambda$. We report the value of the zero-utility premium for firm i relative to the date-0 value of the risk profile prior to redistribution, i.e.,

$$\%ZU_i = \frac{p_i}{\mathbb{E}_{\mathbb{P}_i}[Y_i]}, \text{ for } i \in \{PF, INS\}. \tag{28}$$

Table 2 summarizes the benefits for the pension fund and for the insurer, for three values of the risk aversion parameter λ, distinguishing the case of homogeneous and heterogeneous beliefs.

We first discuss the results for the case with homogeneous beliefs (first two columns in Table 2). The redistribution implies that in each state of the world, either the pension fund makes a net payment to the insurer on date $T = 1$, or the insurer makes a net payment to the pension fund on date $T = 1$. As can be seen from Table 2, in expectation the pension fund makes a net payment to the insurer on date $T = 1$; for each λ, the expected liabilities of the pension fund increase (negative reduction $\%RedEV_i$) and the expected liabilities of the insurer decrease (positive reduction $\%RedEV_i$). However, even though the pension fund loses in expectation and the pension fund gains in expectation, both the pension fund and the insurer benefit from the redistribution in expected utility terms. For both firms, the relative zero-utility premium corresponding to the redistribution is monotonically increasing in the degree of risk aversion, and varies from 0.1% to 3.7% for the pension fund, and from 0.7% to 18.4% for the insurer. The relative zero-utility premium of the insurer is relatively high as compared to the relative zero-utility premium of the pension fund. This occurs because the Nash bargaining solution in (22) strives to equate the utility gains $\Delta U_i(Y_i^{\text{post}})$ for both parties. The relative impact is bigger for the insurer because the best estimate value of its prior liabilities is lower.

The last two columns in Table 2 show that when the two parties have heterogeneous beliefs, the effect of the degree of risk aversion is no longer monotone. The utility gains when $\lambda = 0.01$ and when $\lambda = 0.0001$ are both larger than the utility gains when $\lambda = 0.001$. This occurs because when risk aversion is low ($\lambda = 0.0001$), the firms can both increase their utility significantly if, in each scenario, they shift risk to the firm that assigns the lowest probability to the scenario, because this decreases the (perceived) aggregate expected liabilities for both parties. When $\lambda = 0.0001$, the percentage reductions in the expected value of the liabilities are much higher than in the case of homogeneous beliefs. Moreover, they are positive for both firms, indicating that both firms believe that they can benefit in expectation. When the degree of risk aversion increases, the effect of heterogeneity on the risk redistribution becomes smaller (see Section 3.2). Therefore, when firms are sufficiently risk averse the (perceived) utility gains from risk reduction increase when risk aversion increases, as is the case with homogeneous beliefs.

Moreover, Table 2 also shows that as compared to the case where the two parties agree regarding the probability distribution of future mortality rates, disagreement regarding this probability distribution may, but need not, increase the utility gains from the redistribution.

6 Conclusion

We have analyzed the effects of heterogeneous beliefs regarding the probability distribution of future mortality rates on the redistributions of risk that will arise if a limited number of firms bargain over a redistribution that benefits all. The goal of the firms is to reduce the impact of longevity risk on the net asset value at a future date by exploiting the natural hedge potential that arises from combining liabilities with different sensitivities to longevity risk. We find that, as compared to the case where the firms have homogeneous beliefs regarding the probability distribution of mortality rates, with heterogeneous beliefs they face an additional incentive to shift, in each scenario, payments to the firm that assigns the lowest probability to that scenario. This in turn has important implications for the structure of the redistributions that will arise if all parties are rational, and, hence, choose a Pareto optimal risk redistribution. In particular, the additional incentive faced by the firms implies that the perceived benefits from risk redistribution are no longer necessarily increasing in the degrees of risk aversion of the firms. This result is confirmed in numerical study in which we determine the redistribution of risk between an insurer with a portfolio of term assurance contracts and a pension fund with a portfolio of life annuities.

We conclude by discussing an interesting direction for future research. Given our focus on the shape of risk redistributions that arise from bargaining, we have not explicitly modeled costs associated with the risk redistribution. While we do not expect that costs will affect our qualitative results (i.e., the characteristics of redistributions that arise via bargaining), they may affect the relative attractiveness of risk redistribution via bargaining as compared to risk redistribution via market-based solutions. That horse race will not only depend on the degree of market completeness, but also on the costs involved in either type of redistribution and on the extent to which these costs are transparent. At present, market incompleteness implies that opportunities for risk redistribution via the market are still limited. Moreover, while costs involved in redistributions that arise via bargaining can be fully transparent, market-based solutions currently often lack cost transparency. Recent literature, however, devotes significant attention to the development of insurance products with enhanced cost transparency (see, e.g., Donnelly et al., 2014). Our results suggest that even when cost transparency is enhanced and the market becomes more complete, parties may still prefer risk redistribution via bargaining over market solutions as long as there is no consensus regarding the probability distribution of future survival rates. This occurs because parties can then benefit from redistributing risk with a counterparty with different beliefs. A more detailed investigation of this issue is left for future research.

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the perceived benefits will be lower and . If the market for longevity-linked products expands and becomes more complete, market-based solutions may become an attractive alternative and

A Proofs

For any (sub)set of firms $S \subseteq N$, we let $\mathcal{F}(S)$ be the set of posterior risk profiles for the firms in S that they can reach if they redistribute their risks amongst each other, not involving the other firms, i.e.,

$$\mathcal{F}(S) = \left\{ (Y_i^{\text{post}})_{i \in S} : \sum_{i \in S} Y_i^{\text{post}} = \sum_{i \in S} Y_i \right\}. \tag{29}$$

In order to prove Proposition 1, we introduce the correspondence V that assigns to each set of firms $S \subseteq N$ the set of potential expected utility gains from feasible redistributions of risk, allowing for "free disposal", i.e., for all $S \subseteq N$:

$$V(S) = \left\{ a \in \mathbb{R}^S : \exists (Y_i^{\text{post}})_{i \in S} \in \mathcal{F}(S) \text{ s.t. } a \leq (\Delta U_i(Y_i^{\text{post}}))_{i \in S} \right\}. \tag{30}$$

For any $S \subseteq N$, we define the set $\partial V(S)$ as the boundary of $V(S)$. Moreover, for any $S \subseteq N$, we let $\mathcal{PO}(S)$ be the set of Pareto optimal redistributions of risk when the firms in S redistribute their risk, i.e., $\mathcal{PO}(S)$ is given by (12) with N replaced by S. Then we have the following lemma.

Lemma 1 *For every $S \subseteq N$, it holds that:*

(i) $V(S)$ *is convex;*

(ii) $\partial V(S) = \left\{ (\Delta U_i(Y_i^{post}))_{i \in S} : (Y_i^{post})_{i \in S} \in \mathcal{PO}(S) \right\}$;

(iii) for every $x \in \partial V(S)$, *there exists unique* $(Y_i^{post})_{i \in S} \in \mathcal{PO}(S)$ *such that* $x = (\Delta U_i(Y_i^{post}))_{i \in S}$.

Proof (i) The proof is a straightforward generalization of the proof of Riddell (1981), who showed this result in case of two firms. Let $S \subseteq N$, $a, b \in V(S)$ and $\gamma \in (0,1)$. Then, there exist $(Y_i^{\text{post},a})_{i \in S}$ and $(Y_i^{\text{post},b})_{i \in S}$ such that $\sum_{i \in S} Y_i^{\text{post},a} = \sum_{i \in S} Y_i$, $\sum_{i \in S} Y_i^{\text{post},b} = \sum_{i \in S} Y_i$, $a \leq \left(\Delta U_i(Y_i^{\text{post},a}) \right)_{i \in S}$ and $b \leq \left(\Delta U_i(Y_i^{\text{post},b}) \right)_{i \in S}$. Clearly, we have

$$\sum_{i \in S} \left(\gamma Y_i^{\text{post},a} + (1-\gamma) Y_i^{\text{post},b} \right) = \sum_{i \in S} Y_i.$$

Moreover, by concavity of $u_i(\cdot)$, it follows that

$$\Delta U_i(\gamma Y_i^{\text{post},a} + (1-\gamma) Y_i^{\text{post},b}) \geq \gamma \Delta U_i(Y_i^{\text{post},a}) + (1-\gamma) \Delta U_i(Y_i^{\text{post},b}) \geq \gamma a_i + (1-\gamma) b_i,$$

for all $i \in S$. Hence, $\gamma a + (1-\gamma) b \in V(S)$ and, therefore, (i) holds true.

(ii) From "free disposal" of $V(S)$ and monotonicity of u_i, it follows that

$$\partial V(S) = \{x \in V(S) : \nexists y \in V(S) \text{ s.t. } y > x\} \tag{31}$$

$$= \{x \in V(S) : \nexists y \in V(S) \text{ s.t. } y \gneqq x\}\,. \tag{32}$$

Note that $a \in \partial V(S)$ if and only if there does not exist an $(\widetilde{Y}_i^{\text{post}})_{i \in S} \in \mathcal{F}(S)$ such that $(\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in S} \gneqq a$. From this and $\partial V(S) \subset V(S)$, it follows that for every $a \in \partial V(S)$, there exist feasible posterior risk profiles $(Y_i^{\text{post}})_{i\in S} \in \mathcal{F}(S)$ such that $a = (\Delta U_i(Y_i^{\text{post}}))_{i\in S}$. Moreover, it is verified immediately that $(Y_i^{\text{post}})_{i\in S} \in \mathcal{PO}(S)$ implies that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \in \partial V(S)$. Hence, (ii) holds true.

(iii) Let $x \in \partial V(S)$ be given, and suppose that there exist $(Y_i^{\text{post},a})_{i\in S}$, $(Y_i^{\text{post},b})_{i\in S}$ $\in \mathcal{PO}(S)$ with $(Y_i^{\text{post},a})_{i\in S} \neq (Y_i^{\text{post},b})_{i\in S}$ and $(\Delta U_i(Y_i^{\text{post},a}))_{i\in S} = (\Delta U_i(Y_i^{\text{post},b}))_{i\in S} = x$. Then, by strict concavity of u_i, for $i \in S$, we have that $\Delta U_i(\frac{1}{2}Y_i^{\text{post},a} + \frac{1}{2}Y_i^{\text{post},b}) \geq \frac{1}{2}\Delta U_i(Y_i^{\text{post},a}) + \frac{1}{2}\Delta U_i(Y_i^{\text{post},b}) = \Delta U_i(Y_i^{\text{post},a}) = \Delta U_i(Y_i^{\text{post},b})$ for all $i \in S$ with at least one strict inequality. Because $\frac{1}{2}(Y_i^{\text{post},a})_{i\in S} + \frac{1}{2}(Y_i^{\text{post},b})_{i\in S} \in \mathcal{F}(S)$, this contradicts the fact that $(Y_i^{\text{post},a})_{i\in S}$, $(Y_i^{\text{post},b})_{i\in S} \in \mathcal{PO}(S)$. Hence, for every $x \in \partial V(S)$, there exists a unique $(Y_i^{\text{post}})_{i\in S} \in \mathcal{PO}(S)$ such that $x = (\Delta U_i(Y_i^{\text{post}}))_{i\in S}$. Hence, (iii) holds true. □

Proof of Proposition 1 Scarf (1967) considers the correspondence $\widehat{V}$ defined as

$$\widehat{V}(S) = \left\{a \in \mathbb{R}^S : \exists (Y_i^{\text{post}})_{i\in S} \in \widehat{\mathcal{F}}(S) \text{ s.t. } a \leq (\widehat{u}_i(Y_i^{\text{post}}))_{i\in S}\right\}, \tag{33}$$

where $\widehat{\mathcal{F}}(S)$=$\left\{(Y_i^{\text{post}})_{i\in S} \in \mathbb{R}^S : \sum_{i\in S} Y_i^{\text{post}} = \sum_{i\in S} Y_i\right\}$, and for each $i \in N$, $\widehat{u}_i : \mathbb{R} \to \mathbb{R}$ is monotone and concave. He shows that the core of the corresponding NTU-game, i.e., the set

$$C(N, \widehat{V}) = \left\{x \in \widehat{V}(N) : \nexists S \subseteq N \text{ s.t. } (x_i)_{i\in S} \in \widehat{V}(S) \backslash \partial \widehat{V}(S)\right\} \tag{34}$$

is non-empty. First, note that the correspondence V defined in (30) follows from (33) by setting $\widehat{u}_i = \Delta U_i$, for all $i \in N$, and by replacing $\widehat{\mathcal{F}}(S)$ by $\mathcal{F}(S)$ as defined in (29), i.e., by allowing the domain D_i, $i \in N$, to be a convex subset of $\mathbb{R}$. Using the fact that, for all $i \in N$, concavity of u_i implies concavity of ΔU_i, that $\lim_{x\to a_i} u_i'(x) = \infty$, $\lim_{x \to b_i} u_i'(x) = 0$, and that $u_i''(\cdot) < 0$, it is verified immediately that the proof in Scarf (1967) extends to the correspondence V. Hence, it follows that the core of the corresponding NTU-game, which is given by

$$C(N, V) = \{x \in V(N) : \nexists S \subseteq N \text{ s.t. } (x_i)_{i\in S} \in V(S) \backslash \partial V(S)\}\,, \tag{35}$$

is non-empty. Next, we show that

$$C(N, V) \subseteq \left\{(\Delta U_i(Y_i^{\text{post}}))_{i\in N} : (Y_i^{\text{post}})_{i\in N} \in \mathcal{S}\right\}. \tag{36}$$

Let $a \in C(N,V)$ be given. This implies that $a \in \partial V(N)$. It therefore follows from Lemma 1(ii) that there exists an $(Y_i^{\text{post}})_{i\in N} \in \mathcal{PO}(N) = \mathcal{PO}$ such that $a = (\Delta U_i(Y_i^{\text{post}}))_{i\in N}$.

To show that $(Y_i^{\text{post}})_{i\in N} \in \mathcal{S}$, we show that if there exist $S \subseteq N$ and $(\widetilde{Y}_i)_{i\in S} \in \mathcal{F}(S)$ such that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \leq (\Delta U_i(\widetilde{Y}_i))_{i\in S}$, then $(\Delta U_i(\widetilde{Y}_i))_{i\in S} = (\Delta U_i(Y_i^{\text{post}}))_{i\in S}$. Suppose there exist $S \subseteq N$ and $(\widetilde{Y}_i)_{i\in S} \in \mathcal{F}(S)$ such that

$$(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \leq (\Delta U_i(\widetilde{Y}_i))_{i\in S}.$$

This implies that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \in V(S)$. Because $(\Delta U_i(Y_i^{\text{post}}))_{i\in N} = a \in C(N,V)$, it follows from (35) that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \in \partial V(S)$. It then follows from Lemma 1(iii) that there exists a $(\widehat{Y}_i)_{i\in S} \in \mathcal{PO}(S)$ such that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} = (\Delta U_i(\widehat{Y}_i))_{i\in S}$. Because $(\widehat{Y}_i)_{i\in S} \in \mathcal{NI}(S)$, $(\widetilde{Y}_i)_{i\in S} \in \mathcal{F}(S)$, and $(\Delta U_i(\widehat{Y}_i))_{i\in S} = (\Delta U_i(Y_i^{\text{post}}))_{i\in S} \leq (\Delta U_i(\widetilde{Y}_i))_{i\in S}$, it follows from (10) that $(\Delta U_i(\widetilde{Y}_i))_{i\in S} = (\Delta U_i(\widehat{Y}_i))_{i\in S}$. Hence, we can conclude that

$$(\Delta U_i(\widetilde{Y}_i))_{i\in S} = (\Delta U_i(Y_i^{\text{post}}))_{i\in S}.$$

Hence, there do not exist $S \subseteq N$ and $(\widetilde{Y}_i)_{i\in S} \in \mathcal{F}(S)$ such that $(\Delta U_i(Y_i^{\text{post}}))_{i\in S} \lneqq (\Delta U_i(\widetilde{Y}_i))_{i\in S}$. This implies that $(Y_i^{\text{post}})_{i\in N} \in \mathcal{S}$. Because $a = (\Delta U_i(Y_i^{\text{post}}))_{i\in N}$, we can conclude that the inclusion in (36) holds true. Because $C(N,V)$ is non-empty, this concludes the proof. $\square$

Proof of Proposition 2 Let $\overline{k} \in \mathbb{R}^N_{++}$ be given, and let $(Y_i^{\text{post}})_{i\in N}$ be the corresponding Pareto optimal posterior risk profiles from (14) and (15). Without loss of generality, let $i = 1$. It is sufficient to show that $Y_1^{\text{post}}(\omega)$ is increasing in $\mathbb{P}_1(\omega)$, and that $Y_j^{\text{post}}(\omega)$ is decreasing in $\mathbb{P}_1(\omega)$ for all $j \neq 1$. Suppose $\mathbb{P}_1(\omega)$ increases. Because $u_j'(\cdot)$ is continuous and strictly decreasing for all $j \in N$, it follows from (14) and (15) that $Y_1^{\text{post}}(\omega)$ increases. Because $Y_1^{\text{post}}(\omega)$ increases, it follows from (15) that $k_1\mathbb{P}_1(\omega)u_1'\left((1+r)^T\left(A_1 - Y_1^{\text{post}}(\omega)\right)\right)$ increases. It then follows from (14) that $k_j\mathbb{P}_j(\omega)u_j'\left((1+r)^T\left(A_j - Y_j^{\text{post}}(\omega)\right)\right)$ increases for all $j \neq 1$, which implies that $Y_j^{\text{post}}(\omega)$ decreases for all $j \neq 1$. This concludes the proof. $\square$

Proof of Proposition 3 First note that it follows from (14) and (15) that *NCRMU* is satisfied iff $(Y_i)_{i\in N} \notin \mathcal{PO}$.
(i) Suppose condition *NCRMU* is not satisfied. Then, $(Y_i)_{i\in N} \in \mathcal{PO}$, and it follows from (12) that there do not exist feasible posterior risk profiles $(\widetilde{Y}_i^{\text{post}})_{i\in N} \in \mathcal{F}(N)$ such that $(\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in N} \gneqq (\Delta U_i(Y_i))_{i\in N} = 0$. Because $\mathcal{S} \subset \mathcal{F}(N)$, this concludes the proof.

(ii) Let $N = \{1,2\}$. Recall that when $|N| = 2$, it holds that $\mathcal{S} = \mathcal{PO} \cap \mathcal{IR}$. Therefore, it follows from (i) that it is sufficient to show that condition *NCRMU* implies the existence of a $(Y_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ with $\Delta U_i(Y_i^{\text{post}}) > 0$ for all $i \in N$. We first show that *NCRMU* implies the existence of an $(\widetilde{Y}_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ with $(\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in N} \gneqq 0$. Suppose there does not exist an $(\widetilde{Y}_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ with $\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in N} \gneqq 0$. Then, $(Y_i)_{i\in N} \in \mathcal{PO}$, and so condition *NCRMU* is not satisfied. Hence, condition *NCRMU* implies the existence of an $(\widetilde{Y}_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ with $(\Delta U_i(\widetilde{Y}_i^{\text{post}}))_{i\in N} \gneqq 0$. It remains to show that there also exists an $(Y_i^{\text{post}})_{i\in N} \in \mathcal{PO}$ with $\Delta U_i(Y_i^{\text{post}}) > 0$ for all $i \in N$. If $\Delta U_i(\widetilde{Y}_i^{\text{post}}) > 0$ for all $i \in \{1, 2\}$, the proof is concluded. If not, we can without loss of generality assume that $\Delta U_1(\widetilde{Y}_1^{\text{post}}) > 0$ and $\Delta U_2(\widetilde{Y}_2^{\text{post}}) = 0$. Then, let $(\hat{k}_1, \hat{k}_2) \in \mathbb{R}^2_{++}$ be such that $(\widetilde{Y}_i^{\text{post}})_{i\in N}$ satisfies (14) and (15), and let $(Y_i^{\text{post}})_{i\in N}$ be the alternative Pareto optimal posterior risk profiles satisfying (14) and (15) for $(\hat{k}_1, \hat{k}_2 + \varepsilon)$ for some $\varepsilon > 0$. Using the same arguments as in the proof of Proposition 2, it can be shown that for all $\omega \in \Omega$, it holds that $Y_1^{\text{post}}(\omega)$ is strictly decreasing in ε, and $Y_2^{\text{post}}(\omega)$ is strictly increasing in ε. Hence, there exists an ε small enough such that $\Delta U_1(Y_1^{\text{post}}) > 0$ and $\Delta U_2(Y_2^{\text{post}}) > 0$. This concludes the proof. $\square$

Proof of Proposition 4 Let $C(N, V)$ be as defined in (35). First, we show that

$$A = \{(\Delta U_i(Y_i^{\text{post}}))_{i\in N} : (Y_i^{\text{post}})_{i\in N} \in \mathcal{S}\} = C(N, V). \tag{37}$$

The fact that $A \supseteq C(N, V)$ is shown in the proof of Proposition 1, so we only need to show that $A \subseteq C(N, V)$. Let $x \in A$ and suppose that $x \notin C(N, V)$. Because $x \in A$, there exists an $(\hat{Y}_i)_{i\in N} \in \mathcal{S}$ such that $x = (\Delta U_i(\hat{Y}_i))_{i\in N}$. Because $x \notin C(N, V)$, there exists an $S \subseteq N$ such that $(x_i)_{i\in S} \in V(S)\backslash\partial V(S)$. Then, there exists an $(\widetilde{x}_i)_{i\in S} \gneqq (x_i)_{i\in S}$ such that $(\widetilde{x}_i)_{i\in S} \in \partial V(S)$. According to Lemma 1(ii), there exists an $(\widetilde{Y}_i)_{i\in S} \in \mathcal{PO}(S)$ such that $(\widetilde{x}_i)_{i\in S} = (\Delta U_i(\widetilde{Y}_i))_{i\in S}$. So, it holds that $(\hat{Y}_i)_{i\in S} \notin \mathcal{NI}(S)$. This is a contradiction, so $x \in C(N, V)$ and (37) holds.
So, it holds that

$$\mathcal{CNB} = \left\{ (Y_i^{\text{post}})_{i\in N} \in \mathcal{S} : (\Delta U_i(Y_i^{\text{post}}))_{i\in N} \in \underset{x\in C(N,V)}{\operatorname{argmax}} \prod_{i\in N} x_i \right\}, \tag{38}$$

where $C(N, V)$ is given by

$$C(N, V) = V(N) \cap \bigcap_{S\subseteq N} \left\{ x \in \mathbb{R}^N : (x_i)_{i\in S} \in \mathbb{R}^S\backslash(V(S)\backslash\partial V(S)) \right\}. \tag{39}$$

Since the set $V(N)$ is closed and the set $\left\{x \in \mathbb{R}^N : (x_i)_{i\in S} \in V(S)\backslash\partial V(S)\right\}$ is open for every $S \subseteq N$, it follows that $C(N, V)$ is closed. Since $C(N, V) \subset V(N) \cap \mathbb{R}^N_+$ and $V(N) \cap \mathbb{R}^N_+$ is bounded, the set $C(N, V)$ is compact. Moreover, in the proof of Proposition 1, we show that $C(N, V)$ is non-empty. So, in (38), there exists an $x \in C(N, V)$ such that $x = \operatorname{argmax}_{\widehat{x}\in C(N,V)} \prod_{i\in N} \widehat{x}_i$. Since $C(N, V) \subset \partial V(N)$, it follows that $x \in \partial V(N)$. Hence, it follows from Proposition 1(ii) that there exists an $(Y_i^{\text{post}})_{i\in N} \in \mathcal{PO}(N) = \mathcal{PO}$ such that $x = (\Delta U_i(Y_i^{\text{post}}))_{i\in N}$. This concludes the proof. $\square$

Proof of Proposition 5 (i) Consider

$$\widehat{\mathcal{CNB}} = \underset{x \in V(N), x \geq 0}{\text{argmax}} \left\{ \prod_{i \in N} x_i \right\}, \tag{40}$$

where $V(N)$ is as defined in (30). We know from Proposition 1 that $V(N)$ is convex. Moreover, it is easily verified that $V(N)$ is comprehensive, and that $V(N) \cap \mathbb{R}_+^N$ is non-empty and compact. It therefore follows from Nash (1950) that $\widehat{\mathcal{CNB}}$ is non-empty, single-valued, and satisfies $\widehat{\mathcal{CNB}} \subset \partial V(N) \cap \mathbb{R}_+^N$. It then follows from Lemma 1(ii) that for every $x \in \widehat{\mathcal{CNB}}$, there exists a $(Y_i^{\text{post}})_{i \in N} \in \mathcal{PO}$ such that $x = \Delta U_i(Y_i^{\text{post}}))_{i \in N} \geq 0$, i.e.,

$$\widehat{\mathcal{CNB}} \subseteq \left\{ (\Delta U_i(Y_i^{\text{post}}))_{i \in N} : (Y_i^{\text{post}})_{i \in N} \in \mathcal{PO} \cap \mathcal{IR} \right\}.$$

Therefore,

$$\begin{aligned} \widehat{\mathcal{CNB}} &= \left\{ (\Delta U_i(Y_i^{\text{post}}))_{i \in N} : (Y_i^{\text{post}})_{i \in N} \in \underset{(\widehat{Y}_i)_{i \in N} \in \mathcal{PO} \cap \mathcal{IR}}{\text{argmax}} \left\{ \prod_{i \in N} \Delta U_i(\widehat{Y}_i) \right\} \right\} \\ &= \left\{ (\Delta U_i(Y_i^{\text{post}}))_{i \in N} : (Y_i^{\text{post}})_{i \in N} \in \mathcal{CNB} \right\}, \end{aligned}$$

where the second equality follows from the fact that $\mathcal{S} = \mathcal{PO} \cap \mathcal{IR}$ when $|N| = 2$. Because $\widehat{\mathcal{CNB}}$ is non-empty, it follows that $\mathcal{CNB}$ is non-empty. Moreover, because $\widehat{\mathcal{CNB}}$ is single-valued, it follows from Proposition 1(iii), that $\mathcal{CNB}$ is single-valued.

(ii) For any given $k > 0$, let $(f_1(k), f_2(k))$ be the unique solution of (14) and (15) for $\overline{k} = (1, k)$. Because $\mathcal{S} = \mathcal{PO} \cap \mathcal{IR}$, it holds that $(Y_1^{\text{post}}, Y_2^{\text{post}}) \in \mathcal{S}$ iff there exists a $k > 0$ such that $(Y_1^{\text{post}}, Y_2^{\text{post}}) = (f_1(k), f_2(k))$, $\Delta U_1(f_1(k)) \geq 0$, and $\Delta U_2(f_2(k)) \geq 0$. Let the range of the utility function u_i be given by $(\underline{y}_i, \overline{y}_i)$, allowing for $\underline{y}_i = -\infty$ and/or $\overline{y}_i = +\infty$. Because by assumption, $u_i''(\cdot) < 0$, $\lim_{x \to -\infty} u_i'(x) = \infty$, and $\lim_{x \to +\infty} u_i'(x) = 0$, it holds that $(u_i')^{-1}$ exists, is strictly decreasing, and satisfies $\lim_{y \to \underline{y}_i} (u_i')^{-1}(y) = +\infty$, and $\lim_{y \to \overline{y}_i} (u_i')^{-1}(y) = 0$. Therefore, it follows from (14) and (15) that f_1 is strictly increasing with $\lim_{k \to 0} f_1(k) \overset{d}{=} -\infty$ and $\lim_{k \to +\infty} f_1(k) \overset{d}{=} +\infty$. Likewise, f_2 is strictly decreasing, with $\lim_{k \to 0} f_2(k) \overset{d}{=} -\infty$ and $\lim_{k \to +\infty} f_2(k) \overset{d}{=} -\infty$. Because $u_i'(\cdot) > 0$, this implies that $U_1(f_1(k))$ is strictly decreasing in k with $\lim_{k \to 0} U_1(f_1(k)) > U_1(Y_1)$ and $\lim_{k \to +\infty} U_1(f_1(k)) < U_1(Y_1)$. Hence, there exists a $k_{\max} \in \mathbb{R}_+$ such that $\Delta U_1(f_1(k)) \geq 0$ iff $k \leq k_{\max}$. Likewise, $U_2(f_2(k))$ is strictly increasing in k with $\lim_{k \to 0} U_2(f_2(k)) < U_2(Y_2)$, and $\lim_{k \to +\infty} U_2(f_2(k)) > U_2(Y_2)$. Hence, there exists a $k_{\min} \in \mathbb{R}_+$ such that $\Delta U_2(f_2(k)) \geq 0$ iff $k \geq k_{\min}$. Hence, $\mathcal{S} = \{(f_1(k), f_2(k)) : k \geq k_{\min}, k \leq k_{\max}\}$. Because $\mathcal{S} \neq \emptyset$, it follows that $k_{\min} \leq k_{\max}$.

(iii) This follows from Proposition 3(ii) and the fact that $\mathcal{CNB} \in \mathcal{S} = \mathcal{PO} \cap \mathcal{IR}$. □

B Age composition portfolios

The age composition of the pension fund and the accrued rights of the participants as a function of their age are displayed in Figure 2. The age composition of the insurer's portfolio is displayed in Figure 3.

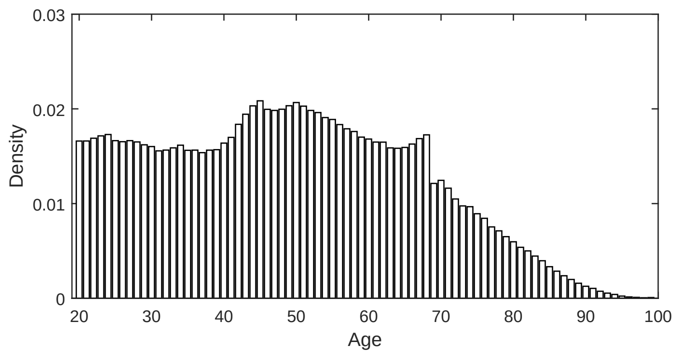

Figure 2: The age composition of the participants of a pension fund.

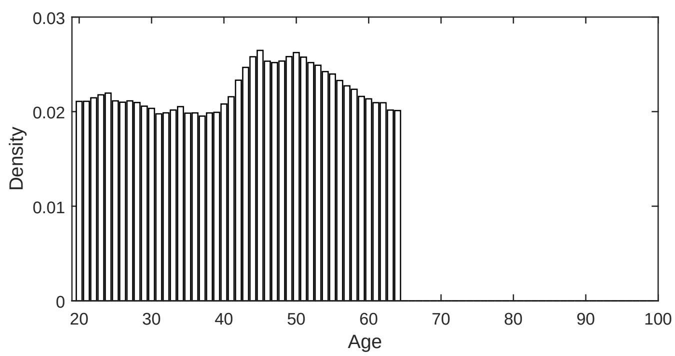

Figure 3: The age composition of the participants of an insurer.

C Lee-Carter model

In this appendix, we present a brief description of the Lee-Carter (1992) model. The probability that an individual of age x at time t survives the next year is modeled as

$$p_{x,t} = \exp(-m_{x,t}), \tag{41}$$

where $m_{x,t}$ represents the central death rate of a man with age x at time t (see, e.g., Pitacco et al., 2009). The central death rate is given by $m_{x,t} = D_{x,t}/E_{x,t}$, where $D_{x,t}$ is the observed number of deaths in year t in the cohort aged x at the beginning of year t, and $E_{x,t}$ is the corresponding exposure to death. Lee and Carter (1992) propose the following log-bilinear relationship:

$$\log(m_{x,t}) = a_x + b_x\kappa_t + \varepsilon_{x,t}, \qquad \varepsilon_{x,t} \sim^{i.i.d.} N(0,\sigma_x^2), \tag{42}$$

for all $t = t_0, t_0+1, \ldots, 0$ and $x = 1, 2, \ldots, 100$, where $\mathcal{K}_t = \{\kappa_{\widetilde{t}} : \widetilde{t} = t_0, t_0+1, \ldots, t\}$ and $t_0 < 0$. Here, t_0 is the first year in the dataset, and $t = 0$ is the last year in the dataset. The following normalizations are imposed: $\sum_{x=1}^{100} b_x = 1$ and $\sum_{t=t_0}^{0} \kappa_t = 0$. The estimates of a_x, b_x and κ_t are obtained via Singular Value Decomposition.

Future values of κ_t are forecasted using an ARIMA(0,1,1) model:

$$\kappa_t = \kappa_{t-1} + c + e_t + \theta e_{t-1}, \tag{43}$$

for all $t \geq 1$, where we impose the following distribution of the errors:

$$e_t \sim N(0,\sigma^2). \tag{44}$$

We use a standard bootstrap approach (see, e.g., Koissi et al., 2006) to include parameter uncertainty in the simulations.

We estimate the model based on mortality data for Dutch males from the HMD database[21], for the period 1977-2009 in case of LC(1977-2009), and for the period 1987-2009 in case of LC(1987-2009). The parameter estimates of the ARIMA(0,1,1) model are presented in Table 3.

	$\hat{c}$	$\hat{\theta}$	$\hat{\sigma}$
LC(1977-2009)	-2.00	-0.27	2.29
LC(1987-2009)	-2.13	-0.11	2.65

Table 3: Estimates of c, θ and σ in the ARIMA(0,1,1) model for male mortality rates, corresponding to (43) and (44), using HMD data from 1977 to 2009 (first row), and using HMD data from 1987 to 2009 (second row).

D Simulation of (Y_{PF}, Y_{INS})

To generate scenarios for Y_i as defined in (5), we use the following procedure:

- We simulate S trajectories for ${}_1p_{x,0}$, using (41)-(44) in Appendix C.
- For every simulated trajectory of ${}_1p_{x,0}$:
  - we determine the corresponding value of $\widetilde{L}_{i,1}$, using (24) and (25);
  - we re-estimate the model and determine the corresponding best-estimate values of $({}_\tau p_{x,1})_{\tau\geq 1}$, which we denote $({}_\tau p_{x,1}^{(\mathrm{BE}(1))})_{\tau\geq 1}$, by setting $\varepsilon_{x,t} = 0$ for all x and all $t > 1$ in (42), and $e_t = 0$ for all $t > 1$ in (43);

[21]See http://www.mortality.org/.

– the value of $\widetilde{L}_{i,\tau}^{(\mathrm{BE}(1))}$ for $\tau \geq 2$ is given by

$$\widetilde{L}_{PF,\tau}^{(\mathrm{BE}(1))} = \sum_{j=1}^{50{,}000} \delta_j \cdot \ {}_1p_{x_j,0} \cdot \ {}_{\tau-1}p_{x_j+1,1}^{(\mathrm{BE}(1))} \cdot 1_{\{x_j+\tau \geq 65\}}, \tag{45}$$

$$\widetilde{L}_{INS,\tau}^{(\mathrm{BE}(1))} = \sum_{j=1}^{N} 10 \cdot \ {}_1p_{x_j,0} \cdot \left({}_{\tau-2}p_{x_j+1,1}^{(\mathrm{BE}(1))} - \ {}_{\tau-1}p_{x,j+1,1}^{(\mathrm{BE}(1))} \right) \cdot 1_{\{x_j+\tau<65\}}. \tag{46}$$

E Discretization of the joint distribution of (Y_{PF}, Y_{INS})

In this appendix, we describe the method that we used to discretize the joint probability distribution of (Y_{PF}, Y_{INS}), distinguishing the homogeneous and the heterogeneous case. For the heterogeneous case, we use the following procedure:

- We simulate $S = 300,000$ scenarios for (Y_{PF}, Y_{INS}) under model LC(1977-2009), and $S = 300,000$ scenarios for (Y_{PF}, Y_{INS}) under model LC(1987-2009). We denote:
  - $\hat{\mathcal{S}} = \{1, 2, \ldots, S\}$;
  - $\mathcal{M} = \{LC(1987-2009), LC(1977-2009)\}$.

  Moreover, we denote the corresponding simulated values of $Y = Y_{PF} + Y_{INS}$ by $Y_{s,M}$ for $s \in \hat{\mathcal{S}}$, and $M \in \mathcal{M}$.

- To determine the state space Ω of the discretized distribution, we partition the interval $[\min_{s\in\hat{\mathcal{S}},M\in\mathcal{M}} \{Y_{s,M}\}, \max_{s\in\mathcal{S},M\in\mathcal{M}} \{Y_{s,M}\}]$ in 1,000 equally-sized sub-intervals, indexed by $\omega \in \Omega$ with $|\Omega| = 1,000$. In the discretized distribution, every state of the world corresponds with a sub-interval ω.

- It remains to determine realizations in each state of the world of the discretized version of Y, which we denote $\widehat{Y}(\omega)$, and the probabilities of the states under both models $M \in \mathcal{M}$, which we denote $\mathbb{P}_M(\{\omega\})$:
  - For every state $\omega \in \Omega$, we let $\widehat{Y}(\omega)$ be the average of all values of $Y_{s,M}$ that fall into sub-interval ω.
  - For every state $\omega \in \Omega$, we let $\mathbb{P}_M(\{\omega\})$ for model $M \in \mathcal{M}$ be equal to the fraction of simulated scenarios s for which $Y_{s,M}$ falls into sub-interval ω. If this results in $\mathbb{P}_M(\{\omega\}) = 0$, we set this probability equal to $\zeta := 10^{-10}$.

The optimal risk redistributions and the corresponding welfare gains in the homogeneous case are also calculated via a discretization of the state space. The procedure to obtain the posterior risk profiles and welfare gains is as described above for the heterogeneous case, but now with $\mathcal{M} = \{LC(1977-2009)\}$.