Non-Cooperative Dynamic Games for General Insurance Markets

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Abstract

In the insurance industry, the number of product-specific policies from different companies has increased significantly. The strong market competition has boosted the demand for a competitive premium. In actuarial science, scant literature still exists on how competition actually affects the calculation and the cycles of company's premiums. In this paper, we model premium dynamics via differential games, and study the insurers' equilibrium premium dynamics in a competitive market. We apply an optimal control theory methodology to determine the open-loop Nash equilibrium premium strategies. The market power of each insurance company is characterized by a price sensitive parameter, and the business volume is affected by the solvency ratio. We study two models. Considering the average market premiums, the first model studies an exponential relation between premium strategies and volume of business. The second model initially characterizes the competition between any selected pair of insurers, and then aggregates all the paired competitions in the market. Numerical examples illustrate the premium dynamics, and show that premium cycles may exist in equilibrium.

Keywords: Insurance Market Competition; Premium Cycles; Solvency Ratio; Open-loop Nash Equilibrium

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1 Introduction

1.1 Motivation

This paper constructs two models for determining the premium of general policies in competitive, non-cooperative, insurance markets. In the corresponding literature, there is still little research done on how the insurance premium follows from competition, and responds to changes initiated by competitors (Taylor, 1986; Daykin and Hey, 1990; Emms, 2012; Pantelous and Passalidou, 2015; Wu and Pantelous, 2017). Moreover, despite the fact that in many lines of insurance the presence of *underlying cycles* has been observed empirically, there is a constant endeavour to understand the dynamics of insurance premiums (Cummins and Outreville, 1987; Rantala, 1988; Doherty and Kang, 1988; Daykin et al., 1994; Winter, 1994; Cummins and Danzon, 1997; Lamm-Tennant and Weiss, 1997; Taylor, 2008; Malinovskii, 2010).

Furthermore, as it is observed in practice, the competition among insurance companies is getting stronger. Moreover, the domination of oligopoly markets is reflected often in the determination of insurance premiums (Friedman, 1982). Thus, a fair, but also commercially attractive premium is not any more a product of a simple risk assessment exercise, but a highly challenging decision-making process. Consequently, the demand of mathematical models is more essential than ever to investigate the *interconnectivity* among the competitors in the corresponding markets[1] and to understand the formulation of premium cycles.

1.2 Developments in Competitive Insurance Markets

Over the last three decades, academics have been interested in investigating on how competition might affect insurance premiums and how insurers respond to changes in the premium levels that being offered by competitors.

In actuarial science, Taylor (1986) pointed out that competition is a key component

[1] In this paper, we are interested in *local* markets. Thus, we should emphasize that competition is formulated at a local (national) level. Moreover, every insurer is regulated by the same competition and market's authority.

in insurance premium pricing. He embedded the law of demand to analyse the change of exposure volume through a comparison between the insurer's and market average premiums. Later, Taylor (1987) noted that the optimum underwriting strategies might be substantially affected by the expense rates.[2] Emms and his co-authors were able to extend significantly Taylor (1986, 1987)'s ideas developing a series of models in continuous time by implementing a variety of optimal control theory techniques (Emms and Haberman, 2005; Emms, 2007a,b; Emms et al., 2007; Emms and Haberman, 2009). In this paper, we defer from these approaches by modelling the competition via the open-loop Nash equilibrium.

1.3 Game-theoretic approaches for insurance markets

In a control theory framework as presented in Section 1.2, a single insurer's objective function is optimized considering the market information as inputs. However, as a key assumption, a premium strategy does not cause any reaction to the rest of the market's competitors.

Instead of focussing on a single insurer, we propose to model the entire insurance markets' premium competition by constructing an insurance game. Then, each insurer is a player in the game, with its premium as a strategy. The entire market's premium profile can be obtained by solving the Nash Equilibrium (hereafter referred to as NE) of the corresponding game.

The use of game theory in actuarial science has a long history. The first attempts go back to Borch (1962, 1974), Bühlmann (1980, 1984), and Lemaire (1984, 1991), who applied cooperative games to model insurer and reinsurer risk transfer; see also other extensions and reviews (Aase, 1993; Brockett and Xia, 1995; Tsanakas and Christofides, 2006). Two models were applied in non-life insurance markets for non-cooperative games: a) the Bertrand oligopoly in which insurers set premiums and b) the Cournot oligopoly in which insurers choose the volume of business.[3] Boonen (2016) also proposed a way to

[2] In discrete-time, Taylor's approach has been revisited recently, and linear as well as non-linear stochastic demand functions were considered (Pantelous and Passalidou, 2013, 2015, 2017).

[3] For the Bertrand model see (Polborn, 1998; Rees et al., 1999; Dutang et al., 2013), and for the Cournot model see (Powers et al., 1998; Powers and Shubik, 1998).

optimally regulate bargaining for risk redistributions. He studied the strategic interaction between two insurance companies that trade risk over-the-counter in a one-period model.

Emms (2012) considered a model by applying a differential game-theoretic methodology for non-cooperative markets. Under his framework, each insurer's premium depends on other insurers' premium strategies, assuming that each market participant chooses an optimal premium strategy. Insurers' premium strategies are obtained by calculating the NE of the game. Recently, Wu and Pantelous (2017) introduced the concept of aggregate games based on a one-period framework which aggregates market competition by initially characterizing any selected paired insurers' payoff. The existence of NE was guaranteed by proving that the constructed game is a potential game (Monderer and Shapley, 1996; Jensen, 2010).

1.4 Generalized finite-time differential game models: A new approach

In this paper, generalized finite-time differential games with a finite number of insurers are constructed. The formulation allows us to investigate the mechanism for the premium cycles by solving NE premium profiles. When the market reaches a NE, no insurer can increase its profit by modifying its strategy given the optimal strategies of the other insurers.

As in Emms (2012), the optimal control theory methodology is incorporated. Under a continuous-time framework, the number of new contracts is modelled considering competition, while the loss of exposure due to policy termination is assumed to be proportional to the current volume of exposure. In this direction, two competition-related models are proposed, studied and compared: (1) *Model I* adopts the exponential demand function of Taylor (1986, 1987) and Emms (2012) considering the market average premium; while, (2) *Model II* is formulated based on the aggregate game developed in Wu and Pantelous (2017).

The main innovations between the new approach, and respectively Emms (2012) for Model I, and Wu and Pantelous (2017) for Model II can be summarized as follows. First, we analyse the effect of solvency ratio to exposure changes, which is developed further in Model II. Second, the exposure change is separated between the marginal number of new and the expired holding policies following Emms and Haberman (2005). Third, the exponential function proposed by Taylor (1986) in discrete framework is adopted to model the marginal number of new policies at a selected time point in Model I. Finally, in Model II, the concept of aggregate games is adopted, and a generalized finite-time differential game is formulated extending the one-period framework proposed by Wu and Pantelous (2017).[4]

Modelling price sensitivity by a single parameter was initially proposed by Taylor (2008), and it was further explored by Wu and Pantelous (2017). It uses the concept of price elasticity of demand. In this study, a price sensitivity parameter is formulated as the market power factor for optimization purposes. The solvency ratio is the capital per unit of premium, and it is taken into consideration in the competition between each pair of insurers, as it is observable by the policyholders. In Taylor (2008), it is stated that the management department will adjust its actuarial premium with respect to the current solvency ratio. Considering historical data, when the capital amount is relatively high compared with actuarial premium value, insurance companies prefer to increase their premium value. The reason is that the insurers are more confident to pay the claims under this condition. Solvency ratios are observable due to rating agencies. In this paper, we implement the concept of solvency ratio in the competition, and develop an insurance game where the solvency ratio is embedded. In particular, we assume that if the (observable) solvency ratio is high, the number of new contracts sold will be affected less by other insurers' premium strategies. Interestingly, premium cycles are observed in the numerical example of Model II, even without the consideration of any stochastic variables.[5]

[4]In another words, Model I is the natural extension of Taylor (1986, 1987) in continuous time and under a game theoretical approach. On the other hand, Model II inherits all the good features of Model I (and also the corresponding to it literature) in continuous time, and uses the concept of aggregation games which was developed in Wu and Pantelous (2017).

[5]As it will be clear in the numerical example, Model II appears to capture well the presence of underlying cycles, which was *not* the case in Emms (2012). According to the authors' knowledge, this is the first time that the underlying cycles of the premium are captured by a mathematical model which considers competition in the market.

This paper is organized as follows. Section 2 introduces the construction of the two insurance market competition models. In Section 3, the optimization problem is formulated for the two models, and the Hamiltonians are presented. Section 4 presents numerical examples of those two models. Section 5 concludes.

2 Model Construction

2.1 Basic Notation

In this part of section, the necessary notation and parameters involved in this paper are introduced in Table 1. Next, the definition of the key parameters is provided for a better understanding of the remaining paper:

[Insert Table 1 somewhere here]

2.2 Baseline model

Let $\mathbf{N} = \{1, \ldots, n\}$ be the finite set of insurers in the insurance market, and T a given future time. At every point in time $t \in [0, T]$, every insurer $i \in \mathbf{N}$ makes a decision to set the *premium* $p_i(t)$ per unit of exposure. The decisions of all insurers in the market lead to the state variable $\theta_i(t) = (k_i(t), q_i(t))$, where $k_i(t) > 0$ is the capital per unit of exposure of insurer i, and $q_i(t) > 0$ is the volume of exposure of insurer i, which represents the number of policies. Denote $P_i(t) = \{p_i(t') : t' \in [0, t]\}$ and $\Theta_i(t) = \{\theta_i(t') : t' \in [0, t]\}$. Moreover, we write $M_{-i}(t) = \{M_j(t) : j \in \mathbf{N} \backslash \{i\}\}$ for any $\mathbf{N}$-valued function M, $\dot{k}_i(t) = \frac{d}{dt}k_i(t)$ and $\dot{q}_i(t) = \frac{d}{dt}q_i(t)$.

In line with Emms and Haberman (2005), we assume that there is a fixed length $\tau > 0$ of insurance policies, and all new policyholders are required to pay the current premium rate $p_i(t)$. We illustrate in Figure 1 how the underwriting of policies affects the exposure volume $q_i(t)$ of the insurer. The dashed line displays the volume of exposure (policies) at time t, which is $q(t)$. Jumps up are due to newly sold policies, and jumps down are due to policy terminations. Moreover, for every t, the top solid line displays the aggregate number of policies that is accrued since time 0.

[Insert Figure 1 somewhere here]

The change in exposure at any time t can be split up into the exposure generated by selling new contracts and the exposure lost due to policy termination. Next, we identify the details of those two effects. To obtain a Markov model and use conventional control theory, we follow Emms and Haberman (2005) by assuming that the loss due to policy termination is given by $\frac{1}{\tau}\, q_i(t)$ for any insurer i. This implies that the policies expire at a rate that is proportional to the number of policies in the portfolios of insurer i. In other words, if insurer i has $q_i(t)$ policies that each have a duration of length τ, then the policies terminate according to a uniform distribution, yielding a rate of policy termination that is given by $\frac{1}{\tau}\, q_i(t)$. Then, the state equation of exposure for insurer i is given by

$$\dot{q}_i(t) = m_i(t) q_i(t) - \frac{1}{\tau}\, q_i(t), \tag{2.1}$$

where $m_i(t)$ is the marginal number of new policies sold at time t per unit of exposure. Typically, the value of $m_i(t)$ depends on the premiums in the market $(p_i(t), p_{-i}(t))$ and the state variable $k_i(t)$.

Define $I_i(t, t+\Delta t)$ as the premium income of insurer i in period $[t, t+\Delta t)$ and $C_i(t, t+\Delta t)$ as the cost of holding capital. Here, we assume that the premiums are paid at the beginning of each contract and all insurance policies have a fixed length τ, and so $I_i(t, t+\Delta t)$ is the premium income of the new contracts generated. We get from a Taylor series approximation that:

$$I_i(t, t + \Delta t) = p_i(t)\, m_i(t)\, q_i(t)\, \Delta t + O(\Delta t^2),$$

where $f(\Delta t) = O(\Delta t^2)$ is such that $\limsup_{\Delta t \to 0} |\frac{f(\Delta t)}{\Delta t^2}| < \infty$.

Define $\beta \in (0, 1)$ as the depreciation of capital. The cost of holding capital $C_i(t, t+\Delta t)$ during the period $[t, t + \Delta t)$ is given by

$$C_i(t, t + \Delta t) = \beta\, K_i(t)\, \Delta t + O(\Delta t^2).$$

Moreover, define $\pi_i > 0$ as the break-even premium of insurer i, that is the deterministic insurance claim that needs to be paid per unit of exposure (Taylor, 1986; Emms and Haberman, 2005; Emms, 2012; Wu and Pantelous, 2017). So, the insurer needs to pay $\pi_i\, q_i(t)\, \Delta t + O(\Delta t^2)$ for insurance claims during the period $[t, t + \Delta t)$. The total capital for insurer i at time $t + \Delta t$ is given by

$$\begin{aligned} K_i(t + \Delta t) &= K_i(t) + I_i(t, t + \Delta t) - C_i(t, t + \Delta t) - \pi_i\, q_i(t)\, \Delta t + O(\Delta t^2) \\ &= K_i(t) + \big(p_i(t)\, m_i(t) - \pi_i - \beta\, k_i(t)\big)\; q_i(t)\Delta t + O(\Delta t^2). \end{aligned}$$

The volume of the insurer's exposure is also modified considering the entry of new business and the expiration of existing policies. The difference of capital per unit of exposure in period $[t, t + \Delta t)$ equals to

$$\begin{aligned} \Delta k_i(t) &= k_i(t + \Delta t) - k_i(t) \\ &= \frac{K_i(t + \Delta t)}{q_i(t + \Delta t)} - k_i(t) \\ &= \frac{K_i(t + \Delta t)}{q_i(t) + \dot{q}_i(t)\Delta t} - k_i(t) + O(\Delta t^2) \\ &= \frac{k_i(t) + \big(p_i(t)\, m_i(t) - \pi_i - \beta\, k_i(t)\big)\; \Delta t}{1 + \big(m_i(t) - \frac{1}{\tau}\big)\; \Delta t} - k_i(t) + O(\Delta t^2) \\ &= \left(p_i(t)\, m_i(t) - \pi_i - k_i(t)\, \left(\beta + m_i(t) - \frac{1}{\tau}\right)\right)\, \Delta t + O(\Delta t^2), \end{aligned}$$

by using a Taylor series approximation. Therefore, the state equation of capital per exposure for insurer i is given by

$$\dot{k}_i(t) = p_i(t)\, m_i(t) - \pi_i - k_i(t)\, \left(\beta + m_i(t) - \frac{1}{\tau}\right). \tag{2.2}$$

For a given time period $[0, T]$, we assume that every insurer $i \in \mathbf{N}$ aims to maximize the *net present value* of its profit. This objective is in line with Emms (2012), and the objective function for insurer $i \in \mathbf{N}$ is given by

$$u_i(P_i(T); \Theta_i(T)) = \int_0^T e^{-\zeta t} F_i(p_i(t); \theta_i(t))dt. \tag{2.3}$$

Here, ζ is the discount factor, and

$$F_i(p_i(t);\theta_i(t)) = (p_i(t)m_i(t) - \pi_i - \beta k_i(t))q_i(t). \tag{2.4}$$

A set of control functions $t \mapsto (p_1^*(t), p_2^*(t), \ldots, p_n^*(t))$ is a NE for the game within the class of open-loop strategies if the following holds. For any insurer i, the control $p_i^*(\cdot)$ provides a solution to the optimal control problem:

$$\textit{maximize} \quad u_i(P_i(T);\Theta_i(T)), \tag{2.5}$$

over the set of controllers, $P_i(T)$, where the set of controllers of other insurers, $P_{-i}(T)$, is feasible, and the system has dynamics: $\{p_1(0), \ldots, p_n(0)\}$ and $\{\theta_1(0), \ldots, \theta_n(0)\}$ given, and (2.1) and (2.2) hold:

$$\begin{aligned} \dot{k}_i(t) &= p_i(t)\, m_i(t) - \pi_i - k_i(t)\left(\beta + m_i(t) - \frac{1}{\tau}\right), \\ \dot{q}_i(t) &= m_i(t)q_i(t) - \frac{1}{\tau}\, q_i(t), \end{aligned}$$

for all $i \in \mathbf{N}$.

The rate of selling policies m_i is affected by market competition, and not yet specified. To model m_i explicitly, we propose two models in Sections 3.1 and 3.2 that are denoted by *Model I* and *Model II*, respectively. Inspired by Taylor (1986, 1987) and Emms (2012), Model I investigates exponential relations between exposure volume and premium competition. On the other hand, Model II characterizes exposure volume regarding the aggregation of competition among all the pairs of insurers. The price elasticity function concept is adopted to investigate the exposure volume change in Model II, as an extension of Wu and Pantelous (2017) to a dynamic setting.

2.3 Model I Formulation

Model I adopts the exponential demand function proposed in Taylor (1986, 1987) and Emms (2012) for modelling the competition between any pair of insurers. Let us define the function ρ_i of insurer i at time t by

$$\rho_i(t) = -(p_i(t) - \bar{p}_{-i}(t)),$$

where $\bar{p}_{-i}(t)$ is the average premium of all the other insurers in insurance market except i. When $\rho_i(t)$ is positive, insurer i's premium $p_i(t)$ is less than $\bar{p}_{-i}(t)$; then we assume that insurer i gains exposure in the insurance market. Policies flow in a reverse manner when ρ_i is negative. We model the rate of selling new policies $m_i(t)$ for insurer i at time t as

$$m_i(t) = \frac{1}{\tau}\, r_i\, e^{b_i\, \rho_i(t) - \frac{p_i(t)}{k_i(t)}}, \tag{2.6}$$

where $b_i > 0$ is the price sensitivity parameter of insurer $i \in \mathbf{N}$, and $r_i > 0$ is a benchmark solvency ratio for insurer i.

In line with the exponential demand function in Taylor (1986, 1987), we initially model $m_i(t)$ as $\frac{1}{\tau}\, e^{b_i \rho_i(t)}$. When $p_i(t) < \bar{p}_{-i}(t)$, then $\frac{1}{\tau}\, e^{b_i \rho_i(t)}$ is larger than $\frac{1}{\tau}$. We further augment this effect with an influence of the solvency ratio on competition, which is a new concept in our paper. The solvency ratio is defined as $\frac{k_i(t)}{p_i(t)}$. Since Taylor (2008) assumed that the management department adjusts the actuarial premium by comparing the insurer's current solvency ratio and a benchmark solvency ratio, we augment m with the factor $r_i e^{-\frac{p_i(t)}{k_i(t)}}$.[6] When the solvency ratio $\frac{k_i(t)}{p_i(t)}$ increases, the rate of selling new policies $m_i(t)$ increases, which describes the effect that insurers with larger solvency ratios can obtain more policies.

2.4 Model II Formulation

As proposed by Wu and Pantelous (2017), Model II initially specifies the flow of policies between any pair of insurers. The entire insurance market competition can be evaluated by aggregating among the different pairs of insurers. For any insurer j, let us define the

[6]An augmentation with $r_i e^{\frac{k_i(t)}{p_i(t)}}$ would lead to an ill-posed problem, and it is therefore avoided.

transfer function $\rho_{j\to i}(t)$ from insurer j to insurer i at time t as follows

$$\rho_{j\to i}(t) = 1 - \frac{p_i(t)}{p_j(t)}.$$

At time t, when insurer i's premium is less than insurer j's premium, insurer j's policies tend to flow to insurer i. Policies flow in a reverse direction when $p_i(t) > p_j(t)$. We assume that the exposure flow from insurer j to insurer i is given by

$$q_{j\to i}(t) \quad = \quad a_i\, \rho_{j\to i}(t)\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\, q_i(t), \tag{2.7}$$

where $a_i > 0$ is the price sensitivity parameter of insurer i, and $r_i > 0$ is a benchmark solvency ratio for insurer i. Typically, we have $q_{j\to i}(t) \neq -q_{i\to j}(t)$. The exposure changes over time follow from the competition in the entire market. It is obtained by summing up all the bilateral policies' gains or losses. The aggregate exposure gain or loss for insurer i is then given by

$$\dot{q}_i(t) = \sum_{j\in\mathbf{N}, j\neq i} q_{j\to i}(t). \tag{2.8}$$

We allow $\sum_{i\in\mathbf{N}} \dot{q}_i(t)$ to be *not* equal to zero, since potential customers may enter (leave) the insurance market when the premiums are low (high).

Substituting (2.7) and (2.8) in (2.1) yields that the rate of selling new policies for insurer i at time t in Model II is given by

$$m_i(t) = \frac{1}{\tau} + a_i\, \frac{1}{r_i}\, e^{\frac{p_i(t)}{k_i(t)}} \sum_{j\in\mathbf{N}, j\neq i} \left(1 - \frac{p_i(t)}{p_j(t)}\right). \tag{2.9}$$

3 Theoretical Results

As it was discussed in Section 2, the NE premium strategy for the ith insurer follows from a maximization problem over the set of feasible premium strategies given the feasible premium strategies of the other insurers (Dockner et al., 2000). Thus, in this section, the corresponding Hamiltonians and related results for Models I and II are presented.

3.1 Optimisation Problem for Model I

From (2.4)-(2.6), we derive that the marginal profit for insurer i is given by

$$F_i(p_i(t);\theta_i(t)) = \left(p_i(t)\,\frac{1}{\tau}\,r_i\,e^{b_i\left(\bar{p}_{-i}(t)-p_i(t)\right)-\frac{p_i(t)}{k_i(t)}} - \pi_i - \beta\,k_i(t)\right)\,q_i(t). \tag{3.1}$$

We obtain the following dynamics for the state variables of insurer i:

$$\begin{aligned}\dot{k}_i(t) &= p_i(t)\,\frac{1}{\tau}\,r_i\,e^{b_i\left(\bar{p}_{-i}(t)-p_i(t)\right)-\frac{p_i(t)}{k_i(t)}} - \pi_i \\ &\quad -k_i(t)\,\left(\beta+\frac{1}{\tau}\,r_i\,e^{b_i\left(\bar{p}_{-i}(t)-p_i(t)\right)-\frac{p_i(t)}{k_i(t)}} - \frac{1}{\tau}\right),\end{aligned} \tag{3.2}$$

and

$$\dot{q}_i(t) = \left(r_i\,e^{b_i\left(\bar{p}_{-i}(t)-p_i(t)\right)-\frac{p_i(t)}{k_i(t)}} - 1\right)\,\frac{1}{\tau}q_i(t). \tag{3.3}$$

With the objective function and state equations, the Hamiltonian for the ith insurer is given by[7]

$$\begin{aligned}H_i &= e^{-\zeta t}\,\left(p_i(t)\,\frac{1}{\tau}\,r_i\,e^{b_i\left(\bar{p}_{-i}(t)-p_i(t)\right)-\frac{p_i(t)}{k_i(t)}} - \pi_i - \beta\,k_i(t)\right)\,q_i(t) \\ &\quad +\sum_{j\in\mathbf{N}}\mu_{ij}(t)\,\Bigg[p_j(t)\,\frac{1}{\tau}\,r_j\,e^{b_j\left(\bar{p}_{-j}(t)-p_j(t)\right)-\frac{p_j(t)}{k_j(t)}} - \pi_j \\ &\quad -k_j(t)\,\left(\beta+\frac{1}{\tau}\,r_j\,e^{b_j\left(\bar{p}_{-j}(t)-p_j(t)\right)-\frac{p_j(t)}{k_j(t)}} - \frac{1}{\tau}\right)\Bigg] \\ &\quad +\sum_{j\in\mathbf{N}}\lambda_{ij}(t)\,\left(r_j\,e^{b_j\left(\bar{p}_{-j}(t)-p_j(t)\right)-\frac{p_j(t)}{k_j(t)}} - 1\right)\,\frac{1}{\tau}q_j(t).\end{aligned} \tag{3.4}$$

For any $i,j\in\mathbf{N}$, the adjoint equations are given by

$$\frac{d\lambda_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial q_j(t)},\quad \lambda_{ij}(T)=0, \tag{3.5}$$

$$\frac{d\mu_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial k_j(t)},\quad \mu_{ij}(T)=0. \tag{3.6}$$

The next lemma provides the solution of (3.5) and (3.6) for $j\neq i$.

[7] With slight abuse of notation, we do not explicitly write that the Hamiltonian depends on $(p_i(t), p_{-i}(t), \theta_i(t))$.

Lemma 1. *For any $t \in [0,T]$ and $i, j \in \boldsymbol{N}$ such that $j \neq i$, it holds that $\lambda_{ij}(t) = 0$ and $\mu_{ij}(t) = 0$.*

Proof. For $j \neq i$, it holds that

$$\frac{d\lambda_{ij}(t)}{dt} = -\lambda_{ij}(t) \left(r_j \, e^{b_j \left(\bar{p}_{-j}(t) - p_j(t)\right) - \frac{p_j(t)}{k_j(t)}} - 1 \right) \frac{1}{\tau}. \tag{3.7}$$

Let $A = \left(r_j \, e^{b_j \left(\bar{p}_{-j}(t) - p_j(t)\right) - \frac{p_j(t)}{k_j(t)}} - 1 \right) \frac{1}{\tau}$. When $A \neq 0$, we have

$$\left| \frac{d\lambda_{ij}(t)}{A\lambda_{ij}(t)} \right| = dt \Leftrightarrow \int \left| \frac{d\lambda_{ij}(t)}{A\lambda_{ij}(t)} \right| = \int dt \Leftrightarrow \ln \left| \lambda_{ij}(t) \right| = |At| + c$$

and, hence, we get $\left|\lambda_{ij}(t)\right| = e^{|A \cdot t| + c}$. With $\lambda_{ij}(T) = 0$, it is a contradiction. When $A = 0$, we have $\lambda_{ij}(t) = 0$ by construction. Hence $\lambda_{ij}(t) = 0$, when $j \neq i$. We can prove that $\mu_{ij}(t) = 0$ for all $j \neq i$ in a similar way. □

Using Lemma 1, the Hamiltonian in (3.4) simplifies to

$$\begin{aligned} H_i &= e^{-\zeta t} \left(p_i(t) \, \frac{1}{\tau} \, r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - \pi_i - \beta \, k_i(t) \right) q_i(t) \\ &\quad + \mu_{ii}(t) \left[p_i(t) \, \frac{1}{\tau} \, r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - \pi_i \right. \\ &\quad \left. - k_i(t) \left(\beta + \frac{1}{\tau} \, r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - \frac{1}{\tau} \right) \right] \\ &\quad + \lambda_{ii}(t) \left(r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - 1 \right) \frac{1}{\tau} q_i(t). \end{aligned} \tag{3.8}$$

From the adjoint equations, we have

$$\begin{aligned} \frac{d\lambda_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial q_i(t)} \\ &= -e^{-\zeta t} \left(p_i(t) \, \frac{1}{\tau} \, r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - \pi_i - \beta \, k_i(t) \right) \\ &\quad - \lambda_{ii}(t) \left(r_i \, e^{b_i \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - 1 \right) \frac{1}{\tau}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\frac{d\mu_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial k_i(t)} \\
&= -e^{-\zeta t}\, q_i(t) \left(\frac{p_i(t)^2}{k_i(t)^2}\, \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} - \beta \right) \\
&\quad -\mu_{ii}(t) \Bigg[-\beta - \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} + \frac{1}{\tau} \\
&\quad + \big(p_i(t) - k_i(t)\big)\, \frac{p_i(t)}{k_i(t)^2} \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \Bigg] \\
&\quad -\lambda_{ii}(t)\, \frac{p_i(t)}{k_i(t)^2}\, \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}}\, q_i(t).
\end{aligned} \tag{3.10}$$

The first-order conditions of the Hamiltonian, defined in (3.8), are given by

$$\begin{aligned}
\frac{\partial H_i}{\partial p_i(t)} &= e^{-\zeta t}\, q_i(t) \left(p_i(t) \left(-b_i - \frac{1}{k_i(t)} \right) + 1 \right) \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \\
&\quad + \mu_{ii}(t) \Bigg[\big(p_i(t) - k_i(t)\big) \left(-b_i - \frac{1}{k_i(t)} \right) \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \\
&\quad + \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \Bigg] \\
&\quad + \lambda_{ii}(t)\, q_i(t) \left(-b_i - \frac{1}{k_i(t)} \right) \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}}.
\end{aligned} \tag{3.11}$$

which must equal zero for all $t \in [0, T]$ and $i \in \mathbf{N}$. The second-order conditions of the Hamiltonians are given by

$$\begin{aligned}
\frac{\partial^2 H_i}{\partial p_i(t)^2} &= e^{-\zeta t}\, q_i(t) \left(2 \left(-b_i - \frac{1}{k_i(t)} \right) + \left(b_i + \frac{1}{k_i(t)} \right)^2 \right) \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \\
&\quad + \mu_{ii}(t) \Bigg[2 \left(-b_i - \frac{1}{k_i(t)} \right) \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \\
&\quad + \big(p_i(t) - k_i(t)\big) \left(b_i + \frac{1}{k_i(t)} \right)^2 \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}} \Bigg] \\
&\quad + \lambda_{ii}(t)\, q_i(t) \left(b_i + \frac{1}{k_i(t)} \right)^2 \frac{1}{\tau}\, r_i\, e^{b_i\, \left(\bar{p}_{-i}(t) - p_i(t)\right) - \frac{p_i(t)}{k_i(t)}}.
\end{aligned} \tag{3.12}$$

It is well-known in optimal control theory that the solution of the first-order conditions is a NE when the second-order conditions of Hamiltonians in (3.12) are non-positive for all $t \in [0, T]$ and $i \in \mathbf{N}$ (Friesz, 2010).

3.2 Optimisation Problem for Model II

For Model II, the marginal profit for insurer i is given by

$$F_i(p_i(t); \theta_i(t)) = \left(p_i(t) \left(\frac{1}{\tau} + \sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) - \pi_i - \beta\, k_i(t) \right) q_i(t). \tag{3.13}$$

Similarly to (3.2)-(3.3), we derive the following state equations

$$\begin{aligned} \dot{k}_i(t) &= p_i(t) \left(\frac{1}{\tau} + \sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) \\ &\quad - \pi_i - k_i(t) \left(\beta + \sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right), \qquad (3.14) \\ \dot{q}_i(t) &= \sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\, q_i(t). \qquad (3.15) \end{aligned}$$

With the objective function and state equations, the Hamiltonian for insurer i is given by

$$\begin{aligned} H_i &= e^{-\zeta t} \left(p_i(t) \left(\frac{1}{\tau} + \sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) - \pi_i - \beta\, k_i(t) \right) q_i(t) \\ &+ \sum_{j \in \mathbf{N}} \mu_{ij}(t) \Bigg[p_j(t) \left(\frac{1}{\tau} + \sum_{\ell \in \mathbf{N}, \ell \neq j} a_j \left(1 - \frac{p_j(t)}{p_\ell(t)} \right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j} \right) \\ &\quad - \pi_j - k_j(t) \left(\beta + \sum_{\ell \in \mathbf{N}, \ell \neq j} a_j \left(1 - \frac{p_j(t)}{p_\ell(t)} \right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j} \right) \Bigg] \\ &+ \sum_{j \in \mathbf{N}} \lambda_{ij} \left(\sum_{\ell \in \mathbf{N}, \ell \neq j} a_j \left(1 - \frac{p_j(t)}{p_\ell(t)} \right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j}\, q_j(t) \right). \qquad (3.16) \end{aligned}$$

For any $i, j \in \mathbf{N}$, the adjoint equations are given by

$$\frac{d\lambda_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial q_j(t)}, \qquad \lambda_{ij}(T) = 0, \tag{3.17}$$

$$\frac{d\mu_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial k_j(t)}, \qquad \mu_{ij}(T) = 0. \tag{3.18}$$

The next lemma provides the solution of (3.17) and (3.18) for $j \neq i$.

Lemma 2. *For any $t \in [0, T]$ and $i, j \in \boldsymbol{N}$ such that $j \neq i$, it holds that $\lambda_{ij}(t) = 0$ and $\mu_{ij}(t) = 0$.*

Proof. The proof is similar to the proof of Lemma 1, and so it is omitted. □

Due to Lemma 2, the Hamiltonian in (3.16) simplifies to

$$\begin{aligned} H_i &= e^{-\zeta t}\left(p_i(t)\left(\frac{1}{\tau} + \sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right) - \pi_i - \beta\, k_i(t)\right) q_i(t) \\ &+ \mu_{ii}(t)\left[p_i(t)\left(\tau^{-1}\frac{1}{\tau} + \sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right)\right. \\ &\qquad \left. -\pi_i - k_i(t)\left(\beta + \sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right)\right] \\ &+ \lambda_{ii}\left(\sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\, q_i(t)\right). \end{aligned} \tag{3.19}$$

From the adjoint equations, we get

$$
\begin{aligned}
\frac{d\mu_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial k_i(t)} \\
&= -e^{-\zeta t}\, q_i(t), \Bigg[p_i(t) \left(\sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-(k_i(t))^2\, p_i(t)\right) \right) \\
&\quad - \sum_{j\in\mathbf{N}, j\neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right) \right) \Bigg] \\
&\quad -\mu_{ii}(t) \Bigg[\left(p_i(t) - k_i(t)\right) \left(\sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-(k_i(t))^2\, p_i(t)\right) \right) - \beta \\
&\quad - \sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \Bigg] \\
&\quad -\lambda_{ii}\, q_i(t) \left(\sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-(k_i(t))^2\, p_i(t)\right) \right),
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
\frac{d\lambda_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial q_i(t)} \\
&= -e^{-\zeta t} \left(p_i(t) \left(\frac{1}{\tau} + \sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) - \pi_i - \beta\, k_i(t) \right) \\
&\quad -\lambda_{ii} \left(\sum_{j\in\mathbf{N}, j\neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right).
\end{aligned}
\tag{3.21}
$$

The first-order conditions of the Hamiltonian in (3.19) are given by

$$\begin{aligned}
\frac{\partial H_i}{\partial p_i(t)} &= e^{-\zeta t}\, q_i(t) \Bigg[\frac{1}{\tau} + \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) \\
&\quad + p_i(t) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \Bigg] \\
&\quad + \mu_{ii}(t) \Bigg[\frac{1}{\tau} + \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) \\
&\quad - \big(p_i(t) - k_i(t)\big) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \Bigg] \\
&\quad + \lambda_{ii}(t)\, q_i(t) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right), \qquad (3.22)
\end{aligned}$$

which must equal zero for all $t \in [0, T]$ and $i \in \mathbf{N}$.

The first-order conditions of the Hamiltonians yield a NE if the second order conditions are satisfied. The second-order conditions of the Hamiltonians in (3.19) are given by

$$\begin{aligned}
\frac{\partial^2 H_i}{\partial r_i(t)^2} &= e^{-\zeta t}\, q_i(t) \Bigg[2 \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \\
&\quad + p_i(t) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(\frac{-2k_i(t) + p_j(t) - p_i(t)}{k_i(t)^2\, p_j(t)} \right)\right) \Bigg] \\
&\quad + \mu_{ii}(t) \Bigg[2 \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \\
&\quad + \big(p_i(t) - k_i(t)\big) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(\frac{-2k_i(t) + p_j(t) - p_i(t)}{k_i(t)^2\, p_j(t)} \right)\right) \Bigg] \\
&\quad + \lambda_{ii}(t)\, q_i(t) \left(\sum_{j \in \mathbf{N}, j \neq i} a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-(k_i(t))^2\, p_i(t)\right)\right), \qquad (3.23)
\end{aligned}$$

which must be non-positive for all $t \in [0, T]$ and $i \in \mathbf{N}$.

4 Numerical Application

In this section, a numerical application for both models is formulated in a competitive insurance market environment. In the Appendix, the relevant steps using the Matlab programming language for Model I (Section 4.1) and Model II (Section 4.2) are presented in details. For simplicity and better understanding of the numerous parameters involved, an insurance market with two competitors is considered.

4.1 Model I

A two insurers' game is constructed with Insurer 1 and Insurer 2. In Section 3.1, we obtained ten variables (p_1, p_2, k_1, k_2, q_1, q_2, μ_{11}, μ_{22}, λ_{11}, and λ_{22}), two first-order conditions of the Hamiltonian, and eight ODEs ($\dot{k}_1$, $\dot{k}_2$, $\dot{q}_1$, $\dot{q}_2$, $\dot{\mu}_{11}$, $\dot{\mu}_{22}$, $\dot{\lambda}_{11}$, and $\dot{\lambda}_{22}$).

When (3.11) equals zero, the two players' equilibrium premiums are not correlated. That is, the premium of is decoupled. This implies that marginal premium $\dot{p}_1$ for Insurer 1 can be obtained by differentiating the corresponding solution of p_1 from (3.11) with respect to time t, where we do not need to consider the correlation with the price changes of Insurer 2.[8] Likewise, we obtain $\dot{p}_2$. Under this framework, we have ten variables which correspond to ten ODEs, accordingly. Considering the initial and terminal conditions, a Bounded Value Problem (hereafter referred to as BVP) is formulated, which can be solved numerically. In the following subsection, we describe the numerical steps to calculate of the NE premium strategies.

4.1.1 Algorithm of Calculating Equilibrium Premium Strategy for Model I

In this section, the main steps of the algorithm are presented, whereas the Appendix provides the details of the algorithm.

Step 1: Calculate p_1, when the first-order conditions of the Hamiltonian for Insurer 1 in (3.11) are satisfied.

Step 2: Differentiate p_1 obtained in Step 1 with respect to time t to calculate $\dot{p}_1$.

[8]See Section 4.1.1 and Appendix for more details.

Step 3: Repeat steps 1 and 2 with respect to Insurer 2 to calculate $\dot{p}_2$.

Step 4: We have ten variables, $p_1, p_2, k_1, k_2, q_1, q_2, \mu_{11}, \mu_{22}, \lambda_{11}$, and λ_{22} and the corresponding ODEs. This is a BVP with six conditions from the initial information of both insurers and four terminal conditions from (3.5) and (3.6), which we solve using the "bvp45" function in Matlab.

Step 5: Test whether the second-order conditions of Hamiltonians for both insurers in (3.12) are non-positive during the time interval $[0, T]$.[9]

4.1.2 Numerical Example of Model I

Here, we illustrate an insurance game considering a period of three years. We investigate the competition among two candidates: one player represents a relatively small insurer (Insurer 1), while the other one is regarded as a large insurer with substantial market power (Insurer 2).

Table 2 shows the parameter values of our insurance game, and Table 3 the initial information, including the initial premium, volume of exposure and capital per exposure regarding both insurers. The insurance company with a greater market power (Insurer 2) has a larger price sensitivity parameter b.

[Insert Table 2 somewhere here]

[Insert Table 3 somewhere here]

With the algorithm introduced in Section 4.1.1, a premium profile of both insurers is calculated, which is presented in Figure 2, with non-positive second-order Hamiltonian profiles occurred for both insurers. Since the second-order conditions of the Hamiltonians are satisfied, the premium profiles that follow from the first-order conditions constitutes a NE. Figures 3 and 4 describe the exposure volume and capital per exposure, respectively.

[Insert Figure 2 somewhere here]

[9] The strategy profiles with positive second-order conditions will be neglected.

[Insert Figure 3 somewhere here]

[Insert Figure 4 somewhere here]

Although there is a slight decrease in the value of premium, the larger market power of Insurer 2 yields that the equilibrium premium of Insurer 2 is kept at a relatively high level over the whole time horizon. Insurer 1 adopts a relatively low premium level with the purpose of absorbing more policies. Insurer 2 slightly lower its capital per exposure through the competition while insurer's capital per exposure inappreciably increased. No cycles appear in the equilibrium premiums of Model I. The equilibrium strategies of the two insurers keep stable over the 3-years period. Different sets of parameters are tested for Model I, and none of the parameters yield any cycles.

4.2 Model II

Similarly to Section 4.1, we consider a game with Insurer 1 and Insurer 2. In Section 3.2, we obtained ten variables (p_1, p_2, k_1, k_2, q_1, q_2, μ_{11}, μ_{22}, λ_{11}, and λ_{22}), two first-order conditions of the Hamiltonian for the two insurers, and eight ODEs ($\dot{k}_1$, $\dot{k}_2$, $\dot{q}_1$, $\dot{q}_2$, $\dot{\mu}_{11}$, $\dot{\mu}_{22}$, $\dot{\lambda}_{11}$, and $\dot{\lambda}_{22}$).

We can eliminate two variables p_2 and λ_{22}[10], and substitute $\dot{\lambda}_{22}$ to the differential equation of p_1. Under these circumstances, we take eight variables which correspond to eight ODEs, accordingly. The backward integration considers the standard Mean Value Theorem, which is adopted in this section in order to solve the BVP.

4.2.1 Algorithm of Calculating Equilibrium Premium Strategy for Model II

In this section, the main steps of the algorithm are presented, whereas the Appendix provides the details of the algorithm.

Step 1: Calculate p_2 when the first-order condition of the Hamiltonian in (3.22) for Insurer 1 is satisfied.

[10] See Step 1 in the algorithm presented in Section 4.2.1 and Appendix. We get an expression for p_2, which will be substituted from the solution of Insurer 1's first order condition of Hamiltonian in (3.22).

Step 2: Get an expression of λ_{22} from the first-order condition of insurer 2's Hamiltonian, with p_2 replaced by the solution of Insurer 1's first order condition of Hamiltonian (3.22) calculated in Step 1.

Step 3: Differentiate the expression of λ_{22} with respect to time t. Generate an ODE for p_1.

Step 4: Apply a backward iteration of the system with the first-order conditions of Hamiltonian for insurer 1 and 2. Terminal values of ten variables are required to be used as inputs. From (3.17) and (3.18), it follows that $\mu_{11}(T) = \mu_{22}(T) = \lambda_{11}(T) = \lambda_{22}(T) = 0$. For the other 6 variables, $p_1(T)$, $p_2(T)$, $k_1(T)$, $k_2(T)$, $q_1(T)$, $q_2(T)$ need to satisfy (3.22) in order to be used as inputs.

Since $\mu_{11}(T) = \mu_{22}(T) = \lambda_{11}(T) = \lambda_{22}(T) = 0$, (3.22) does not depend on $q_1(T)$ and $q_2(T)$ at time T. We use the Matlab solver 'fsolve' to provide $p_1(T)$, $p_2(T)$ when $k_1(T)$, $k_2(T)$ are fixed, via (3.22).

Then, we guess $q_1(T)$ and $q_2(T)$. Terminal values of the 10 variables are used as inputs in the backward iteration.

Step 5: Stop until the initial values of p_1, p_2, k_1, k_2, q_1, and q_2 from backward iteration equals to the initial data value. Otherwise, we adjust the guess of $k_1(T)$, $k_2(T)$, $q_1(T)$, and $q_2(T)$.[11]

Step 6: From the previous step, we collect the terminal values that yield the correct initial values. We check whether the second-order conditions of the Hamiltonians of both insurers are non-positive during the time interval $[0, T]$.[12]

Remark 1. A NE of a game with more insurers can also be obtained applying similar algorithms to those in Sections 4.1.1 and 4.2.1. However, as it is expected that more loops are required and the corresponding computational cost will increase in order to calculate the NE premium strategy.

[11] The Algorithmic Steps for Model II in the Appendix shows the guess only of $k_1(T)$.

[12] Similarly to the algorithm for Model I, we will neglect the strategy profiles with positive second-order conditions.

4.2.2 Numerical Example of Model II

We introduce in this section a 3-year insurance game. Similarly to Model I, we study numerically the competition among a large insurer with market power and a relatively small insurer. The insurance company with greater market power has a lower price sensitivity parameter a. As it was the case in Wu and Pantelous (2017), the Lerner index is considered for a. The Lerner index is a market power measurement which describes the inverse correlation between market performance and profit margin for monopolist environments (Lerner, 1934).

Table 4 states the chosen parameter values. Table 5 illustrates the initial information of the two insurers.

[Insert Table 4 somewhere here]

[Insert Table 5 somewhere here]

With the algorithm introduced in Section 4.2.1, a NE premium profile for the two insurers is calculated, which is shown in Figure 5. Similarly as for Model I, verification that it is a NE follows from the fact that the second-order Hamiltonians are non-positive for both insurers. Figures 6 and 7 describe the exposure volume and capital per exposure, accordingly. Premium cycles are observed in the whole time period. Figure 5 supports the empirical evidence in the insurance literature (Cummins and Outreville, 1987; Doherty and Kang, 1988; Winter, 1994; Cummins and Danzon, 1997; Lamm-Tennant and Weiss, 1997) that premium cycles in insurance markets are caused by market competition. Although the premiums between the two insurers are not proportional, the shape of premium cycle profiles is similar. Figure 5 suggests that Insurer 1 follows Insurer 2's premium strategy. The premium of Insurer 1 even falls below the break-even premium level from the 3rd month to the 6th month in order to remain competitive and attract more policies. The two insurers' total capital, displayed in Figure 8, remains stable for the first two years. Due to the increment of premium, both insurers gain massive capital in the third year, which is particularly true for Insurer 1.

[Insert Figure 5 somewhere here]

[Insert Figure 6 somewhere here]

[Insert Figure 7 somewhere here]

[Insert Figure 8 somewhere here]

Different sets of parameters are tested for Model II, and premium cycles appear to be related to the profit margins. When the break-even premium is relatively low compared to the premium, insurers intend to compete with their rivals, and more cycles appear. Figure 9 shows the equilibrium premium profiles where the break-even premiums are increased to $\pi_1 = 0.8$ and $\pi_2 = 0.85$, while the other parameters remain the same as they appear in Tables 4 and 5. In Figure 9, we find premium cycles, but with longer periods; while both insurers' equilibrium premium slightly increased smoothly over the time horizon. For instance, Figure 10 shows the NE premium profiles where the break-even premiums are given by $\pi_1 = 1.2$ and $\pi_2 = 1.1$.

[Insert Figure 9 somewhere here]

[Insert Figure 10 somewhere here]

5 Conclusion

This paper studies two finite-time differential games in an insurance market. It provides how competition impacts the premium process of non-life insurance products. An optimal control theory approach is applied to determine premiums in the open-loop Nash Equilibrium. The first model (Model I) adopts the exponential demand function proposed by Taylor (1986, 1987) and Emms (2012), and the second model (Model II) is formulated based on the aggregate exposure proposed by Wu and Pantelous (2017). Numerical examples illustrate the premium dynamics, and show that premium cycles do exist in equilibrium for the Model II.

As a future research, we are interested in applying stochastic models to dynamic games. Another possible extension would be to study feedback Nash equilibria for dynamic games.

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Algorithmic Steps using Matlab Programming for Model I

The steps 1, 2 and 4 are presenting using Matlab:

Matlab - Step 1:

% Type in (3.11) with respect to Insurer 1, denoted as *firstorderH1*.

1: x1=solve(*firstorderH1*==0, p_1).

Matlab - Step 2:

% Create symbolic variables with respect to t;

1: $odex1$=diff$(x1(t), t)$.

% $odex1$ includes *diff* $(k_1(t), t)$, *diff* $(q_1(t), t)$, *diff* $(\lambda_{11}(t), t)$, diff $(\mu_{11}(t), t)$.
% Replace the differential equations in $odex1$ with their expressions,
see (3.2), (3.3), (3.9), and (3.10). $\dot{p}_1$ is obtained.

Matlab - Step 4:

1: init=bvpinit(linspace(0,3,1000),@*bc_init*);
2: sol=bvp4c(@*rhs_bvp*, @*bc_bvp*,init);
3: t=linspace(0,3,1000);
4: BS=deval(sol,t);
5: plot(t,BS(1,:));

6: function rhs=*rhs_bvp*(t,y);
7: rhs=[$\dot{p}_1$; $\dot{p}_2$;$\dot{k}_1$;$\dot{k}_2$;$\dot{q}_1$;$\dot{q}_2$;$\dot{\mu}_{11}$;$\dot{\mu}_{22}$;$\dot{\lambda}_{11}$;$\dot{\lambda}_{22}$] ;

8: function bc=*bc_bvp*(yl, yr);
9: bc=[yl(1)-0.88; yl(2)-1.05; yl(3)-0.6; yl(4)-1; yr(5)-5225; yr(6)-13700; yr(7); yr(8); yr(9); yr(10)];
10: end.

% @*bc_init* is the initial guess of this BVP.[13]

[13]See matlab code '*solinit*' for more details.

Algorithmic Steps using Matlab Programming for Model II

The steps 1 to 5 are presenting using Matlab:

Matlab - Step 1:
% Type in (3.22) with respect to Insurer 1, denoted as *FirstorderH1*. 1: x2=solve(*FirstorderH1*==0, p_2).

Matlab - Step 2:
% Type in (3.22) for Insurer 2, denoted as *FirstorderH2* . 1: *FirstorderH2_fh*=matlabFunction(*FirstorderH2*) ; 2: *FirstorderH2_new*=*FirstorderH2*_fh(a_2,r_2,τ^{-1},ζ,t,p_1,**x2**,k_2,q_2,μ_{22},λ_{22}) ; 3: x10=solve(*FirstorderH2_new*==0,λ_{22}).

Matlab - Step 3:
% Create symbolic variables with respect to t; 1: $dx10$=diff($x10(t), t$); % $dx10$ includes *diff* $(p_1(t), t)$, *diff* $(k_1(t), t)$, *diff* $(k_2(t), t)$, *diff* $(q_1(t), t)$, *diff* $(q_2(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\mu_{22}(t), t)$, *diff* $(\lambda_{11}(t), t)$. (3.15), (3.16), (3.20), and (3.21) provide all the above differential equations, except *diff* $(p_1(t), t)$. % Replace *diff* $(k_1(t), t)$, *diff* $(k_2(t), t)$, *diff* $(q_1(t), t)$, *diff* $(q_2(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\mu_{22}(t), t)$, *diff* $(\lambda_{11}(t), t)$ with the corresponding differential equations in $dx10$, which is denoted as *x_10*. *x_10* includes ten variables p_1, p_2, k_1, k_2, q_1, q_2, μ_{11}, μ_{22}, λ_{11}, and λ_{22} and *diff* $(p_1(t), t)$; 2: *x_10* - *diff* $(\lambda_{22}(t), t) == 0$ % *x_10* is the derivative of $\lambda_{22}(t)$ with respect to t which satisfies both two insurers' first-order conditions of Hamiltonians. Regarding *diff* $(\lambda_{22}(t), t)$ from (3.21), an equation of *diff* $(p_1(t), t)$ is obtained through the above equation.

Matlab - Step 4:

1: x0=[0.000001 0.000001];
2: p=fsolve(@premium,x0);
3: function F = premium(u);

% Replace p_1 as $u(1)$, p_2 as $u(2)$ in $\frac{\partial H_1}{\partial p_1}$ and $\frac{\partial H_2}{\partial p_2}$.

4: F = [$\frac{\partial H_1}{\partial p_1}$; $\frac{\partial H_2}{\partial p_2}$] ;
5: end.

% Given by the terminal values k_1^T, k_2^T, q_1^T, q_2^T, the terminal value of p_1 and p_2 are calculated by solving F with variable vector u regarding $\mu_{11}^T = \mu_{22}^T = \lambda_{11}^T = \lambda_{22}^T = 0$,

6: for i=N:-1:1;
7: t = (i-1) * T/(N-1) ;

% 8 ODE systems with 8 variables:

8: $k_1(i-1) = -dt * \mathit{diff}\ (k_1(t), t) + k_1(i);$
$k_2(i-1) = -dt * \mathit{diff}\ (k_2(t), t) + k_2(i);$
$q_1(i-1) = -dt * \mathit{diff}\ (q_1(t), t) + q_1(i);$
$q_2(i-1) = -dt * \mathit{diff}\ (q_2(t), t) + q_2(i);$
$\mu_{11}(i-1) = -dt * \mathit{diff}\ (\mu_{11}(t), t) + \mu_{11}(i);$
$\mu_{22}(i-1) = -dt * \mathit{diff}\ (\mu_{22}(t), t) + \mu_{22}(i);$
$\lambda_{11}(i-1) = -dt * \mathit{diff}\ (\lambda_{11}(t), t) + \lambda_{11}(i);$
syms v

% Replace $\mathit{diff}\ (p_1(t), t)$ with an approximation of the first derivative $\frac{p_1(i)-v}{dt}$. $p_1(i-1) = vpasolve(x\_10 - \mathit{diff}\ (\lambda_{22}(t), t) == 0, v);$
% Replace p_2, λ_{22} with $x2$, $x10$ correspondingly in all the above equations.

9: end.

Matlab - Step 5:

k_1^T=linspace(1.6,2,9) ;
x0=[0.000001 0.000001];
Y=zeros(length(k_1^T),2);
for z=1:length(k_1^T) ;
  fun= @(x)premium(x,$k_1^T(z)$) ;
  Y(z,:)=fsolve(fun,x0);
end.

for z=1:length(k_1);
$p_1(N, z) = Y(:, 1)$; ...

% "..." symbolises the assignment of terminal values, like $k_1(N, z) = k_1^T(z)$, $k_2(N, z) = 3$, $q_1(N, z) = 5000$. Unnecessary repetitions are omitted. See also Tables.

  for i=N:-1:1;
    W = [0,1];
  ...

% "..." means that we have to use here the 8 ODE systems with 8 variables mentioned in Step 4,

    if $(p_1(1, z) > 0.885)$ && $(p_1(1, z) < 0.895)$ && $(p_2(1, z) > 1.535)$ && $(p_2(1, z) < 1.545)$ && ...

% "..." describes all the initial values of insurers, such as premium, capital and exposure. p_2 at each stage i can be calculated through x2,

      W(index,1)=[$k_1(z)$,1];
      index = index +1;
  end.
end.

TABLES

$\mathbf{N}$	Set of *insurers* in the insurance market, $\mathbf{N} = \{1, \ldots, n\}$, $n \in \mathbb{N}$;
T	Terminal time in the future;
t	A moment in time, $t \in [0, T]$;
p_i	*Premium value* (per unit of exposure) for insurer $i \in N$;
q_i	*Exposure* (volume of business) for insurer $i \in N$,
	which represents the number of policies (quantity) undertaken by insurer $i \in N$;
k_i	*Capital* (per unit of exposure) for insurer $i \in N$;
π_i	*Break-even premium* for insurer $i \in N$;
m_i	Marginal number of new policies (at time t);
I_i	Premium Income for insurer $i \in N$;
K_i	Total capital for insurer $i \in N$;
C_i	Total cost of holding capital for insurer $i \in N$;
$\bar{p}_{-i}$	Average premium of all the other insurers in insurance market except $i \in N$;
β	Depreciation of capital;
τ	Fixed length of insurance policies;
ζ	Discount rate, for the calculation of the net present value;
θ	State variable, that includes k_i, q_i;
μ, λ	Adjoint variables for q and k, respectively;
a_i, b_i	*Price sensitivity* (positive) parameter of insurer $i \in N$;
u_i	Objective function for insurer $i \in N$.

Table 1: Notation for the Model I & II.

Number of market participants	n	2
Break-even premium of Insurer 1	π_1	0.6
Break-even premium of Insurer 2	π_2	0.609
Price sensitivity parameter of Insurer 1	b_1	0.2
Price sensitivity parameter of Insurer 2	b_2	0.28
Benchmark solvency ratio of Insurer 1	r_1	3.3
Benchmark solvency ratio of Insurer 2	r_2	2.2
Depreciation of capital	β	0.03
Discount factor	ζ	0.02

Table 2: Parameter values for Model I.

Initial premium of Insurer 1	$p_1(0)$	0.88
Initial premium of Insurer 2	$p_2(0)$	1.05
Initial exposure volume of Insurer 1	$q_1(0)$	5,225
Initial exposure volume of Insurer 2	$q_2(0)$	13,700
Initial capital per exposure of Insurer 1	$k_1(0)$	0.6
Initial capital per exposure of Insurer 2	$k_2(0)$	1

Table 3: Initial information of Insurer 1 and Insurer 2.

Number of market participants	n	2
Break-even premium of Insurer 1	π_1	0.6
Break-even premium of Insurer 2	π_2	0.61
Price sensitivity parameter of Insurer 1	a_1	1.16
Price sensitivity parameter of Insurer 2	a_2	0.87
Benchmark solvency ratio of Insurer 1	r_1	2
Benchmark solvency ratio of Insurer 2	r_2	3.5
Depreciation of capital	β	0.03
Discount factor	ζ	0.02

Table 4: Parameter values for Model II.

Initial premium of Insurer 1	$p_1(0)$	0.89
Initial premium of Insurer 2	$p_2(0)$	1.54
Initial exposure volume of Insurer 1	$q_1(0)$	3,240
Initial exposure volume of Insurer 2	$q_2(0)$	5,240
Initial capital per exposure of Insurer 1	$k_1(0)$	1.28
Initial capital per exposure of Insurer 2	$k_2(0)$	1.96

Table 5: Initial information of Insurer 1 and Insurer 2.

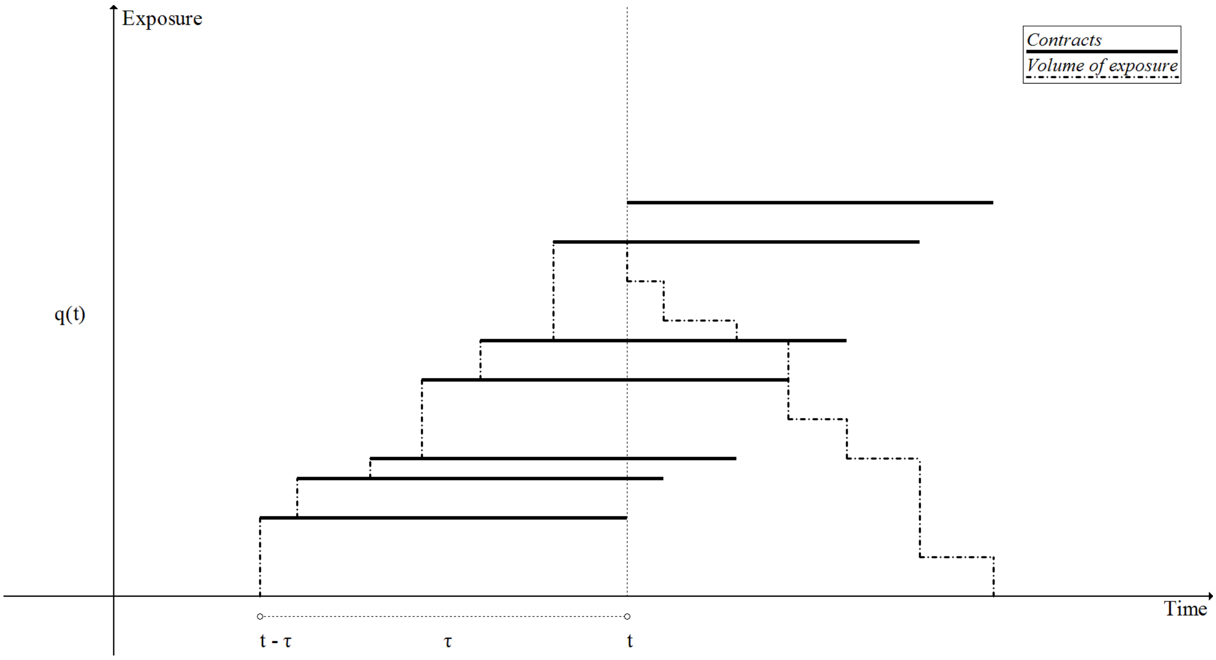

Figure 1: The dashed line is the volume of exposure $q(t)$ for time t, and the solid lines denote the duration of policies with the same start date. Moreover, for every t, the top solid line displays the aggregate number of policies that is accrued from time 0.

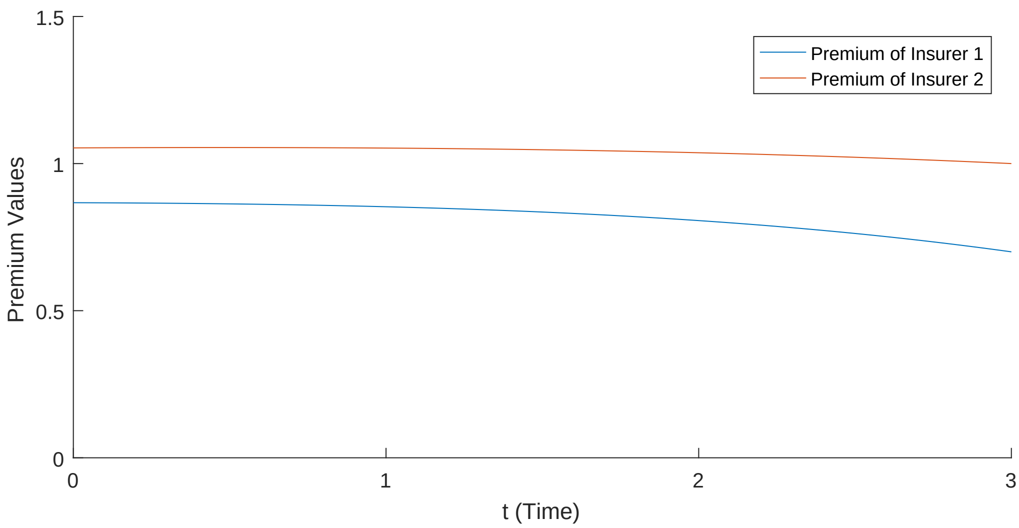

Figure 2: Equilibrium premium profiles of both insurers over three years time in Model I. The blue line is Insurer 1's premium and the red line is Insurer 2's premium. The premium is given on the y-axis while the corresponding time is on the x-axis.

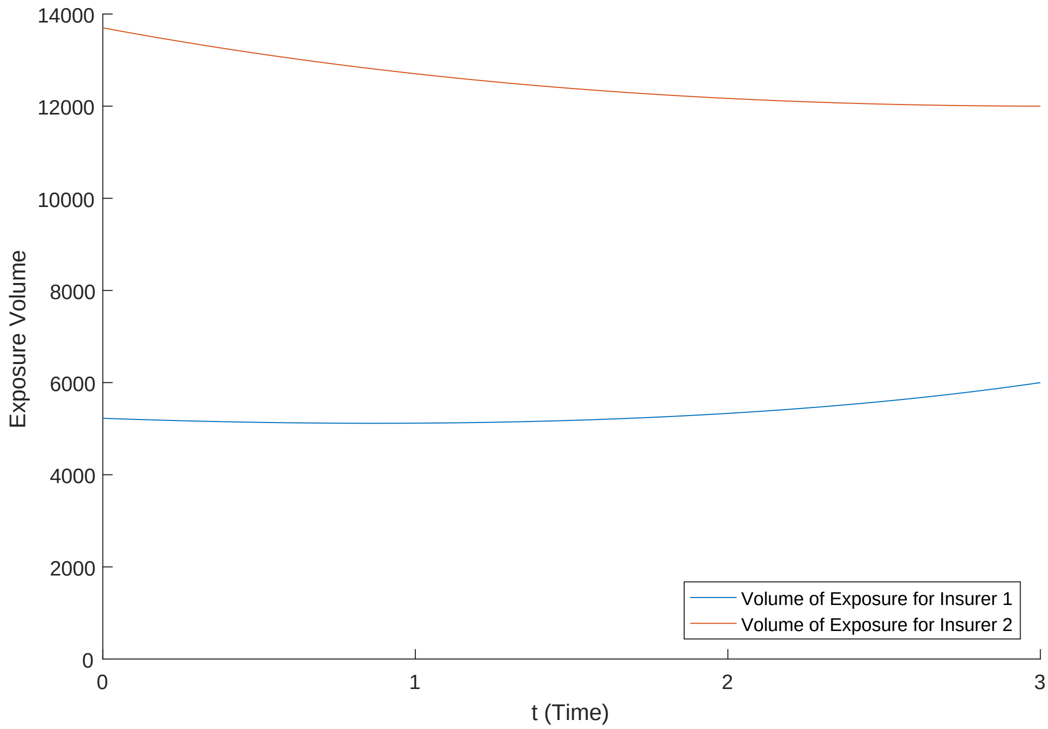

Figure 3: Volume of exposure profiles regarding both insurers over three years time in Model I. The blue line is Insurer 1's volume of exposure and the red line is Insurer 2's volume of exposure. The volume of exposure is given on the y-axis while the corresponding time is on the x-axis.

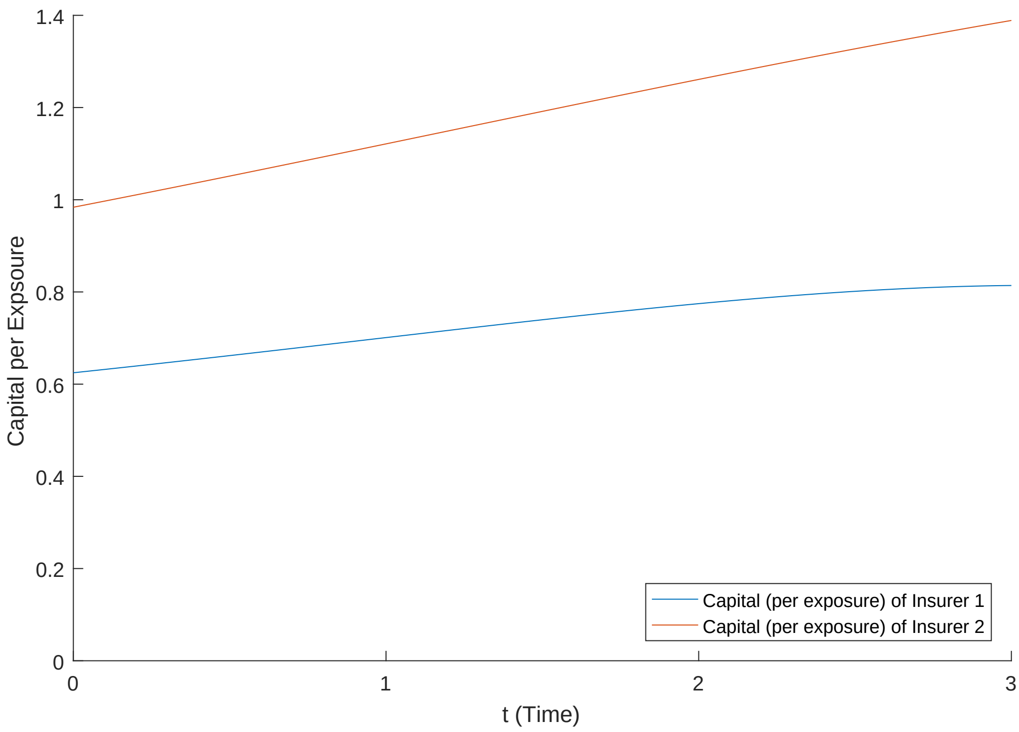

Figure 4: Exposure per capital profiles for both insurers over three years time in Model I. The blue line is Insurer 1's capital per exposure and the red line is Insurer 2's capital per exposure. The capital per exposure is given on the y-axis while the corresponding time is on the x-axis.

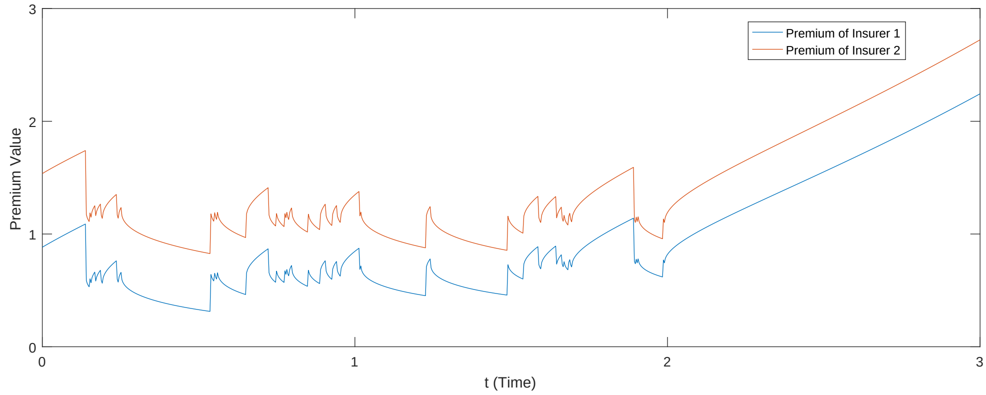

Figure 5: Equilibrium premium profiles of both insurers over three years time in Model II. The blue line is Insurer 1's premium and the red line is Insurer 2's premium. The premium is given on the y-axis while the corresponding time is on the x-axis.

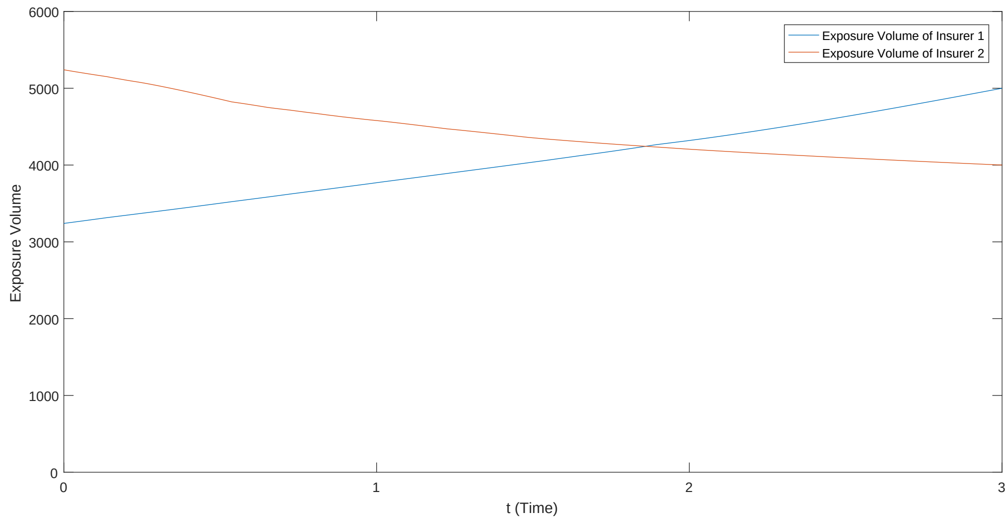

Figure 6: Volume of exposure profiles regarding both insurers over three years time in Model II. The blue line is Insurer 1's volume of exposure and the red line is Insurer 2's volume of exposure. The volume of exposure is given on the y-axis while the corresponding time is on the x-axis.

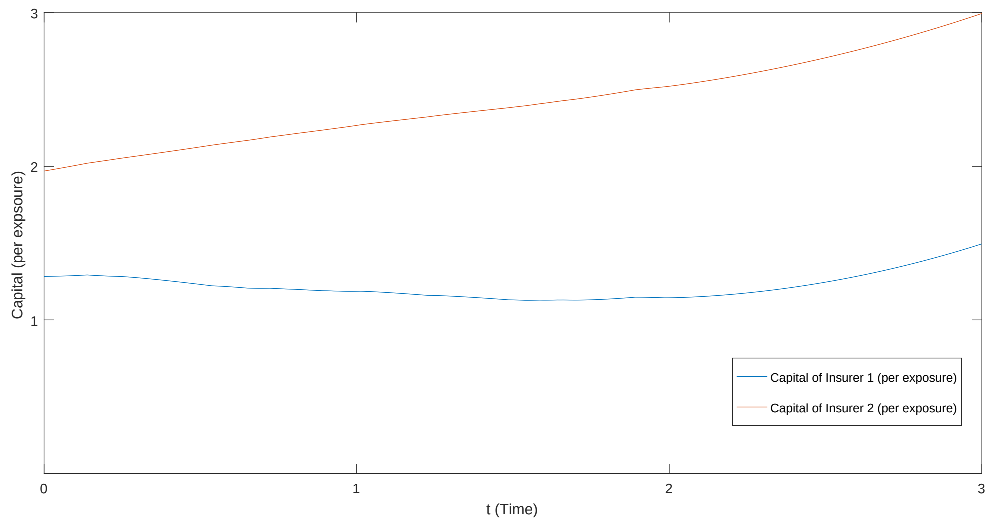

Figure 7: Exposure per capital profiles regarding both insurers over three years time in Model II. The blue line is Insurer 1's capital per exposure and the red line is Insurer 2's capital per exposure. The capital per exposure is given on the y-axis while the corresponding time is on the x-axis.

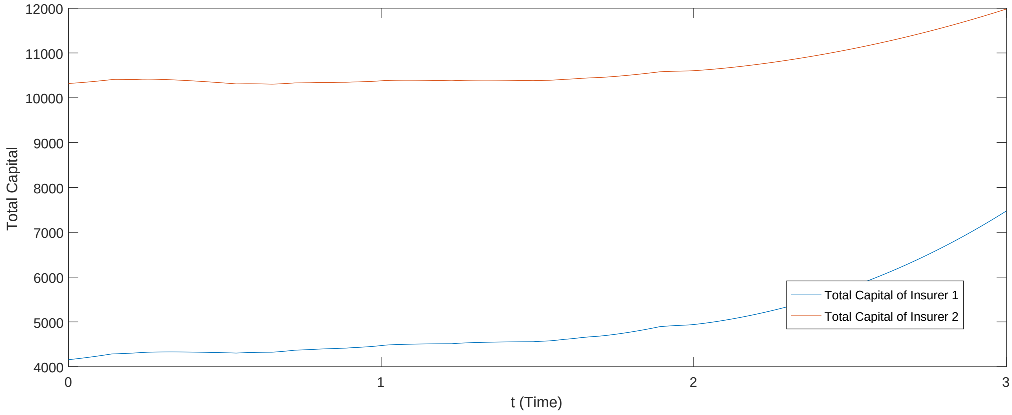

Figure 8: Total capital profiles regarding both insurers over three years time in Model II. The blue line is Insurer 1's capital and the red line is Insurer 2's capital. The capital is given on the y-axis while the corresponding time is on the x-axis.

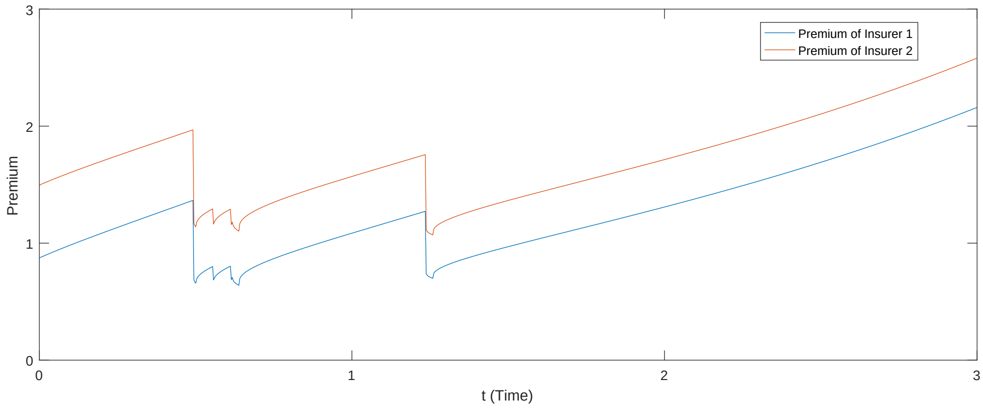

Figure 9: Equilibrium premium profiles of both insurers over three years time in Model II, where $\pi_1 = 0.8$, and $\pi_2 = 0.85$. The blue line is Insurer 1's premium and the red line is Insurer 2's premium. The premium is given on the y-axis while the corresponding time is on the x-axis.

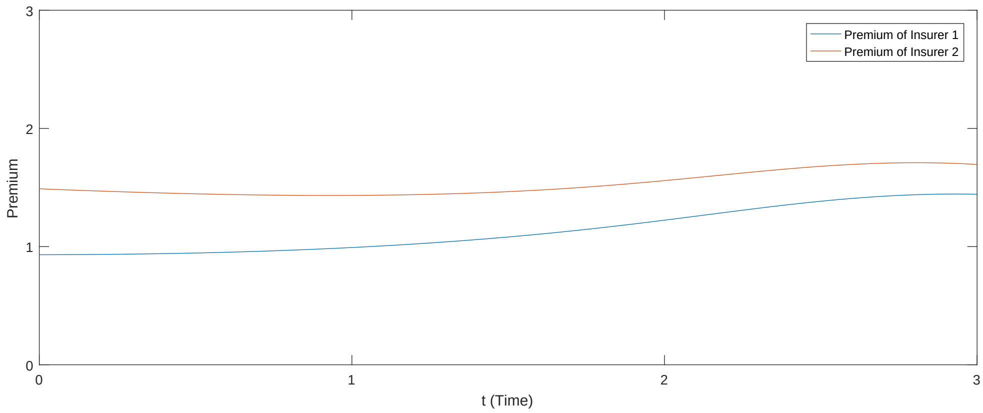

Figure 10: Equilibrium premium profiles of both insurers over three years time in Model II, where $\pi_1 = 1.2$, and $\pi_2 = 1.1$. The blue line is Insurer 1's premium and the red line is Insurer 2's premium. The premium is given on the y-axis while the corresponding time is on the x-axis.