Nash equilibria of Over-The-Counter Bargaining for Insurance Risk Redistributions: the Role of a Regulator*

Tim J. Boonen†
University of Amsterdam

October 1, 2015

Abstract

This paper proposes a way to optimally regulate bargaining for risk redistributions. We discuss the strategic interaction between two firms, who trade risk Over-The-Counter in a one-period model. Novel to the literature, we focus on an incomplete set of possible risk redistributions. This keeps the set of feasible contracts simple. We consider catastrophe and longevity risk as two key examples. The reason is that the trading of these risks typically occurs Over-The-Counter, and that there are no given pricing functions. If the set of feasible strategies is unconstrained, we get that all Nash equilibria are such that no firm benefits from trading. A way to avoid this, is to restrict the strategy space a priori. In this way, a Nash equilibrium that is interesting for both firms may exist. The intervention of a regulator is possible by restricting the set of feasible strategies. For instance, a firm has to keep a deductible on its prior risk. We characterize optimal regulation by means of Nash bargaining solutions.

JEL-classification: C72, C78

Keywords: cooperative bargaining, Nash equilibria, non-cooperative bargaining, regulation

1 Introduction

This paper studies bilateral bargaining for optimal risk redistributions. Firms can benefit from sharing risk if their risks are not perfectly correlated. We present a model that can be used in several fields, but its application is prominent for insurance risk. We focus on bilateral trading of Over-The-Counter (OTC) contracts. The firms bargain for a fair risk redistribution of the risk in their liabilities. Here, prices of derivatives do not exist. In this paper, we determine the prices implicitly via a bargaining process. This bargaining process reflects each firm's alternatives to trade. When two counterparties meet, their bilateral relationship is strategic. Of key interest for a firm is how to determine their bargaining strategy.

Firms are often enforced to hedge their risk by a regulator. A worst-case scenario should have relatively low consequences for the firm. This makes holding risky portfolios undesirable.

*The author thanks Anja De Waegenaere, Yong Li, Henk Norde, anonymous referees, and participants of the 8th conference in Actuarial Science and Finance in Samos, 2014, and the Gerber-Shiu Workshop, Hong Kong, 2014, for useful comments.

†Email address: t.j.boonen@uva.nl, Amsterdam School of Economic, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

So, firms want to reinsure themselves for losses in the tail of their risk distribution. This means that a part of the losses in the worst-case scenarios is sold to a reinsurer. A reinsurer typically asks a high risk premium to hedge the risk, meaning that the firms pay more than the expected payment that is determined by the contract. Therefore, firms could trade Over-The-Counter with each other. In this way, there is an opportunity to redistribute a part of their risk without paying a high risk premium.

We consider catastrophe and longevity risk as two key examples. For both classes of insurance risk, firms cannot find an exchange market to buy a hedge against their exposure. Reinsurance contracts do exist, but the capacity of reinsurance is limited (see, e.g., Basel Committee on Banking Supervision, 2013, for longevity risk). Also, the pricing of CAT-bonds and longevity-linked securities is very debatable. For this reason, these contracts are mainly traded Over-The-Counter. In this paper, we allow firms to have heterogenous beliefs regarding the underlying probability distribution. This might increase the potential to benefit (see Boonen et al., 2012).

Cox and Lin (2007) suggest that pension funds and life insurers have a natural hedge in their systematic longevity risk exposure. This risk arises from the fact that the total population becomes older in a stochastic way. The liabilities of pension funds and life insurers are directly linked with longevity. For systematic risk, there is no diversification that weakens this risk, i.e., increasing the number of indemnity contracts is not sufficient. Blake et al. (2006) provide a detailed discussion on this. Exposure to systematic longevity risk can be rather substantial for pension funds as shown by e.g. Hári et al. (2008). Firms can find a counterparty for trading a part of their longevity risk. If people systematically become older than expected, life insurers benefit while pension funds lose and vice versa. Therefore, their risks are typically negatively correlated (see, e.g., Tsai et al., 2010; Wang et al., 2010), and, so, there might exist risk redistributions that are beneficial for both firms.

We focus on the question how firms come to an agreement in bargaining for risk redistributions. We focus on the strategic behavior of the firms. We parameterize the strategic behavior of firms to redistribute their own risk. Firms use a specific standardized instrument as, for instance, CAT bonds (e.g., Cummins, 2008) for catastrophe risk and q-forwards (Coughlan et al., 2007) for longevity risk. Alternatively, firms can design tailored instruments on the risk in their portfolio. As an example, we suggest that firms use the stop-loss rule, which is supported by e.g. Denuit and Vermandele (1998) and Goovaerts et al. (1990). This rule implies that the firm bears the risk up to a deductible. The rest is borne by the counterparty. Then, firms reinsure their worst-case stochastic losses.

Using an insurance principle, firms are simultaneously proposing a redistribution of their own risk to the counterparty. The risk will be redistributed if both firms decide to accept after having observed each other's strategy. Firms use an expected utility function to evaluate their future losses. We determine all Nash equilibria (Nash, 1950b) of this bargaining problem. Since there is complete information, one could expect that bargaining will result in a Pareto optimal outcome. However, we show that this does not need to be true. It is even the case that almost all Nash equilibria are such that the status quo will be retained.

We provide a cooperative bargaining model to characterize optimal regulation. The regulator is allowed to make binding agreements regarding the possible strategies of the firms. A solution concept selects a strategy, which is enforced by mandating a firm to keep the corresponding part of its prior risk. We characterize the Nash bargaining solution (Nash, 1950a).

There is also a stream in the literature that focus on non-cooperative sequential bargaining (Rubinstein, 1982). In an insurance-customer context as considered by Viaene et al. (2002)

and Huang et al. (2013), there is a natural order. In the first period, the insurance company makes an offer that involves charging the client a price for a coverage. The client then decides whether to accept the offer or not. If the client rejects, he/she can make a counter-proposal and so on. For bilateral bargaining for risk redistribution between firms, there is not a natural order of making a proposal. Also, as parties meet each other for OTC trading, the cost of waiting as in Rubinstein (1982), Viaene et al. (2002), and Huang et al. (2013) are negligible. Therefore, we assume that firms are participating in a simultaneous-move bargaining situation. Dutang et al. (2013) study competition for insurance companies by Nash equilibria. They focus on the competition for policyholders with other insurance companies. We, however, focus on an insurer that reinsures its risk in its portfolio with another insurer, which could be a reinsurer.

The remainder of this paper is organized as follows. In Section 2, we define the assumptions of the risk redistribution model. The Nash equilibria of this model are discussed in Section 3. In Section 4, we characterize optimal regulation. Section 5 concludes this paper with a remark to generalize the assumptions of the bargaining problem.

2 Description of the risk redistribution model

We consider a single-period economy. Throughout this paper, we impose the following assumptions.

- The state space, denoted by Ω, is discrete. Note that we still allow Ω to be infinite.
- Both firms use a von Neumann-Morgenstern expected utility function $u_i : \mathbb{R} \to \mathbb{R}$ to evaluate their risk, with $u_i'(\cdot) > 0$ and $u_i''(\cdot) < 0$ for all $i \in \{1, 2\}$. Firm i uses (subjective) probability measure $\mathbb{P}_i$ on the state space Ω. Since we consider a one-period model, discounting is redundant.
- Firm $i \in \{1, 2\}$ possesses the risk $X_i \in L^\infty(\Omega)$, that is X_i is measurable and bounded. It represents the future loss at a given future time.
- There is complete information about the risks, utility functions, and subjective probability measures.

We use a static bargaining problem. The risks are evaluated at a given future time period. If firms bargain for Over-The-Counter trades in insurance risk, a potential drawback of this approach is that firms aim to agree on a redistribution of their liability risk over the complete run-off. Considering instead a shorter period to realize the trade, however, allows for the possibility to negotiate a new contract in the future. For example, firms can re-evaluate their portfolio according to new data, new regulations, and a new liability portfolio due to attrition and new insurance contracts.

We allow the firms to use subjective probability measures; the subjective probability measure of firm $i \in \{1, 2\}$ is given by $\mathbb{P}_i$, where we assume that $\mathbb{P}_i(\omega) > 0$ for all $i \in \{1, 2\}$ and $\omega \in \Omega$. Different subjective probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ lead to asymmetry in the beliefs of the underlying probability distribution. So, firms are allowed to "agree to disagree" on the probability measure. Wilson (1968) and Riddell (1981) discuss the use of heterogeneous subjective probability measures. Heterogeneous beliefs regarding the underlying probability distribution are due to the fact that there might be little consensus regarding the underlying distribution of catastrophe risk (see, e.g., Kunreuther et al., 1995, for earthquake insurance) and longevity risk (see, e.g., Blake et al., 2006). In Section 5, we provide more general assumptions on the preference functions such that the results in this paper remain valid.

A way to redistribute risk is to add both risks and, then, redistribute this aggregated risk in a Pareto optimal way. Derived from Borch's theorem (Borch, 1962), Gerber and Pafumi (1998) characterize all Pareto optimal outcomes for a wide range of possible expected utility functions. Depending on the expected utility functions, there is a single parameter to bargain for. This parameter corresponds with a specific risk redistribution of the aggregate risk. It depends on individual risks only via their sum. Pooling the risk and then redistributing may lead to a complex reinsurance contract as it is written on the aggregate risk only. In contrast to this approach of Gerber and Pafumi (1998), we propose a structure in which we restrict the set of possible redistributions to account for the prior risks that firms possess. This contract is described by a parametric indemnity on the prior risk for every firm. A risk redistribution follows from parts of the prior risks that will be transferred to the other firm. We allow for a wide range of possible indemnity contracts.

We parameterize (and so restrict) the functional form of posterior risks. A strategy for a firm is given by a proposal how to redistribute its own risk. In return, the firm expects a proposal from the other firm. We assume that a strategy of a firm is represented by a real number that directly corresponds with a redistribution of its own risk. We assume that Firm 1 can choose a parameter c and Firm 2 can choose a parameter d. So, parameter $c \in \mathcal{C}$ indicates the part of X_1 that is offered to Firm 2, and parameter $d \in \mathcal{D}$ indicates the part of X_2 that is offered to Firm 1. The joint strategy space $\mathcal{C} \times \mathcal{D}$ will be specified in Sections 3 and 4. A larger value of c or d corresponds to a smaller part of the risk that is offered to the other firm. Before bargaining, the functional form of posterior risks is common knowledge. The posterior risks are given by $X_1^{\text{post}}(c,d), X_2^{\text{post}}(c,d) \in L^\infty(\Omega)$, where $(c,d) \in \mathbb{R}^2$, and are such that there is market clearing, i.e.,

$$X_1^{\text{post}}(c,d) + X_2^{\text{post}}(c,d) = X_1 + X_2, \tag{1}$$

for all $(c,d) \in \mathbb{R}^2$. We impose the following assumptions on the parametric form of posterior risks.

- There exists a strategy profile $(\overline{c}, \overline{d}) \in \mathbb{R}^2_+$ such that $X_1^{\text{post}}(\overline{c}, \overline{d}) = X_1$ and $X_2^{\text{post}}(\overline{c}, \overline{d}) = X_2$.
- The functions $X_1^{\text{post}}(\cdot,\cdot)(\omega)$ and $X_2^{\text{post}}(\cdot,\cdot)(\omega)$ are locally Lipschitz continuous, uniformly for all $\omega \in \Omega$.[1]
- For any given $(\hat{c}, \hat{d}) \leq (\overline{c}, \overline{d})$, $X_1^{\text{post}}(\cdot, \hat{d})(\omega)$ and $X_2^{\text{post}}(\hat{c}, \cdot)(\omega)$ are increasing for all $\omega \in \Omega$ and strictly increasing for some $\omega \in \Omega$ on $(-\infty, \overline{c}]$ resp. $(-\infty, \overline{d}]$.
- For any given $(\hat{c}, \hat{d}) \in \mathbb{R}^2$, there exist states $\omega, \omega' \in \Omega$ such that $\lim_{c \to -\infty} X_1^{\text{post}}(c, \hat{d})(\omega) = \lim_{d \to -\infty} X_2^{\text{post}}(\hat{c}, d)(\omega') = -\infty$.

For vectors $a, b \in \mathbb{R}^2$, we define $a \leq b$ as an element-wise inequality, i.e., $a_1 \leq b_1$ and $a_2 \leq b_2$. Moreover, $a < b$ implies $a_1 < b_1$ and $a_2 < b_2$, and $a \lneqq b$ implies $a_1 \leq b_1$ and $a_2 \leq b_2$ where at least one inequality is strict. Moreover, we define $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x \geq 0\}$ and $\mathbb{R}^2_{++} = \{x \in \mathbb{R}^2 : x > 0\}$.

A market is called complete if every contract on the state space Ω can be traded. In this paper, we do not allow any possible reinsurance contract. The parametrization of the posterior risks generates market incompleteness. However, (re)insurance markets are typically incomplete, as argued by Schlesinger and Doherty (1985).

[1] Locally Lipschitz continuity of $X_1^{\text{post}}(\cdot,\cdot)(\omega)$, uniformly for all $\omega \in \Omega$, is defined as follows. For all $(c,d) \in \mathbb{R}^2$, there exists a neighborhood U of (c,d) and an $m < \infty$ such that $|X_1^{\text{post}}(c,d)(\omega) - X_1^{\text{post}}(c',d')(\omega)| < m\|(c,d) - (c',d')\|$ for all $(c',d') \in U$ and for all $\omega \in \Omega$.

The expected utility functions are given by:

$$U_1(c,d) = E^{\mathbb{P}_1}[u_1(-X_1^{\text{post}}(c,d))], \tag{2}$$

and

$$U_2(c,d) = E^{\mathbb{P}_2}[u_2(-X_2^{\text{post}}(c,d))], \tag{3}$$

for all $(c,d) \in (-\infty,\overline{c}] \times (-\infty,\overline{d}]$, where $E^{\mathbb{P}_i}$ denotes the expectation under the probability distribution $\mathbb{P}_i$. So, the utility level $U_i(\overline{c},\overline{d})$ corresponds to the prior expected utility level for firm i. We show in the appendix (Lemma B.1 and Lemma B.2) that the expected utility functions U_1 and U_2 are strictly monotonic and continuous.

The parametric form to redistribute risk can be formulated via specific investment instruments (e.g., CAT-bonds for catastrophe risk and q-forwards for longevity risk). In the following example, we provide three key ways to construct tailored posterior risks $X_1^{\text{post}}(c,d)$ and $X_2^{\text{post}}(c,d)$ that satisfy the properties stated above.

Example 2.1 In this example, we assume that for all $i \in \{1,2\}$ there exists a state $\omega \in \Omega$ such that $X_i(\omega) > 0$.

A commonly used principle in insurance is the stop-loss principle. Stop-loss reinsurance is popular in practice and is supported by, e.g., Denuit and Vermandele (1998) and Goovaerts et al. (1990). For the strategy profile $(c,d) \in \mathbb{R}^2$, we have that Firm 1 keeps $\min\{X_1, c\}$ and offers $\max\{X_1 - c, 0\}$ to Firm 2, and Firm 2 keeps $\min\{X_2, d\}$ and offers $\max\{X_2 - d, 0\}$ to Firm 1. Thus, the posterior risks are given by

$$X_1^{\text{post}}(c,d) = \min\{X_1, c\} + \max\{X_2 - d, 0\},$$

and

$$X_2^{\text{post}}(c,d) = \min\{X_2, d\} + \max\{X_1 - c, 0\},$$

for all $(c,d) \in \mathbb{R}^2$. Locally Lipschitz continuity of $X_1^{\text{post}}(\cdot,d)$ and $X_2^{\text{post}}(c,\cdot)$ is shown in Appendix A. It is easy to show that the stop-loss rule satisfies all properties that are introduced for the posterior risks. If $c > \sup X_1$, the proposal of Firm 1 is always to keep its risk and, so, this strategy is identical to $\overline{c} = \sup X_1$. Similarly, we have $\overline{d} = \sup X_2$. Moreover, as retaining less than the original risk is generally not allowed, the strategy profile $(0,0)$ is a natural minimum of the joint strategy space.

It is also possible that firms use a proportional rule. This rule states that the firms keep a given percentage of their risk. Lampaert and Walhin (2005) show that stop-loss treaties in reinsurance are often hard to price and might lead to moral hazard behavior from the insurer. Therefore, these authors support the proportional treaty.

For every strategy profile $(c,d) \in \mathbb{R}^2$, we have that Firm 1 keeps $\min\{X_1, 0\} + c\max\{X_1, 0\}$ and offers $(1-c)\max\{X_1, 0\}$ to Firm 2, and Firm 2 keeps $\min\{X_2, 0\} + d\max\{X_2, 0\}$ and offers $(1-d)\max\{X_2, 0\}$ to Firm 1. So, the posterior risks are given by

$$X_1^{\text{post}}(c,d) = \min\{X_1, 0\} + c\max\{X_1, 0\} + (1-d)\max\{X_2, 0\},$$

and

$$X_2^{\text{post}}(c,d) = \min\{X_2, 0\} + (1-c)\max\{X_1, 0\} + d\max\{X_2, 0\},$$

for all $(c,d) \in \mathbb{R}^2$. Negative losses (i.e., gains) are not traded since, otherwise, $X_1^{\text{post}}(\cdot,d)$ and $X_2^{\text{post}}(c,\cdot)$ may not be increasing. Locally Lipschitz continuity is again shown in Appendix A. It is easy to show that the proportional rule satisfies all properties that are introduced above.

Retaining one's own risk corresponds with the strategy profile $(\overline{c}, \overline{d}) = (1, 1)$. The strategy profile $(0, 0)$ is a natural minimum of the joint strategy space.

Alternatively, firms can also use a mixture of parametric forms to redistribute their own risk. Moreover, we allow firms to use different parametric forms. For instance, we allow that one firm uses a stop-loss rule and the other firm uses a proportional rule. Alternatively, one can let the parametric form of one firm be a risk-free payment, i.e., $X_1^{\text{post}}(c, \overline{d}) = X_1 - (\overline{c} - c)$ for all $c \in \mathbb{R}$. Then, Firm 1 is buying risk from Firm 2. If risks X_1 and X_2 are independent, it is optimal for Firm 2 to use a stop-loss parametric form (Goovaerts et al., 1990). Then, the corresponding bargaining problem is in line with Kihlstrom and Roth (1982), Schlesinger (1984), and Quiggin and Chambers (2009), who all study cooperative bargaining for an insurance contract between a client and an insurer. We can normalize $\overline{c} \geq 0$ in any way. $\triangledown$

3 Non-cooperative bargaining

In this section, we show the Nash equilibria of the risk redistribution bargaining problem. We fix the joint strategy space $\mathcal{C} \times \mathcal{D} \subset (-\infty, \overline{c}] \times (-\infty, \overline{d}]$ with $(\overline{c}, \overline{d}) \in \mathcal{C} \times \mathcal{D}$. So, for a given parametric form of the posterior risks, we fix the upper bound of the joint strategy space $\mathcal{C} \times \mathcal{D}$. For a firm, this upper bound strategy corresponds with proposing to keep its own risk. The strategy space might be based on requirements of a regulator, which we discuss in Section 4.

In this paper, both firms agree on a risk redistribution after non-cooperative bargaining. When firms bargain, they maximize their own expected utility, and, so, a Pareto optimal risk redistribution does not need to be agreed upon (see, e.g., the Prisoner's Dilemma). Firms select a risk redistribution based on a two-stage bargaining process. In the first stage, firms make simultaneously a proposal for a redistribution of their own risk. Then, the firms observe each other's bid and decide individually whether the deal is accepted. A deal goes through only if both firms accept. A firm accepts the deal if its expected utility weakly increases. If one firm rejects, the deal is canceled and both firms keep their prior risks. So, the objective of firm $i \in \{1, 2\}$ is to maximize $U_i(c, d)$ subject to the constraints $U_1(c, d) \geq U_1(\overline{c}, \overline{d})$ and $U_2(c, d) \geq U_2(\overline{c}, \overline{d})$. If a constraint is violated, one firm will cancel the deal in the second stage. This leads to the following ex post expected utility functions:

$$\widetilde{U_1}(c, d) = \begin{cases} U_1(c, d) & \text{if } U_1(c, d) \geq U_1(\overline{c}, \overline{d}) \text{ and } U_2(c, d) \geq U_2(\overline{c}, \overline{d}), \\ U_1(\overline{c}, \overline{d}) & \text{otherwise,} \end{cases} \tag{4}$$

and

$$\widetilde{U_2}(c, d) = \begin{cases} U_2(c, d) & \text{if } U_2(c, d) \geq U_2(\overline{c}, \overline{d}) \text{ and } U_1(c, d) \geq U_1(\overline{c}, \overline{d}), \\ U_2(\overline{c}, \overline{d}) & \text{otherwise,} \end{cases} \tag{5}$$

for all $(c, d) \in \mathcal{C} \times \mathcal{D}$.

The best response correspondence for Firm 1, denoted by $br_1 : \mathcal{D} \to \mathcal{C}$, is given by

$$br_1(d) = \operatorname*{argmax}_{c \in \mathcal{C}} \widetilde{U_1}(c, d), \tag{6}$$

for all $d \in \mathcal{D}$. For Firm 2, we have

$$br_2(c) = \operatorname*{argmax}_{d \in \mathcal{D}} \widetilde{U_2}(c, d), \tag{7}$$

for all $c \in \mathcal{C}$. Note that br_1 and br_2 can be empty.

In a Nash equilibrium, both firms have selected one of the best responses to the other firm's strategy (Nash, 1950b). So, $(\hat{c}, \hat{d}) \in \mathcal{C} \times \mathcal{D}$ is a Nash equilibrium if and only if it holds that $\hat{c} \in br_1(\hat{d})$ and $\hat{d} \in br_2(\hat{c})$, i.e., the set of Nash equilibria is given by:

$$NE = \{(c, d) \in \mathcal{C} \times \mathcal{D} | c \in br_1(d), d \in br_2(c)\}. \tag{8}$$

In order to show some Nash equilibria, we first define the set of strategies of a firm for which there is no possibility to strictly benefit for the counterparty. We define the set $N^{\mathcal{C}} \subseteq \mathcal{C}$ as

$$N^{\mathcal{C}} = \{c \in \mathcal{C} | \forall d \in \mathcal{D} : \widetilde{U_2}(c, d) = U_2(\overline{c}, \overline{d})\}. \tag{9}$$

Similarly, we define the set $N^{\mathcal{D}} \subseteq \mathcal{D}$ as

$$N^{\mathcal{D}} = \{d \in \mathcal{D} | \forall c \in \mathcal{C} : \widetilde{U_1}(c, d) = U_1(\overline{c}, \overline{d})\}. \tag{10}$$

For all $c \in N^{\mathcal{C}}$, there is no response $d \in \mathcal{D}$ for Firm 2 that yields a strict improvement for itself. This leads to the following result.

Proposition 3.1 *Every strategy profile in $N^{\mathcal{C}} \times N^{\mathcal{D}}$ is a Nash equilibrium.*

Note that for every $(\hat{c}, \hat{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$, we have that $\widetilde{U_1}(\hat{c}, \hat{d}) = U_1(\overline{c}, \overline{d})$ and $\widetilde{U_2}(\hat{c}, \hat{d}) = U_2(\overline{c}, \overline{d})$. So, all strategy profiles in $N^{\mathcal{C}} \times N^{\mathcal{D}}$ are of no interest for the firms. Nevertheless, Proposition 3.1 states that all strategy profiles in $N^{\mathcal{C}} \times N^{\mathcal{D}}$ are Nash equilibria. In the proof of the next corollary, we show that $(\overline{c}, \overline{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$ and, so, the subsequent result follows from Proposition 3.1.

Corollary 3.2 *The strategy profile $(\overline{c}, \overline{d})$ is a Nash equilibrium.*

This corollary states that keeping your risk is a Nash equilibrium.

Theorem 3.3 *If the strategy space $\mathcal{C}$ or $\mathcal{D}$ has no minimum, the set of Nash equilibria is given by $NE = N^{\mathcal{C}} \times N^{\mathcal{D}}$.*

Theorem 3.3 states that it is necessary to agree a priori on that the strategy spaces $\mathcal{C}$ and $\mathcal{D}$ have no minima. Otherwise, both firms do not benefit in any Nash equilibrium. For instance, if the firms use the joint strategy space $\mathcal{C} \times \mathcal{D} = (-\infty, \overline{c}] \times (-\infty, \overline{d}]$, every Nash equilibrium is in $N^{\mathcal{C}} \times N^{\mathcal{D}}$. The Nash equilibria are in line with the Prisoner's Dilemma, in the sense that both firms do not benefit optimally. By maximizing their own expected utility, the firms are likely to retain the status quo. This is in line with the Bertrand paradox (Bertrand, 1883) in which firms set the price for a product equal to the marginal cost in the only Nash equilibrium. So, both firms do not benefit in the market in order to obtain the whole market share.

In the rest of this paper, we assume that $\mathcal{C}$ and $\mathcal{D}$ are closed and bounded intervals, hence have a minimum. Then, we can define $\underline{c} = \min \mathcal{C}$ and $\underline{d} = \min \mathcal{D}$. Since $\mathcal{C}$ and $\mathcal{D}$ are compact, we have that br_1 and br_2 map every element of their domain to a non-empty subset. We will show that it might be possible for both firms to strictly benefit in a Nash equilibrium if the lower bounds $\underline{c}$ and $\underline{d}$ are wisely chosen.

Before we provide the best response correspondences, we need the following result.

Lemma 3.4 *For all $d \in \mathcal{D}$, there exists a unique $c^* \in (-\infty, \overline{c}]$ such that $U_2(c^*, d) = U_2(\overline{c}, \overline{d})$. Likewise, for all $c \in \mathcal{C}$, there exists a unique $d^* \in (-\infty, \overline{d}]$ such that $U_1(c, d^*) = U_1(\overline{c}, \overline{d})$.*

According to Lemma 3.4, we are allowed to define the functions $c^* : \mathcal{D} \to (-\infty, \overline{c}]$ and $d^* : \mathcal{C} \to (-\infty, \overline{d}]$ as the functions satisfying $U_2(c^*(d), d) = U_2(\overline{c}, \overline{d})$ for all $d \in \mathcal{D}$, and $U_1(c, d^*(c)) = U_1(\overline{c}, \overline{d})$ for all $c \in \mathcal{C}$. The functions c^* and d^* are the responses of a firm such that the other firm is indifferent compared to the status quo. Note that possibly $c^*(d) \notin \mathcal{C}$. We provide the best response correspondences in the following proposition.

Proposition 3.5 *The best response for Firm 1 to a strategy $d \in \mathcal{D}$ is given by*

$$br_1(d) = \begin{cases} \mathcal{C} & \text{if } d \in N^{\mathcal{D}}, \\ \{\max\{\underline{c}, c^*(d)\}\} & \text{if } d \in (N^{\mathcal{D}})^c, \end{cases}$$

and the best response for Firm 2 to a strategy $c \in \mathcal{C}$ is given by

$$br_2(c) = \begin{cases} \mathcal{D} & \text{if } c \in N^{\mathcal{C}}, \\ \{\max\{\underline{d}, d^*(c)\}\} & \text{if } c \in (N^{\mathcal{C}})^c, \end{cases}$$

where br_1 and br_2 are defined in (6) *and* (7)*, $N^{\mathcal{C}}$ and $N^{\mathcal{D}}$ are defined in* (9) *and* (10)*, and $(N^{\mathcal{C}})^c = \mathcal{C} \backslash N^{\mathcal{C}}$ and $(N^{\mathcal{D}})^c = \mathcal{D} \backslash N^{\mathcal{D}}$.*

If $br_1(d)$ and $br_2(c)$ are singleton-valued, we write them as functions. So, we write $br_1(d) = \max\{\underline{c}, c^*(d)\}$ for $d \in (N^{\mathcal{D}})^c$ and $br_2(c) = \max\{\underline{d}, d^*(c)\}$ for $c \in (N^{\mathcal{C}})^c$.

Proposition 3.6 *The correspondence br_1 is continuous and increasing on $\left(N^{\mathcal{D}}\right)^c$ and the correspondence br_2 is continuous and increasing on $\left(N^{\mathcal{C}}\right)^c$.*

The set of strategies where both firms strictly benefit is given by:

$$\{(c, d) \in \mathcal{C} \times \mathcal{D} | \widetilde{U_1}(c, d) > U_1(\overline{c}, \overline{d}), \widetilde{U_2}(c, d) > U_2(\overline{c}, \overline{d})\}. \tag{11}$$

Next, we present an important result for the Nash equilibria of the risk redistribution bargaining problem.

Theorem 3.7 *There exists at most one Nash equilibrium $(\hat{c}, \hat{d}) \in NE$ such that $\widetilde{U_1}(\hat{c}, \hat{d}) > U_1(\overline{c}, \overline{d})$. If it exists, it satisfies $\hat{d} = \underline{d}$. Likewise, there exists at most one Nash equilibrium $(\hat{c}, \hat{d}) \in NE$ such that $\widetilde{U_2}(\hat{c}, \hat{d}) > U_2(\overline{c}, \overline{d})$. If it exists, it satisfies $\hat{c} = \underline{c}$.*

Note that if Firm 2 uses a stop-loss or proportional rule as parametric form to redistribute its risk and if $\underline{d} = 0$, the strategy $\hat{d} = \underline{d} = 0$ implies that Firm 2 proposes to swap all positive losses of its risk. This strategy might, however, lead to the only Nash equilibrium in which at least one firm strictly benefits. We will show in Example 3.11 a risk redistribution bargaining problem in which there exists a Nash equilibrium in which one firm strictly benefits.

Next, we define the function $f : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ as follows:

$$f(c, d) = \begin{cases} (c, d) & \text{if } (c, d) \in ((N^{\mathcal{C}})^c \times (N^{\mathcal{D}})^c) \cup (N^{\mathcal{C}} \times N^{\mathcal{D}}), \\ (c, d^*(c)) & \text{if } (c, d) \in (N^{\mathcal{C}})^c \times N^{\mathcal{D}}, \\ (c^*(d), d) & \text{if } (c, d) \in N^{\mathcal{C}} \times (N^{\mathcal{D}})^c, \end{cases} \tag{12}$$

for all $(c, d) \in \mathcal{C} \times \mathcal{D}$. The following result is deduced from Theorem 3.7.

Corollary 3.8 *There is at most one Nash equilibrium in which at least one firm strictly benefits. If existent, this Nash equilibrium is given by strategy profile $f(\underline{c}, \underline{d})$, which is defined in* (12).

If both firms strictly benefit in strategy profile $(\underline{c}, \underline{d})$, then $f(\underline{c}, \underline{d}) = (\underline{c}, \underline{d}) \in (br_1(\underline{d}), br_2(\underline{c}))$. We next show that $f(\underline{c}, \underline{d})$ is always a Nash equilibrium. More general, if $(\underline{c}, \underline{d}) \notin (N^{\mathcal{C}})^c \times (N^{\mathcal{D}})^c$, the strategy profile $f(\underline{c}, \underline{d})$ is the unique Nash equilibrium in which at least one firm strictly benefits from OTC trading as shown in the following theorem.

Theorem 3.9 *The set of Nash equilibria is given by*

$$NE = (N^{\mathcal{C}} \times N^{\mathcal{D}}) \cup \{f(\underline{c}, \underline{d})\}.$$

From (12), we directly get $f(c, d) = (\overline{c}, \overline{d})$ if and only if $(c, d) = (\overline{c}, \overline{d})$. From this, Corollary 3.2 and Theorem 3.9, it follows directly that the set NE is singleton-valued if and only if the strategy space $\mathcal{C} \times \mathcal{D}$ is singleton-valued. Generally, the set of Nash equilibria NE can be large. Therefore, we consider a popular refinement of Nash equilibria. If there exist multiple Nash equilibria, firms prefer the Nash equilibria that are Pareto optimal. The notion of Pareto optimality is used in a constrained sense, i.e., Nash equilibria are required to be Pareto optimal among all alternatives that are also Nash equilibria. The set of all Pareto optimal Nash equilibria (Kurz and Hart, 1982) is defined as:

$$PONE = \{(c, d) \in NE | \neg\exists (c', d') \in NE : (U_i(c', d'))_{i\in\{1,2\}} \gneqq U_i(c, d))_{i\in\{1,2\}}\}, \qquad (13)$$

where the set NE is defined in (8). From Corollary 3.8, we get that if there exists a Nash equilibrium in which at least one firm strictly benefits, there is a unique Pareto optimal Nash equilibrium, given by $f(\underline{c}, \underline{d})$. The following corollary follows directly from this and Theorem 3.9.

Corollary 3.10 *The set $PONE$, defined in* (13)*, is given by*

$$PONE = \begin{cases} \{f(\underline{c}, \underline{d})\} & \text{if } (\underline{c}, \underline{d}) \notin N^{\mathcal{C}} \times N^{\mathcal{D}}, \\ N^{\mathcal{C}} \times N^{\mathcal{D}} & \text{otherwise.} \end{cases}$$

Note that if $PONE = N^{\mathcal{C}} \times N^{\mathcal{D}}$, there are no welfare gains in any Nash equilibrium.

Remark that the results in Proposition 3.5, Theorem 3.7, Corollary 3.8, Theorem 3.9, and Corollary 3.10 are sensitive to the assumption of convexity of the joint strategy space $\mathcal{C} \times \mathcal{D}$. For instance, if $\mathcal{C} \times \mathcal{D}$ is finite, it may be possible to find other Nash equilibria in which at least one firm strictly benefits.

The following example illustrates the set $PONE$ in a setting with two catastrophe insurers.

Example 3.11 In this example, we illustrate the risk redistribution model with an application in catastrophe reinsurance. Suppose there are two types of catastrophes, Type 1 and Type 2. Firm 1 sells insurance contracts that depend on catastrophe Type 1, and Firm 2 sells insurance contracts that depend on catastrophe Type 2. If catastrophe Type 1 occurs, Firm 1 bears a cost of 1 unit. Firm 1 believes that this event occurs with probability 2% and Firm 2 believes that this event occurs with probability 1%. Firm 2 bears a cost of 1 unit if catastrophe Type 2 occurs. Moreover, Firm 1 believes that this event occurs with probability 0.5% and Firm 2 believes that this event occurs with probability 1%. The events Type 1 and Type 2 are independent. Let the firms use an exponential utility function with constant absolute risk aversion parameter $\lambda_i > 0$, i.e., for all $i \in \{1, 2\}$, we have $u_i(x) = -\exp(-\lambda_i x)$ for all $x \in \mathbb{R}$. Let $\lambda_1 = \lambda_2 = 1$. Since we can formulate the state space Ω as a set with four states of the world, we can write the probability measures and the risks as vectors:

$$\mathbb{P}_1 = \begin{bmatrix} 0.9751 \\ 0.0049 \\ 0.0199 \\ 0.0001 \end{bmatrix}, \mathbb{P}_2 = \begin{bmatrix} 0.9801 \\ 0.0099 \\ 0.0099 \\ 0.0001 \end{bmatrix}, X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We also assume that the firms use a stop-loss function to redistribute their own risk and, for now, $\underline{c} = \underline{d} = 0$. As shown in Example 2.1, we have $\overline{c} = \overline{d} = 1$.

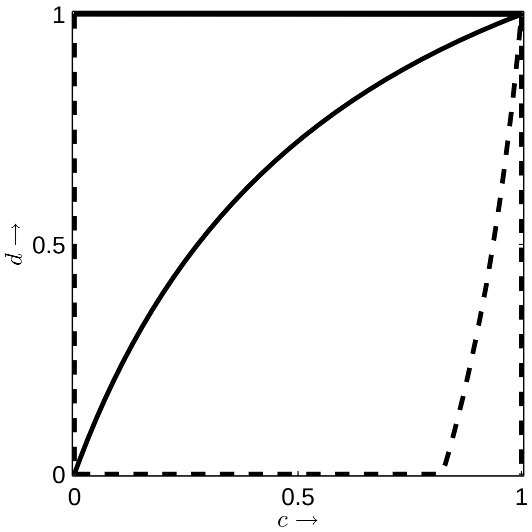

Figure 1: Best response correspondences for Example 3.11. The solid line is br_1 and the dashed line is br_2, which are derived in Proposition 3.5. We get $\{0, 1\} \in N^{\mathcal{C}}$ and $\{1\} \in N^{\mathcal{D}}$ and, so, $br_2(c) = [0, 1]$ for $c \in \{0, 1\}$ and $br_1(1) = [0, 1]$. Moreover, br_1 is a correspondence on the domain $\mathcal{D}$, which is on the y-axis, and that br_2 is a correspondence on the domain $\mathcal{C}$, which is on the x-axis.

Figure 1 displays the best response correspondences. We obtain from this figure that the set of Nash equilibria is given by $NE = \{(0, 0), (0, 1), (1, 1)\}$. Firms are not trading in all except one Nash equilibria. Only in the equilibrium strategy profile $f(0, 0) = (c^*(0), 0) = (0, 0)$, there is a risk redistribution. In this Nash equilibrium, Firm 2 is indifferent compared to the status quo and Firm 1 strictly benefits. So, we get $PONE = \{(0, 0)\}$. Remark that in this equilibrium both firms swap their risks. $\triangledown$

If the firms are forced to obey a given lower bound of the joint strategy space $\mathcal{C} \times \mathcal{D}$, firms can benefit altogether. For instance, in Example 3.11, it follows from Figure 1 that $c^*(0.5) < 0.5$, $d^*(0.5) < 0.5$. From this and strict monotonicity of U_1 and U_2 (see Lemma B.1), it follows that it is beneficial for both firms to set the lower bounds at $\underline{c} = \underline{d} = 0.5$. Then, $\mathcal{C} = \mathcal{D} = [0.5, 1]$, and the strategy profile $f(0.5, 0.5) = (0.5, 0.5)$ is the unique Pareto optimal Nash equilibrium. Characterizing the "optimal" values for $\underline{c}$ and $\underline{d}$ is non-trivial. We discuss this problem in Section 4.

4 The role of the regulator

In this section, we study optimal regulation. A regulator has the point of view of a social planner, aiming to optimize welfare of both firms. It can enforce an attractive Pareto optimal Nash equilibrium by restricting the joint strategy space. In the previous sections we showed that myopically maximizing own utility on the joint strategy space $(-\infty, \overline{c}] \times (-\infty, \overline{d}]$ leads to undesirable Nash equilibria, since firms are trying to trump their opponent. We assume that a regulator can enforce the lower bounds of the strategy spaces $\mathcal{C}$ and $\mathcal{D}$. Firms can be enforced to have a minimal ownership of their own insurance contracts. This prevents moral hazard behavior of a firm. The regulator uses a cooperative bargaining model[2] to determine optimal lower bounds of the strategy spaces.

We assume that the set of possible lower bounds is compact and convex. Without loss of generality, we let the domain of possible lower bounds $(\underline{c}, \underline{d})$ be given by the space $[0, \overline{c}] \times [0, \overline{d}]$. If the firms use a stop-loss or a proportional rule to redistribute their risk, it seems natural to assume that $\underline{c}, \underline{d} \geq 0$ (see Example 2.1).[3]

For every strategy profile $(c, d) \leq (\overline{c}, \overline{d})$, we define

$$\Delta U_i(c, d) = U_i(c, d) - U_i(\overline{c}, \overline{d}), \tag{14}$$

as the excess expected utility for firm $i \in \{1, 2\}$. The regulator enforces the vector of lower bounds $(\underline{c}, \underline{d}) \in [0, \overline{c}] \times [0, \overline{d}]$ such that $\Delta U(\underline{c}, \underline{d}) > 0$. Then, we get from (12) and Corollary 3.10 that

$$PONE = \{f(\underline{c}, \underline{d})\} = \{(\underline{c}, \underline{d})\}.$$

We define the admissible area $\mathcal{A}$ of joint excess expected utility levels by:

$$\mathcal{A} = \left\{ a \in \mathbb{R}^2 \,\middle|\, \exists (c, d) \in [0, \overline{c}] \times [0, \overline{d}] : a \leq \Delta U(c, d) \right\}. \tag{15}$$

In cooperative bargaining, the disagreement point is the vector of utility levels attained when there is no agreement. As normalization, we look at excess expected utilities and, so, the disagreement point is given by the vector $\Delta U(\overline{c}, \overline{d}) = (0, 0)$.

If we compare this with the bargaining set of Nash (1950a), we differ by considering an incomplete market for possible risk redistributions. In a complete market, every market clearing risk redistribution can be obtained (also called an Arrow-Debreu market). In this way, the admissible area of excess expected utility levels would only depend on the individual risks $X_i, i \in \{1, 2\}$ via their sum $X_1 + X_2$. This leads to a contract written on realizations of the risk of a firm and its counterparty jointly (cf. Borch, 1962; Wilson, 1968; Gerber and Pafumi, 1998). The set $\mathcal{A}$ is a subset of the bargaining set defined by Boonen et al. (2012), in which any market clearing risk redistribution is allowed.

Next, we state a popular definition of a cooperative bargaining problem (see, e.g., Van Damme, 1986).

Definition 4.1 *A cooperative bargaining problem is a set $A \subset \mathbb{R}^2$ that satisfies the following conditions:*

(i) A is convex and comprehensive[4]*;*

(ii) $A \cap \mathbb{R}^2_+$ is non-empty and compact.

The class of cooperative bargaining problems is denoted by Σ.

[2]Cooperative bargaining models were first introduced by Nash (1950a) for the case where two agents bargain for a redistribution of goods, using utility levels.

[3]For the stop-loss and proportional parametric form setting the lower bounds at zero implies $X_1^{\text{post}}(0, \overline{d}) = \min\{X_1, 0\}$ and $X_2^{\text{post}}(\overline{c}, 0) = \min\{X_2, 0\}$. Remark that we are still allowed to translate the parametric form in any way; for the stop-loss parametric form we could for example let: $X_1^{\text{post}}(c, \overline{d}) = \min\{X_1, c - \delta\}$ for any $\delta \in \mathbb{R}$ and for all $c \geq 0$.

[4]A set A is called comprehensive if $x \in A$ implies $y \in A$ for all $y \leq x$.

In a cooperative bargaining problem, the aim is to determine an element of the set $\mathcal{A}$, defined in (15), that is perceived as "fair" by both firms.

Proposition 4.2 *The set $\mathcal{A}$ is convex if both firms use the proportional rule as parametric form to redistribute their risk, or if one firm uses the proportional rule and one firm a risk-free payment.*

Also if firms use standardized investment products with non-negative pay-off in all states, we can show that the set $\mathcal{A}$ is convex. For the stop-loss parametric form, it is possible to construct an example where the set $\mathcal{A}$ is not convex. For now, we assume that the set $\mathcal{A}$ is convex.

Proposition 4.3 *If the set $\mathcal{A}$ is convex, it is a cooperative bargaining problem.*

If the strategy profile $(0,0)$ is Pareto optimal, the firms cannot benefit altogether. Then, no trade is Pareto optimal. In the sequel, we exclude the trivial case where the element $(0,0)$ is Pareto optimal.

Proposition 4.4 *If the element $(0,0)$ of the cooperative bargaining problem $\mathcal{A}$ is Pareto dominated, there exists an $a \in \mathcal{A}$ such that $a > 0$.*

A bargaining solution Φ maps every cooperative bargaining problem in Σ to an element of A, i.e., it is such that $\Phi(A) \in A$ for all $A \in \Sigma$. According to Proposition 4.4, there exists a bargaining solution in which both firms strictly benefit.

Definition 4.5 *The Nash bargaining solution for risk redistribution bargaining problems is given by*

$$NB(\mathcal{A}) = \underset{x \in \mathcal{A} \cap \mathbb{R}^2_+}{\operatorname{argmax}}\, x_1 x_2,$$

where the bargaining problem $\mathcal{A}$ is defined in (15).

Singleton-valuedness of the Nash bargaining solution is guaranteed by the fact that the set $\mathcal{A} \cap \mathbb{R}^2_+$ is closed and convex (Nash, 1950a). Since the Nash bargaining solution is Pareto optimal, there exists a strategy profile $(\hat{c}, \hat{d}) \in [0, \overline{c}] \times [0, \overline{d}]$ such that $\Delta U(\hat{c}, \hat{d}) = NB(\mathcal{A})$, where ΔU is defined in (14). This profile is not necessarily unique. Since $\Delta U(\hat{c}, \hat{d}) > 0$ (see Proposition 4.4), the strategy profile $f(\hat{c}, \hat{d}) = (\hat{c}, \hat{d})$ is the unique Pareto optimal Nash equilibrium if the regulator imposes the lower bounds $\underline{c} = \hat{c}$ and $\underline{d} = \hat{d}$ (see Theorem 3.9). Hence, we only consider solutions in the set $\mathcal{A} \cap \mathbb{R}^2_{++}$, which is non-empty for all relevant cases (see Proposition 4.4).

We support the Nash bargaining solution based on the following four properties for a bargaining solution Φ:

- Invariance with respect to positive affine transformations (INV): for any pair $A, \widetilde{A} \in \Sigma$, if there is a strictly increasing linear function h such that $\widetilde{A} = h(A)$, then $\Phi(\widetilde{A}) = h(\Phi(A))$.
- Strict individual rationality (SIR): for any $A \in \Sigma$, $\Phi(A) > 0$.
- Independence of irrelevant alternatives (IIA): for any pair $A, \widetilde{A} \in \Sigma$ such that $A \subset \widetilde{A}$, if $\Phi(\widetilde{A}) \in A$, then $\Phi(A) = \Phi(\widetilde{A})$.
- Symmetry (SYM): let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (y, x)$. Then, $T(\Phi(A)) = \Phi(T(A))$ for every $A \in \Sigma$.

The property (INV) ensures that if utility functions yield exactly the same preferences due to affine transformations of the utility function, the bargaining rule is the same for both utility functions. The property (SIR) ensures that both firms are better off than when thy do not trade. The property (IIA) resembles a gradual elimination of unacceptable outcomes. Eliminated outcomes in the cooperative bargaining problem $\mathcal{A}$ have no effect on the bargaining solution. The property (SYM) ensures that if firms are indistinguishable in a certain cooperative bargaining problem, the bargaining solution should not discriminate between them. For an extensive discussion of the four properties, we refer to Osborne and Rubinstein (1994). The following result is shown by Nash (1950a).

Theorem 4.6 *There exists a unique bargaining solution Φ satisfying the properties (INV), (SIR), (IIA), and (SYM). This rule is given by $\Phi(A) = NB(A)$ for all $A \in \Sigma$.*

Note that we only need convexity of the set $\mathcal{A} \cap \mathbb{R}^2_+$. If the set $\mathcal{A} \cap \mathbb{R}^2_+$ is not convex, there exists a popular generalization of the Nash bargaining solution characterized by Conley and Wilkie (1996). For a discussion of this generalized Nash bargaining solution in insurance bargaining, we refer to Li et al. (2013).

Instead of choosing the lower bounds of the joint strategy space wisely, Rubinstein (1982) designs a bilateral bargaining process in which if firms are perfectly patient, the unique Nash equilibrium converges to the Nash bargaining solution. Moreover, Van Damme (1986) shows that the Nash bargaining solution constitutes the unique equilibrium if two firms have different opinions about the appropriate solution concept. Alternatively, in line with, e.g., Kihlstrom and Roth (1982), Schlesinger (1984), Aase (2009), Quiggin and Chambers (2009), Boonen et al. (2012), Banerjee et al. (2014), and Zhou et al. (2015), we could assume that firms behave cooperatively and, therefore, select the Nash bargaining solution based on abovementioned properties. In this paper, we study non-cooperative bargaining of firms that leads to a cooperative bargaining solution determined by the regulator.

Example 4.7 In this example, we return to the risk redistribution bargaining problem described in Example 3.11. We display the set $\mathcal{A}$, defined in (15), in Figure 2. We get that the set $\mathcal{A}$ is convex and, thus, a cooperative bargaining problem. We obtain that the Nash bargaining solution corresponds uniquely with strategy profile $(\underline{c}, \underline{d}) \approx (0.41, 0)$. So, the regulator should enforce Firm 1 to keep a deductible of 0.41 units. From this it follows that if the joint strategy space is given by $\mathcal{C} \times \mathcal{D} = [0.41, 1] \times [0, 1]$, the posterior risks corresponding to the unique Pareto optimal Nash equilibrium are given by

$$X_1^{\text{post}}(\underline{c}, \underline{d}) \approx \begin{bmatrix} 0 \\ 1 \\ 0.41 \\ 1.41 \end{bmatrix}, \text{ and } X_2^{\text{post}}(\underline{c}, \underline{d}) \approx \begin{bmatrix} 0 \\ 0 \\ 0.59 \\ 0.59 \end{bmatrix}. \tag{16}$$

Firm 2 offers in the Pareto optimal Nash equilibrium its full risk to Firm 1. It gets offered a large part of the risk of Firm 1. For firm $i \in \{1, 2\}$, the zero utility premium relative to the expected value of its prior risk is defined as the value $p_i \in \mathbb{R}$ such that

$$E^{\mathbb{P}_i}\left[u_i(-X_i^{\text{post}}(\underline{c}, \underline{d}) - p_i E^{\mathbb{P}_i}[X_i])\right] = U_i(\overline{c}, \overline{d}). \tag{17}$$

Since u_i is continuous and strictly increasing, and $E^{\mathbb{P}_i}[X_i] > 0$, this value p_i exists, and is unique by the intermediate value theorem. The zero utility premium of the risk redistribution in (16) for Firm 1 is approximately 76% of the expected value of its prior risk. Moreover, for Firm 2,

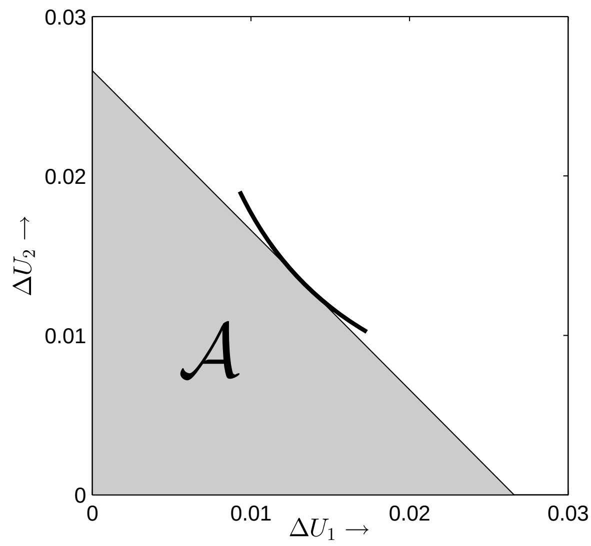

Figure 2: The set $\mathcal{A} \cap \mathbb{R}_+^2$ corresponding to Example 4.7. The intersection of the cooperative bargaining problem $\mathcal{A}$ with the black curvature equals the Nash bargaining solution $NB(\mathcal{A})$. Note that the absolute sizes of $\Delta U_i, i \in \{1,2\}$ have no interpretation.

the zero utility premium is approximately 90% of the expected value of its prior risk. Moreover, we study the reduction in expected value, which is given by

$$\frac{E^{\mathbb{P}_i}\left[X_i\right] - E^{\mathbb{P}_i}\left[X_i^{\text{post}}(\underline{c}, \underline{d})\right]}{E^{\mathbb{P}_i}\left[X_i\right]}, \tag{18}$$

for all $i \in \{1,2\}$. For Firm 1, the expected value of the risk after risk redistribution decreases with approximately 34% of the prior expected value. For Firm 2, the expected value of the risk after risk redistribution decreases with approximately 41% of the prior expected value. Note that these percentages do not need to add up to zero when the beliefs regarding the underlying distribution function are not the same.

In Table 1 and Table 2, we display the effects of different beliefs regarding the underlying probability distribution. We find that if both firms believe that the probability of the Type 1 event is the same as the probability of the Type 2 event, the optimal deductible for both firms equals $\underline{c} = \underline{d} = 0.5$ in the Pareto optimal Nash equilibrium. In this situation, the aggregate risk will be shared with equal proportions, i.e., we have $X_1^{\text{post}}(0.5, 0.5) = X_2^{\text{post}}(0.5, 0.5) = \frac{1}{2}(X_1 + X_2)$. Moreover, we get from Table 1 and Table 2 that if firms have heterogeneous beliefs, they generally believe that they gain in expected value. Moreover, Firm 2 benefits more in terms of the zero-utility premium if Firm 1 assigns a higher probability to the Type 1 event, and a lower probability to the Type 2 event. These two effects both lead to a situation in which Firm 1 is more willing to trade, and it is therefore willing to compensate Firm 2 more.

p_{11}	$\underline{c}$	$\underline{d}$	Zero-utility premium		Reduction in expected value	
			Firm 1	Firm 2	Firm 1	Firm 2
0.5%	0.50	0.50	41%	41%	0%	0%
1%	0.43	0.22	59%	68%	19%	20%
2%	0.41	0	76%	91%	34%	41%
4%	0.46	0	88%	98%	42%	46%

Table 1: Effects of different beliefs of Firm 1 regarding the probability that the Type 1 event occurs, which we here denote by p_{11}. The zero-utility premium is defined in (17), and the reduction in expected value is defined in (18).

p_{12}	$\underline{c}$	$\underline{d}$	Zero-utility premium		Reduction in expected value	
			Firm 1	Firm 2	Firm 1	Firm 2
0.25%	0.46	0	90%	99%	42%	46%
0.5%	0.41	0	76%	91%	34%	41%
1%	0.43	0.23	58%	68%	19%	20%
2%	0.50	0.50	40%	41%	0%	0%

Table 2: Effects of different beliefs of Firm 1 regarding the probability that the Type 2 event occurs, which we here denote by p_{12}. The zero-utility premium is defined in (17), and the reduction in expected value is defined in (18).

In Table 3, we display the effects of risk aversion. We find that when one firm is more risk averse, this firm bears more risk after trading. This observation is in line with Kihlstrom and Roth (1982) and Schlesinger (1984). Moreover, if the two firms are more risk averse altogether, the firms will trade less. In this case, less trade will mitigate the riskiness of the posterior risks. When firms are more risk averse, this effect is stronger compared to the effect of using different probability measures. As expected, the zero-utility premium is significantly higher when firms are more risk-averse.

λ_1	λ_2	$\underline{c}$	$\underline{d}$	Zero-utility premium		Reduction in expected value	
				Firm 1	Firm 2	Firm 1	Firm 2
0.1	0.1	0.38	0	40%	41%	37%	38%
1	1	0.41	0	76%	91%	34%	41%
10	10	0.49	0.45	1988%	3811%	37%	4%
1	10	0.27	0.39	117%	2252%	58%	-12%
10	1	0.62	0.30	941%	89%	21%	32%

Table 3: Effects of different risk aversion parameters of Firm 1 and Firm 2. The zero-utility premium is defined in (17), and the reduction in expected value is defined in (18).

$\triangledown$

5 Concluding remark

In this concluding remark, we propose a generalization of the results in this paper. The results in this paper remain valid if we only impose the some conditions on the utility functions U_1 :

$\mathbb{R}^2 \to \mathbb{R}$ and $U_2 : \mathbb{R}^2 \to \mathbb{R}$. Then, there is not necessarily a risk redistribution problem, but a bargaining problem of finding the strategy profiles $(c, d) \in \mathbb{R}^2$ corresponding to the Pareto optimal Nash equilibria. A strategy of Firm 1 is still a proposal of parameter c, and Firm 2 proposes a parameter d. If a firm $i \in \{1, 2\}$ is worse off than in the status quo, the deal is canceled. We impose the following conditions on the general utility functions $U_1 : \mathbb{R}^2 \to \mathbb{R}$ and $U_2 : \mathbb{R}^2 \to \mathbb{R}$.

- There is complete information about $U_i, i \in \{1, 2\}$.
- There exists a $(\bar{c}, \bar{d}) \in \mathbb{R}^2$ such that $U_i(\bar{c}, \bar{d}), i \in \{1, 2\}$ is the status quo.
- For every $d \leq \bar{d}$, the utility function $U_1(\cdot, d)$ is strictly decreasing and, for every $c \leq \bar{c}$, the utility function $U_1(c, \cdot)$ is strictly increasing. Likewise, for every $d \leq \bar{d}$, the utility function $U_2(\cdot, d)$ is strictly increasing and, for every $c \leq \bar{c}$, the utility function $U_2(c, \cdot)$ is strictly decreasing.
- The utility functions U_1 and U_2 are continuous.
- For every $(\hat{c}, \hat{d}) \in \mathbb{R}^2$, it holds that $\lim_{c \to -\infty} U_2(c, \hat{d}) = \lim_{d \to -\infty} U_1(\hat{c}, d) = -\infty$.

For instance, preferences based on dual utility (see, e.g., Yaari, 1987) satisfy these conditions if we use the same parametrization of the posterior risk profiles as described in Section 2. Also, the results in this paper remain valid if we include ambiguity aversion via, e.g., maxmin expected utility functions as in Gilboa and Schmeidler (1989).

References

Aase, K. (2009). The Nash bargaining solution vs. equilibrium in a reinsurance syndicate. *Scandinavian Actuarial Journal 2009*, 219–238.

Banerjee, S., U. Güçbilmez, and G. Pawlina (2014). Optimal exercise of jointly held real options: A Nash bargaining approach with value diversion. *European Journal of Operational Research 239*, 565–578.

Basel Committee on Banking Supervision (2013). Longevity risk transfer markets: market structure, growth drivers and impediments, and potential risks. Joint Forum.

Bertrand, J. (1883). Review of 'Théorie mathématique de la richesse sociale' and 'Recherches sur les principles mathématiques de la théorie des richesses'. *Journal des Savants 48*, 499–508.

Blake, D., A. J. G. Cairns, and K. Dowd (2006). Living with mortality: longevity bonds and other mortality-linked securities. *British Actuarial Journal 12*, 153–228.

Boonen, T. J., A. De Waegenaere, and H. Norde (2012). Bargaining for Over-The-Counter risk redistributions: The case of longevity risk. CentER Discussion Paper Series No. 2012-090.

Borch, K. (1962). Equilibrium in a reinsurance market. *Econometrica 30*, 424–444.

Conley, J. P. and S. Wilkie (1996). An extension of the Nash bargaining solution to nonconvex problems. *Games and Economic Behavior 13*, 26–38.

Coughlan, G. D., D. Epstein, A. Sinha, and P. Honig (2007). q-forwards: Derivatives for transferring longevity and mortality risks. Technical report, JPMorgan Pension Advisory Group, London.

Cox, S. and Y. Lin (2007). Natural hedging of life and annuity mortality risks. *North American Actuarial Journal 11*, 1–15.

Cummins, J. D. (2008). CAT bonds and other risk-linked securities: State of the market and recent developments. *Risk Management and Insurance Review 11*, 23–47.

Damme, E. van (1986). The Nash bargaining solution is optimal. *Journal of Economic Theory 38*, 78–100.

Denuit, M. and C. Vermandele (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics 22*, 229–233.

Dutang, C., H. Albrecher, and S. Loisel (2013). Competition among non-life insurers under solvency constraints: A game-theoretic approach. *European Journal of Operational Research 231*, 702–711.

Gerber, H. and G. Pafumi (1998). Utility functions: from risk theory to finance. *North American Actuarial Journal 2*, 74–91.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics 18*, 141–153.

Goovaerts, M., R. Kaas, A. Van Heerwaarden, and T. Bauwelinckx (1990). *Effective Actuarial Methods*. Amsterdam: North-Holland.

Hári, N., A. De Waegenaere, B. Melenberg, and T. E. Nijman (2008). Longevity risk in portfolios of pension annuities. *Insurance: Mathematics and Economics 42*, 505–519.

Huang, R. J., Y.-C. Huang, and L. Tzeng (2013). Insurance bargaining under ambiguity. *Insurance: Mathematics and Economics 53*, 812–820.

Kihlstrom, R. E. and A. E. Roth (1982). Risk aversion and the negotiation of insurance contracts. *Journal of Risk and Insurance 49*, 372–387.

Kunreuther, H., J. Meszaros, R. M. Hogarth, and M. Spranca (1995). Ambiguity and underwriter decision processes. *Journal of Economic Behavior and Organization 26*, 337–352.

Kurz, M. and S. Hart (1982). Pareto-optimal Nash equilibria are competitive in a repeated economy. *Journal of Economic Theory 28*, 320–346.

Lampaert, I. and J. Walhin (2005). On the optimality of proportional reinsurance. *Scandinavian Actuarial Journal 2005*, 225–239.

Li, S., H. Xiao, and D. Yao (2013). Contract bargaining with a risk-averse agent. *B.E. Journal of Theoretical Economics 13*, 285–301.

Nash, J. F. (1950a). The bargaining problem. *Econometrica 18*, 155–162.

Nash, J. F. (1950b). Equilibrium points in n-person games. *Proceedings of the National Academy of the USA 36*, 48–49.

Osborne, M. and A. Rubinstein (1994). *A Course in Game Theory*. Cambridge: MIT Press.

Quiggin, J. and R. G. Chambers (2009). Bargaining power and efficiency in insurance contracts. *Geneva Risk and Insurance Review 34*, 47–73.

Riddell, W. C. (1981). Bargaining under uncertainty. *American Economic Review 71*, 579–590.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica 50*, 97–109.

Schlesinger, H. (1984). Two-person insurance negotiation. *Insurance: Mathematics and Economics 3*, 147–149.

Schlesinger, H. and N. A. Doherty (1985). Incomplete markets for insurance: An overview. *Journal of Risk and Insurance 52*, 402–423.

Tsai, J., J. Wang, and L. Tzeng (2010). On the optimal product mix in life insurance companies using conditional value at risk. *Insurance: Mathematics and Economics 46*, 235–241.

Viaene, S., R. Veugelers, and G. Dedene (2002). Insurance bargaining under risk aversion. *Economic Modelling 19*, 245–259.

Wang, J., H. Huang, S. Yang, and J. Tsai (2010). An optimal prodict mix for hedging longevity risk in life insurance companies: The immunization theory approach. *Journal of Risk and Insurance 77*, 473–497.

Wilson, R. (1968). The theory of syndicates. *Econometrica 36*, 119–132.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica 55*, 95–115.

Zhou, R., J. S.-H. Li, and K. S. Tan (2015). Modeling longevity risk transfers as Nash bargaining problems: Methodology and insights. *Economic Modelling 51*, 460–472.

A Locally Lipschitz continuity of any stop-loss and proportional rule

First, we show that the stop-loss rule is locally Lipschitz continuous, uniformly for all $\omega \in \Omega$. It is sufficient to show this only for Firm 1. Local Lipschitz continuity follows from

$$\begin{aligned}|X_1^{\text{post}}(c,d)(\omega) - X_1^{\text{post}}(c',d')(\omega)| &= |\,(\min\{X_1(\omega),c\} + \max\{X_2(\omega)-d,0\}) \\ &\quad - (\min\{X_1(\omega),c'\} + \max\{X_2(\omega)-d',0\})\,| \\ &\leq |\min\{X_1(\omega),c\} - \min\{X_1(\omega),c'\}| \\ &\quad + |\max\{X_2(\omega)-d,0\} - \max\{X_2(\omega)-d',0\}| \\ &\leq |c-c'| + |d-d'| \\ &\leq \sqrt{2}\|(c,d)-(c',d')\|,\end{aligned}$$

for all $(c,d),(c',d') \in \mathbb{R}^2$ and for all $\omega \in \Omega$.

Second, we show that the proportional rule is locally Lipschitz continuous, uniformly for all $\omega \in \Omega$. This follows from

$$\begin{aligned}|X_1^{\text{post}}(c,d)(\omega) - X_1^{\text{post}}(c',d')(\omega)| &= |\min\{X_1(\omega),0\} + c\max\{X_1(\omega),0\} - d\max\{X_2(\omega),0\} \\ &\quad - (\min\{X_1(\omega),0\} + c'\max\{X_1(\omega),0\} - d'\max\{X_2(\omega),0\})\,| \\ &\leq |c\max\{X_1(\omega),0\} - c'\max\{X_1(\omega),0\}| \\ &\quad + |d\max\{X_2(\omega),0\} - d'\max\{X_2(\omega),0\}|\end{aligned}$$

$$
\begin{aligned}
&\leq |c-c'|\max\{X_1(\omega),0\} + |d-d'|\max\{X_2(\omega),0\}\\
&\leq |c-c'|\sup X_1 + |d-d'|\sup X_2\\
&\leq (|c-c'|+|d-d'|)\sup\{X_1,X_2\}\\
&\leq \sqrt{2}\sup\{X_1,X_2\}\|(c,d)-(c',d')\|,
\end{aligned}
$$

for all $(c,d),(c',d') \in \mathbb{R}^2$ and for all $\omega \in \Omega$.

B Proofs

First, we show two properties that the expected utility functions $U_i, i \in \{1,2\}$ satisfy. These properties are widely used in the proofs of Section 3.

Lemma B.1 *For every $d \leq \overline{d}$, the function $U_1(\cdot,d)$ is strictly decreasing and, for every $c \leq \overline{c}$, the function $U_1(c,\cdot)$ is strictly increasing. Likewise, for every $d \leq \overline{d}$, the function $U_2(\cdot,d)$ is strictly increasing and, for every $c \leq \overline{c}$, the function $U_2(c,\cdot)$ is strictly decreasing.*

Proof We show this result only for Firm 1. Let $\hat{d} \leq \overline{d}$ and $\hat{c},\hat{c}'$ such that $\hat{c} < \hat{c}' \leq \overline{c}$. Then, it holds that $X_1^{\text{post}}(\hat{c},\hat{d})(\omega) \leq X_1^{\text{post}}(\hat{c}',\hat{d})(\omega)$ for all $\omega \in \Omega$ with at least one strict inequality for some $\omega \in \Omega$. This implies that for every strictly increasing utility function we have $u_1(-X_1^{\text{post}}(\hat{c},\hat{d})(\omega)) \geq u_1(-X_1^{\text{post}}(\hat{c}',\hat{d})(\omega))$ for all $\omega \in \Omega$ with at least one strict inequality for some $\omega \in \Omega$. Since $\mathbb{P}_1(\omega) > 0$ for all $\omega \in \Omega$, we obtain that $U_1(\hat{c},\hat{d}) > U_1(\hat{c}',\hat{d})$ if we take expectations. Hence, $U_1(\cdot,\hat{d})$ is strictly decreasing. Likewise, we can show that $U_1(\hat{c},\cdot)$ is strictly increasing for every $\hat{c} \leq \overline{c}$. $\square$

Lemma B.2 *The expected utility functions U_1 and U_2 are continuous.*

Proof We show the result only for the expected utility function U_1. Let $(\hat{c},\hat{d}) \leq (\overline{c},\overline{d})$ and $\delta > 0$. Since the risk $X_1^{\text{post}}(\overline{c},\overline{d}) = X_1$ is bounded and since $X_1^{\text{post}}(\cdot,\cdot)(\omega)$ is continuous for all $\omega \in \Omega$, it follows that for any given $(c,d) \leq (\overline{c},\overline{d})$ the risks $X_1^{\text{post}}(c,d)$ is bounded as well. So, there exists an $M_\delta < \infty$ such that $X_1^{\text{post}}(c',d') < M_\delta$ for all $(c',d') \leq (\overline{c},\overline{d})$ such that $\|(c',d')-(\hat{c},\hat{d})\| < \delta$. Let $m_\delta = |u_1'(-M_\delta)| < \infty$. Then, since $u_1'(\cdot) > 0$, we have $|u_1'(-X_1^{\text{post}}(c',d'))| \leq m_\delta$, for all $(c',d') \leq (\overline{c},\overline{d})$ such that $\|(c',d')-(\hat{c},\hat{d})\| < \delta$.

By assumption, it holds that $X_1^{\text{post}}(\cdot,\cdot)(\omega)$ is locally Lipschitz continuous, uniformly for all $\omega \in \Omega$. Consequently, we have that $X_1^{\text{post}}(\cdot,\cdot)(\omega)$ is Lipschitz continuous on every compact subset of $(-\infty,\overline{c}] \times (-\infty,\overline{d}]$. So, for every $\delta > 0$, there exists an $\widetilde{m}_\delta < \infty$ such that $|X_1^{\text{post}}(\hat{c},\hat{d})(\omega) - X_1^{\text{post}}(c',d')(\omega)| < \widetilde{m}_\delta\|(\hat{c},\hat{d})-(c',d')\|$ for all $(c',d') \leq (\overline{c},\overline{d})$ such that $\|(\hat{c},\hat{d})-(c',d')\| < \delta$ and for all $\omega \in \Omega$.

Let $\varepsilon > 0$ and denote $\delta = \min\left\{\frac{\varepsilon}{m_1\widetilde{m}_1},1\right\}$. Then, for all $(c',d') \leq (\overline{c},\overline{d})$ such that $\|(c',d') - (\hat{c},\hat{d})\| < \delta$, we have

$$
\begin{aligned}
|U_1(c',d')-U_1(\hat{c},\hat{d})| &= \left|\sum_{\omega\in\Omega}\mathbb{P}_1(\omega)(u_1(-X_1^{\text{post}}(c',d')(\omega)) - u_1(-X_1^{\text{post}}(\hat{c},\hat{d})(\omega)))\right|\\
&\leq \sum_{\omega\in\Omega}\mathbb{P}_1(\omega)|u_1(-X_1^{\text{post}}(c',d')(\omega)) - u_1(-X_1^{\text{post}}(\hat{c},\hat{d})(\omega))| \qquad (19)\\
&\leq m_1\sum_{\omega\in\Omega}\mathbb{P}_1(\omega)|-X_1^{\text{post}}(c',d')(\omega) + X_1^{\text{post}}(\hat{c},\hat{d})(\omega)| \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
&< m_1 \widetilde{m}_1 \| -(c', d') + (\hat{c}, \hat{d}) \| \sum_{\omega \in \Omega} \mathbb{P}_1(\omega) \\
&= m_1 \widetilde{m}_1 \| -(c', d') + (\hat{c}, \hat{d}) \| \\
&< m_1 \widetilde{m}_1 \delta \\
&\leq \varepsilon,
\end{aligned}
$$

where (19) follows from the triangular inequality and (20) follows from the mean value theorem. This concludes the proof. □

Next, we continue with the proofs of the results in the main text.

Proof of Proposition 3.1 Let $(\hat{c}, \hat{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$. Since $\hat{c} \in N^{\mathcal{C}}$, we get from the definition of $N^{\mathcal{C}}$ in (9) that $\widetilde{U_2}(\hat{c}, d) = U_2(\overline{c}, \overline{d})$ for all $d \in \mathcal{D}$. So, all responses in $\mathcal{D}$ yield the same posterior expected utility for Firm 2. Therefore, every $d \in \mathcal{D}$ is a best response and, so, $br_2(\hat{c}) = \mathcal{D}$. Thus, $\hat{d} \in br_2(\hat{c})$. Likewise, we have $br_1(\hat{d}) = \mathcal{C}$ and, so, $\hat{c} \in br_1(\hat{d})$. Hence, $(\hat{c}, \hat{d})$ is a Nash equilibrium. This concludes the proof. □

Proof of Corollary 3.2 According to Proposition 3.1, it is sufficient to show that $(\overline{c}, \overline{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$. Since $U_2(\cdot, d)$ is strictly increasing (Lemma B.1), we have $U_2(c, \overline{d}) < U_2(\overline{c}, \overline{d})$ for all $c < \overline{c}$. Therefore, every $c \in \mathcal{C}$ such that $c < \overline{c}$ as a response to $\overline{d}$ will be rejected by Firm 2. So, we have $\widetilde{U_2}(c, \overline{d}) = U_2(\overline{c}, \overline{d})$ for all $c \in \mathcal{C}$. Consequently, we get from the definition of $N^{\mathcal{D}}$ in (10) that $\overline{d} \in N^{\mathcal{D}}$. Similarly, we can show that $\overline{c} \in N^{\mathcal{C}}$. This concludes the proof. □

Proof of Theorem 3.3 First, we show that every Nash equilibrium is in $N^{\mathcal{C}} \times N^{\mathcal{D}}$ if the strategy space $\mathcal{D}$ has no minimum.

Let the strategy profile $(\hat{c}, \hat{d})$ be a Nash equilibrium such that $\hat{d} \in N^{\mathcal{D}}$. Then, it holds that $\widetilde{U}_1(\hat{c}, \hat{d}) = U_1(\overline{c}, \overline{d})$. Suppose $\hat{c} \in (N^{\mathcal{C}})^c$. Then, it follows from the definition of $N^{\mathcal{C}}$ in (9) that there exists a response $d^* \in \mathcal{D}$ such that $\widetilde{U}_2(\hat{c}, d^*) > U_2(\overline{c}, \overline{d})$. Then, we get from the definition of $\widetilde{U}_2$ in (5) that $U_1(\hat{c}, d^*) \geq U_1(\overline{c}, \overline{d})$ and $U_2(\hat{c}, d^*) > U_2(\overline{c}, \overline{d})$. Since $U_1(\hat{c}, \cdot)$ is strictly decreasing (Lemma B.1), $U_2(\hat{c}, \cdot)$ is continuous (Lemma B.2) and the strategy space $\mathcal{D}$ has no minimum, there exists a strategy $d' < \hat{d}$ such that $d' \in \mathcal{D}$, $U_1(\hat{c}, d') > U_1(\overline{c}, \overline{d})$ and $U_2(\hat{c}, d') > U_2(\overline{c}, \overline{d})$. Hence, we get from the definition of $\widetilde{U}_1$ in (4) that $\widetilde{U}_1(\hat{c}, d') > U_1(\overline{c}, \overline{d})$. This is a contradiction with $\widetilde{U}_1(\hat{c}, \hat{d}) = U_1(\overline{c}, \overline{d})$. So, $\hat{c} \in N^{\mathcal{C}}$ and, hence, $(\hat{c}, \hat{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$.

Let $(\hat{c}, \hat{d})$ a Nash equilibrium such that $\hat{d} \in \left(N^{\mathcal{D}}\right)^c$. Then, we get from the definition in (10) that there exists a $c \in \mathcal{C}$ such that $\widetilde{U_1}(c, \hat{d}) > U_1(\overline{c}, \overline{d})$. Since $\hat{c} \in br_1(\hat{d})$, we have $\widetilde{U_1}(\hat{c}, \hat{d}) > U_1(\overline{c}, \overline{d})$. Since the expected utility function U_1 is continuous (Lemma B.2) and the strategy space $\mathcal{D}$ has no minimum, there exists a $d' < \hat{d}$ such that $d' \in \mathcal{D}$ and $U_1(\hat{c}, d') > U_1(\overline{c}, \overline{d})$. So, there exists a the strategy d' as response on $\hat{c}$ that will be accepted by Firm 1. According to Lemma B.1, we have that the function $U_2(\hat{c}, \cdot)$ is strictly decreasing. Therefore, $\widetilde{U_2}(\hat{c}, d') > \widetilde{U_2}(\hat{c}, \hat{d})$ and, so, the response d' to strategy $\hat{c}$ yields a strict improvement for Firm 2. Hence, the strategy profile $(\hat{c}, \hat{d})$ is not a Nash equilibrium. This is contradiction. Similarly, we can show that there also does not exist a Nash equilibrium $(\hat{c}, \hat{d})$ such that $\hat{c} \in \left(N^{\mathcal{C}}\right)^c$. Hence, every Nash equilibrium is in $N^{\mathcal{C}} \times N^{\mathcal{D}}$.

The reversed statement that every element of $N^{\mathcal{C}} \times N^{\mathcal{D}}$ is a Nash equilibrium is shown in Proposition 3.1. This concludes the proof. □

Proof of Lemma 3.4 We only show the result for the value c^*. Let $\hat{d} \in \mathcal{D}$. By assumption, it holds that $X_1^{\text{post}}(\cdot, \hat{d})(\omega)$ is decreasing for all $\omega \in \Omega$ and there is a state $\omega \in \Omega$ such that $\lim_{c \to -\infty} X_1^{\text{post}}(c, \hat{d})(\omega) = -\infty$. From the market clearing condition in (1) we get $X_2^{\text{post}}(c, d) = X_1 + X_2 - X_1^{\text{post}}(c, d)$ for all $(c, d) \leq (\overline{c}, \overline{d})$, where the stochastic variable $X_1 + X_2$ is bounded. So, consequently, we have that pay-off $X_2^{\text{post}}(\cdot, \hat{d})(\omega)$ is decreasing for all $\omega \in \Omega$ and there is a state $\omega \in \Omega$ such that $\lim_{c \to -\infty} X_2^{\text{post}}(c, \hat{d})(\omega) = \infty$. Since $u_2'(\cdot) > 0$ and $u_2''(\cdot) < 0$, we have that $\lim_{x \to -\infty} u_2(x) = -\infty$. Combining this with $\mathbb{P}_2(\omega) > 0$ for every $\omega \in \Omega$ yields that we have $\lim_{c \to -\infty} U_2(c, \hat{d}) \to -\infty$. Also, since $U_2(\overline{c}, \cdot)$ is strictly decreasing (Lemma B.1), we have that $U_2(\overline{c}, \hat{d}) \geq U_2(\overline{c}, \overline{d})$. Since $U_2(\cdot, \hat{d})$ is strictly increasing (Lemma B.1) and continuous (Lemma B.2), we have according to the intermediate value theorem that there is a unique value c^* such that $U_2(c^*, \hat{d}) = U_2(\overline{c}, \overline{d})$. This concludes the proof. □

Proof of Proposition 3.5 We prove this result only for the correspondence br_1. If $\hat{d} \in N^{\mathcal{D}}$, we showed that $br_1(\hat{d}) = \mathcal{C}$ in the proof of Proposition 3.1.

Let $\hat{d} \in (N^{\mathcal{D}})^c$. Then, there exists a strategy $c \in \mathcal{C}$ such that $U_1(c, \hat{d}) > U_1(\overline{c}, \overline{d})$. So, the best response for Firm 1 is given by:

$$br_1(\hat{d}) = \underset{c \in \mathcal{C}}{\operatorname{argmax}} \{U_1(c, \hat{d}) | U_2(c, \hat{d}) \geq U_2(\overline{c}, \overline{d})\}. \tag{21}$$

Since the expected utility functions U_1 and U_2 are continuous and the strategy space $\mathcal{C}$ is compact, $br_1(\hat{d})$ is non-empty. Since the function $U_1(\cdot, \hat{d})$ is strictly decreasing and the function $U_2(\cdot, \hat{d})$ is strictly increasing (Lemma B.1), we have that the best response to $\hat{d}$ in (21) is singleton-valued, and is given by:

$$br_1(\hat{d}) = \{\min\{c \in \mathcal{C} | c \geq c^*(\hat{d})\}\}.$$

If $c^*(\hat{d}) \in \mathcal{C}$, then $br_1(\hat{d}) = \{c^*(\hat{d})\}$. If $c^*(\hat{d}) \notin \mathcal{C}$, then $c^*(\hat{d}) < \underline{c} = \min \mathcal{C}$ and, so, the best response is given by $br_1(\hat{d}) = \{\underline{c}\}$. This concludes the proof. □

Proof of Proposition 3.6 We show this result only for br_1. From Proposition 3.5, we get that $br_1(d) = \max\{\underline{c}, c^*(d)\}$ if $d \in \left(N^{\mathcal{D}}\right)^c$ So, br_1 is continuous and increasing on $\left(N^{\mathcal{D}}\right)^c$ if c^* is continuous and increasing. Note that c^* is the solution of $U_2(c^*(d), d) = U_2(\overline{c}, \overline{d})$ for all $d \in (-\infty, \overline{d}]$. We have that U_2 is strictly monotonic (Lemma B.1) and continuous (Lemma B.2). Then, continuity of c^* follows directly from the implicit function theorem.

Next, we show that c^* is increasing. Let $d \in \left(N^{\mathcal{D}}\right)^c$. Since $U_2(c, \cdot)$ is strictly decreasing (Lemma B.1), we have $U_2(c^*(d), \hat{d}) < U_2(c^*(d), d) = U_2(\overline{c}, \overline{d})$ for all $\hat{d} > d$. Since $U_2(\cdot, d)$ is strictly increasing, we have $c^*(\hat{d}) > c^*(d)$. Similarly, we can show that $c^*(d') < c^*(d)$ for all $d' < d$. Hence, c^* is strictly increasing and, so, increasing. □

Proof of Theorem 3.7 This result is in fact an extension of Theorem 3.3. We only show the first result, and the proof of the second result is similar. Suppose that there exists a Nash equilibrium $(\hat{c}, \hat{d}) \in \mathcal{C} \times \mathcal{D}$ such that $\hat{d} > \underline{d}$ and $\widetilde{U_1}(\hat{c}, \hat{d}) > U_1(\overline{c}, \overline{d})$. Then, it follows from (4) that $\widetilde{U_1}(\hat{c}, \hat{d}) = U_1(\hat{c}, \hat{d}) > U_1(\overline{c}, \overline{d})$ and $U_2(\hat{c}, \hat{d}) \geq U_2(\overline{c}, \overline{d})$. Since $U_2(\hat{c}, \cdot)$ is strictly decreasing (Lemma B.1) and $U_1(\hat{c}, \cdot)$ is continuous (Lemma B.2), there exists a $d' < \hat{d}$ such that $U_1(\hat{c}, d') > U_1(\overline{c}, \overline{d})$ and $U_2(\hat{c}, d') > U_2(\hat{c}, \hat{d}) \geq U_2(\overline{c}, \overline{d})$. So, we have $\widetilde{U_2}(\hat{c}, d') > \widetilde{U_2}(\hat{c}, \hat{d})$. Therefore, for Firm 2, d' is a strictly better response than $\hat{d}$ and, so, $\hat{d}$ is not a best response. Hence, $(\hat{c}, \hat{d})$ is not a Nash equilibrium. This is a contradiction. So, if there exists a Nash equilibrium $(\hat{c}, \hat{d}) \in \mathcal{C} \times \mathcal{D}$ such that $\widetilde{U_1}(\hat{c}, \hat{d}) > U_1(\overline{c}, \overline{d})$, we have that $\hat{d} = \underline{d}$.

Suppose that the strategy profile $(\hat{c}, \underline{d})$ is a Nash equilibrium in which Firm 1 strictly benefits. From (10), it follows that $\underline{d} \in (N^{\mathcal{D}})^c$. Therefore, the best response for Firm 1 to strategy $\hat{d} = \underline{d}$ is given by the strategy $br_1(\underline{d}) = \max\{c^*(\underline{d}), \underline{c}\}$ and, so, the strategy profile $(br_1(\underline{d}), \underline{d})$ is the unique Nash equilibrium where Firm 1 strictly benefits. This concludes the proof. □

Proof of Corollary 3.8 Suppose that there exist multiple Nash equilibria where at least one firm strictly benefits. From Theorem 3.7, we get that there exist two Nash equilibria $(\hat{c}_1, \underline{d}), (\underline{c}, \hat{d}_2) \in NE$ in which at least one firm strictly benefits. Suppose $\hat{c}_1 > \underline{c}$ and $\hat{d}_2 > \underline{d}$. Since in both Nash equilibria at least one firm strictly benefits, we have according to Theorem 3.7 that Firm 1 strictly benefits in $(\hat{c}_1, \underline{d})$ and Firm 2 strictly benefits in $(\underline{c}, \hat{d}_2)$. Therefore, $(\underline{c}, \underline{d}) \in (N^{\mathcal{C}})^c \times (N^{\mathcal{D}})^c$, and, so, we have $br_1(\underline{d}) = \hat{c}_1 = c^*(\underline{d})$ and $br_2(\underline{c}) = \hat{d}_2 = d^*(\underline{c})$. Since $c^*(\underline{d}) \in \mathcal{C}$ and $d^*(\underline{c}) \in \mathcal{D}$, we get $c^*(\underline{d}) \geq \underline{c}$ and $d^*(\underline{c}) \geq \underline{d}$. From this and from the fact that $U_2(c, \cdot)$ is strictly decreasing and $U_2(\cdot, d)$ strictly increasing (Lemma B.1), it follows that

$$U_2(\overline{c}, \overline{d}) = U_2(c^*(\underline{d}), \underline{d}) \geq U_2(\underline{c}, d^*(\underline{c})),$$

where the equality follows from the definition of c^*. This is a contradiction, since Firm 2 strictly benefits in the strategy profile $(\underline{c}, \hat{d}_2) = (\underline{c}, d^*(\underline{c}))$. So, we have that $\hat{c}_1 = \underline{c}$ or $\hat{d}_2 = \underline{d}$. Then, from Theorem 3.7 it follows that $\hat{c}_1 = \underline{c}$ and $\hat{d}_2 = \underline{d}$ and, moreover, $(\hat{c}_1, \underline{d}) = (\underline{c}, \hat{d}_2) = (\underline{c}, \underline{d}) = f(\underline{c}, \underline{d})$.

Next, we show that if there exists a firm that strictly benefits in a Nash equilibrium, this equilibrium is given by $f(\underline{c}, \underline{d})$. Suppose that the strategy profile $(\hat{c}_2, \underline{d})$ is the Nash equilibrium in which Firm 1 strictly benefits. Then, we have $\underline{d} \in (N^{\mathcal{D}})^c$ and, therefore, $\hat{c}_2 = br_1(\underline{d})$. If $br_1(\underline{d}) = c^*(\underline{d})$, then $br_1(\underline{d}) \in N^{\mathcal{C}}$ and, so, $(\hat{c}_2, \underline{d}) = (c^*(\underline{d}), \underline{d}) = f(\underline{c}, \underline{d})$. If $br_1(\underline{d}) = \underline{c} > c^*(d)$, then $br_1(\underline{d}) \in (N^{\mathcal{C}})^c$ and, so, $(\hat{c}_2, \underline{d}) = (\underline{c}, \underline{d}) = f(\underline{c}, \underline{d})$. This concludes the proof. □

Proof of Theorem 3.9 Suppose there is a Nash equilibrium $(\hat{c}, \hat{d}) \notin (N^{\mathcal{C}} \times N^{\mathcal{D}}) \cup \{f(\underline{c}, \underline{d})\}$.

Suppose $\hat{c} \in N^{\mathcal{C}}$. The best response to $\hat{c}$ is given by any $d \in br_2(\hat{c}) = \mathcal{C}$, and satisfies $\widetilde{U}_1(\hat{c}, d) = U_1(\overline{c}, \overline{d})$ for all $d \in \mathcal{D}$. If $\hat{d} \in (N^{\mathcal{D}})^c$, there exists a unique best response given by $br_1(\hat{d}) = \max\{\underline{c}, c^*(\hat{d})\}$ with $\widetilde{U}_1(br_1(\hat{d}), \hat{d}) > U_1(\overline{c}, \overline{d}) = \widetilde{U}_1(\hat{c}, \hat{d})$. This is a contradiction with that $(\hat{c}, \hat{d})$ is a Nash equilibrium. Hence, it holds that $\hat{d} \in N^{\mathcal{D}}$, which is a contradiction with $(\hat{c}, \hat{d}) \notin N^{\mathcal{C}} \times N^{\mathcal{D}}$.

Suppose $\hat{c} \in (N^{\mathcal{C}})^c$. From the previous contradiction, it follows that also $\hat{d} \in (N^{\mathcal{D}})^c$. From Proposition 3.5, we then get $\hat{c} = br_1(\hat{d}) = \max\{\underline{c}, c^*(\hat{d})\}$ and $\hat{d} = br_2(\hat{c}) = \max\{\underline{d}, d^*(\hat{c})\}$. Suppose $br_2(\hat{c}) = \underline{d}$. Then, $\hat{c} = br_1(\underline{d}) = \max\{\underline{c}, c^*(\underline{d})\}$ and, so, $(\hat{c}, \hat{d}) = (br_1(\underline{d}), \underline{d}) = f(\underline{c}, \underline{d})$. This is a contradiction. Suppose $br_2(\hat{c}) = d^*(\hat{c}) > \underline{d}$ and, moreover, $\hat{c} > \underline{c}$. According to Theorem 3.7, there are no welfare gains in this Nash equilibrium. Hence, $(\hat{c}, \hat{d}) \in N^{\mathcal{C}} \times N^{\mathcal{D}}$, which is a contradiction. If $\hat{c} = \underline{c}$, then $\hat{d} = br_2(\underline{c})$ and, so, $(\hat{c}, \hat{d}) = (\underline{c}, br_2(\underline{c})) = f(\underline{c}, \underline{d})$. This is a contradiction as well.

Hence, $(\hat{c}, \hat{d}) \in (N^{\mathcal{C}} \times N^{\mathcal{D}}) \cup \{f(\underline{c}, \underline{d})\}$. This concludes the proof. □

Proof of Proposition 4.2 If both firms use the proportional rule as parametric form to redistribute their risk, or if one firm uses the proportional rule and one firm a risk-free payment, we have

$$\lambda X_i^{\text{post}}(c, d) + (1 - \lambda) X_i^{\text{post}}(c', d') = X_i^{\text{post}}(\lambda c + (1 - \lambda)c', \lambda d + (1 - \lambda)d'), \text{ for all } i \in \{1, 2\},$$

and for all $(c, d), (c', d') \leq (\overline{c}, \overline{d})$ and $\lambda \in [0, 1]$. The rest of this proof follows from the same reasoning as in Riddell (1981), but it is shown for completeness.

Let $a, b \in \mathcal{A}$ and $\lambda \in [0,1]$. By definition of the set $\mathcal{A}$ in (15), there exist $(c,d),(c',d') \in [0,\overline{c}] \times [0,\overline{d}]$ such that $a \leq \Delta U(c,d)$ and $b \leq \Delta U(c',d')$. From concavity of the utility function $u_i(\cdot)$, it follows that

$$\begin{aligned}\Delta U_i(\lambda c + (1-\lambda)c', \lambda d + (1-\lambda)d') &= E^{\mathbb{P}_i}[u_i(-\lambda X_i^{\text{post}}(c,d) - (1-\lambda)X_i^{\text{post}}(c',d'))] - U_i(\overline{c},\overline{d}) \\ &\geq \lambda \Delta U_i(c,d) + (1-\lambda)\Delta U_i(c',d') \\ &\geq \lambda a_i + (1-\lambda) b_i,\end{aligned}$$

for all $i \in \{1,2\}$. Since $(\lambda c + (1-\lambda)c', \lambda d + (1-\lambda)d') \in [0,\overline{c}] \times [0,\overline{d}]$, it holds that $\Delta U(\lambda c + (1-\lambda)c', \lambda d + (1-\lambda)d') \in \mathcal{A}$. From this and that the set $\mathcal{A}$ is comprehensive, we get $\lambda a + (1-\lambda)b \in \mathcal{A}$ and, hence, the set $\mathcal{A}$ is convex. This concludes the proof. $\square$

Proof of Proposition 4.3 Comprehensiveness of the set $\mathcal{A}$ holds by construction. Since $(0,0) \in \mathcal{A}$, the set $\mathcal{A} \cap \mathbb{R}_+^2$ is non-empty. Since ΔU_i is continuous (Lemma B.2) and the domain $[0,\overline{c}] \times [0,\overline{d}]$ is compact, we have that the set

$$\widehat{\mathcal{A}} = \left\{ a \in \mathbb{R}^2 \,\middle|\, \exists (c,d) \in [0,\overline{c}] \times [0,\overline{d}] : a = \Delta U(c,d) \right\}$$

is compact. Therefore, the set $\mathcal{A}$, which is the comprehensive hull of $\widehat{\mathcal{A}}$, is closed as well. Therefore, the set $\mathcal{A} \cap \mathbb{R}_+^2$ is the intersection of two closed sets which is closed as well. Moreover, the set $\widehat{\mathcal{A}}$ is bounded from above and the set $\mathbb{R}_+^2$ is bounded from below. Therefore, the set $\mathcal{A} \cap \mathbb{R}_+^2$ is bounded and, hence, compact. So, the set $\mathcal{A}$ is a bargaining problem. This concludes the proof. $\square$

Proof of Proposition 4.4 Suppose that the vector $(a,0)$ is Pareto optimal with $a > 0$. Then, there exists a strategy profile $(c,d) \in [0,\overline{c}] \times [0,\overline{d}]$ such that $(a,0) = \Delta U(c,d)$. Clearly $c < \overline{c}$. Then, since $U_1(\cdot,d)$ is continuous (Lemma B.2) and $U_2(\cdot,d)$ is strictly increasing (Lemma B.1), there exists a $c' > c$ such that $\Delta U(c',d) > 0$. The proof is similar if there exists a Pareto optimal vector $(0,a)$ with $a > 0$. This concludes the proof. $\square$