Bowley Reinsurance with Asymmetric Information: A First-Best Solution

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October 25, 2021

Abstract

Bowley reinsurance solutions are reinsurance contracts for which the reinsurer optimally sets the pricing density, while anticipating that the insurer will chose the optimal reinsurance indemnity given this pricing density. This Bowley solution concept of equilibrium reinsurance strategy has been revisited in the modern risk management framework by Boonen et al. (2021), where the insurer and reinsurer are both endowed with distortion risk measures but there is asymmetric information on the distortion risk measure of the insurer. In this article, we continue to study this framework, but we allow the premium principle to be more flexible. We call this solution the *first-best* Bowley solution. We provide first-best Bowley solutions in closed-form under very general assumptions. We implement some numerical examples to illustrate the findings and the comparisons with the second-best solution. The main result is further extended to the case when both of the reinsurer and the insurers have heterogeneous beliefs on the distribution functions of the underlying risk.

Key words: Bowley reinsurance; Asymmetric information; General premium principle; Distortion risk measure; Heterogeneous beliefs.
JEL classification: C61, G22, G32.

1 Introduction

This paper studies Bowley reinsurance solutions with asymmetric information with a very general pricing principle. In Bowley reinsurance contracts, a monopolistic reinsurer selects the premium principle, and sequentially the insurer selects the optimal reinsurance

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coverage given this premium principle. The reinsurer selects the premium principle such that it maximizes expected profit, while taking into account the insurer's optimal response to buy reinsurance coverage. Bowley solutions are first studied in optimal reinsurance by Chan and Gerber (1985). In the reinsurance market, we assume that there is asymmetric information in the sense that the reinsurer cannot observe the *preferences* of the insurer. More precisely, the insurer is endowed with a distortion risk measure, and the reinsurer does not know the underlying distortion function used by the insurer. The preferences of the insurer may only be revealed by the reinsurance contract that the insurer purchases. The reinsurer communicates the premium principle to the insurer, and the class of admissible premium principles is very flexible. We assume that the premium principle is law invariant, comonotonic additive and the total premium cannot be negative. This is more general than Boonen et al. (2021), as this premium principle may admit negative state prices. Note, however, that we still do not permit arbitrage opportunities due to market-incompleteness. In other words, there does not exist a reinsurance indemnity that is positive somewhere with positive probability and has a non-positive premium. Such premium principles are closely related to distortion riskmetrics (Wang et al., 2020), that generalize distortion risk measures to allow for non-monotone and non-translation-invariant risk measures in the sense of Artzner et al. (2001). We however restrict the underlying distortion function to be non-negative in order to avoid negative prices.

The premium principle that is shown to be optimal is akin to a distortion risk measure, but the underlying "distortion function" may be decreasing somewhere. For this premium principle to be a distortion premium principle (also called Wang's premium principle when the distortion function is concave), the underlying distortion function needs to be non-decreasing. To avoid any confusion, we do not call a function that is not non-decreasing a distortion function, but a *premium generating function*. As seen later from our numerical examples, the first-best Bowley solution in the present paper yields a higher profit for the reinsurer than the second-best solution developed in Boonen et al. (2021).

The literature on Bowley solutions in optimal reinsurance is summarized as follows. First, under expected-utility theory, Chan and Gerber (1985) show the Bowley solutions for many special cases. Recently, also under expected-utility theory, Chi et al. (2020) design as different type of game, and study Bowley solutions therein. In particular, they assume that the reinsurer selects the insurer's budget for reinsurance, and then the insurer maximizes its utility given this premium budget. Distortion risk measures have been studied in many different settings, as they are related to dual utility (Yaari, 1987) and Wang's premium principle (Wang, 1996). Also, it is related to insurance regulation, as the Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) are special cases of distortion risk measures. In the setting of distortion risk measures, Bowley solutions are studied by Cheung et al. (2019) and Boonen et al. (2021). In Cheung et al. (2019), the insurer minimizes a distortion risk measure and the reinsurer is risk-neutral, while the premium principle is also of a distortion risk measure type. The solution of this optimal reinsurance problem is reached by solving two sub-problems in order. The first step is to minimize the distortion risk measure of the retained loss of the insurer with increasing concave distortion function for a given premium functional. This problem is closely related to optimal reinsurance with distortion risk measures (Cui et al., 2013; Assa, 2015). The second step then for the reinsurer is to select the premium functional that maximizes the net expected gain of the reinsurer. Boonen et al. (2021) extend this approach in the second step, where they include asymmetric information of the preferences of the insurer and the reinsurer adopts a distortion premium principle. Henceforth, we refer the results in Boonen et al. (2021) as the *second-best Bowley solution* under asymmetric information.

Bowley solutions are closely related to Stackelberg equilibria in game theory. In Stackelberg equilibria, there is a leader who first discloses their strategy, and the other economic agent - the follower - selects their strategy in response. With symmetric information, the leadership will yield an advantage in welfare compared to the follower. In actuarial science, this is for instance studied by Albrecher and Dalit (2017), Chen and Shen (2018) and Anthropelos and Boonen (2020). In Bowley solutions with asymmetric information, however, it may be possible to mutually strictly benefit in the equilibrium outcome, as we will show in this paper.

The premium principle that we find admits a bang-bang type structure. A feature of this premium principle is that it may also admit negative state prices. This paper explicitly focuses on optimal reinsurance contracts, and we only allow reinsurance indemnity contracts that satisfy a moral hazard condition, implying comonotonicity of the retained and the insured risks. In contrast to complete markets (for instance with Arrow-Debreu securities), this comonotonicity restriction implies that negative state prices cannot be exploited as arbitrage opportunity.[1] Market incompleteness is also a result of the standard assumption that the insurer can only buy reinsurance, and not sell reinsurance.

The results developed in the present paper have the following differences and advantages: (i) we establish the Bowley reinsurance contracts (called the *first-best Bowley solution* in the sequel) under asymmetric information provided that the reinsurer adopts a very general premium generating function and a general cost function when contracting reinsurance contracts with the insurer; (ii) there is no need to consider only special cases for the shape of the distortion functions employed by the insurer, which is in sharp contrast with the basic assumption of the findings in Boonen et al. (2021). The techniques in this paper are very different from the ones in Boonen et al. (2021). More specifically, we use an optimal control method via path-wise optimization instead of a geometric method; (iii) by construction, the reinsurer can gain more profit under our first-best solution compared with the second-best solution in some situations; (iv) the main result can be smoothly extended to the case where the reinsurer and the two types of insurers have heterogeneous beliefs on the probability distributions of the underlying risk.

This paper is set out as follows. Section 2 states the asymmetric information problem studied in this paper. Section 3 solves this problem. In Section 4, we give an example to illustrate the main result when the two distortion functions of the insurer are ordered. Section 5 presents another two numerical examples when the distortion risk measures are (i) VaR and convex distortion risk measure, and (ii) TVaR and convex distortion risk measure. Some comparative analysis is carried out on our first-best Bowley solution and the second-best solution established in Boonen et al. (2021). Section 6 extends the main result of Section 3 to the scenario where the reinsurer and the insurers have asymmetric information and heterogeneous beliefs. Section 7 concludes the paper.

[1] In asset markets, negative pricing kernel realisations appear also in equilibrium markets with mean-variance optimizing agents (Jarrow and Madan, 1997), or in markets with restricted participation (Rahi and Zigrand, 2014).

2 Problem formulation

2.1 Indemnities, premium principles and distortion risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. The insurer is initially endowed with a bounded, non-negative random loss variable X, which is realized at a given future reference time period. Its distribution function F_X is known by both the insurer and the reinsurer.[2]

The insurer cedes the risk $f(X)$ to the reinsurer. We assume that $f \in \mathcal{F}$, with

$$\mathcal{F} = \{f : [0, M] \mapsto [0, M] | f(0) = 0, 0 \leq f(x) - f(y) \leq x - y \text{ for } 0 \leq y \leq x \leq M\},$$

where M is the essential supremum of X. The purpose of restricting the admissible set of ceded loss functions to $\mathcal{F}$ is to avoid moral hazard or insurance swindles; see for instance Huberman et al. (1983), Denuit and Vermandele (1998) and many other recent papers. The function $f \in \mathcal{F}$ is non-decreasing and 1-Lipschitz and hence absolutely continuous. This implies that f is almost everywhere differentiable on $[0, M]$. Moreover, there exists a Lebesgue integrable function $h : [0, M] \mapsto [0, 1]$ such that

$$f(x) = \int_0^x h(z)\mathrm{d}z, \quad x \geq 0, \tag{1}$$

where h is the slope of the ceded loss function f.

In return for the ceding the random loss $f(X)$, the insurer pays a premium to the reinsurer. The premium is determined via a premium principle, which maps every ceded random loss $f(X)$ to non-negative premiums. We assume that the premium principle is comonotonic additive and law invariant, but not necessarily monotone. In particular, for any ceded loss function $f \in \mathcal{F}$, we assume that the reinsurance premium charged is determined by the following general premium principle:

$$\Pi_{g_r}(f(X)) = \int_0^{f(M)} g_r(\overline{F}_{f(X)}(z))\mathrm{d}z = \int_0^M g_r(\overline{F}_X(z))h(z)\mathrm{d}z, \tag{2}$$

where h satisfies (1), $\overline{F}_{f(X)}(z) := 1 - F_{f(X)}(z)$ is the survival function of $f(X)$, $g_r \in \mathcal{G}$, and

$$\mathcal{G} = \{g : [0, 1] \mapsto \mathbb{R}_+ | g(0) = 0, g \text{ is bounded variation}\}.$$

We refer to g_r as a *premium generating function*. In (2), the second equality follows from Cheung and Lo (2017).[3] Recently, Wang et al. (2020) studied a class of functionals that is very similar to (2),[4] and they defined this as a distortion riskmetric. Distortion riskmetrics generalize distortion risk measures as they allow for non-monotone risk measures. The expected premium principle, Wang's premium principle and the VaR are all special cases

[2]In Section 6, we generalize this assumption, and allow for heterogeneous beliefs.

[3]Since any function of bounded variation can be always written as the difference of two non-decreasing functions, the proofs and results of Cheung and Lo (2017) still hold and can be applied here directly. It is worth mentioning that bounded variation is also assumed by Wang et al. (2020).

[4]To be precise, and in contrast to our setting, Wang et al. (2020) allow for functions g_r that can be negative somewhere. Negative values of g_r lead to negative prices for some insurance contracts, and this makes negativity not a desirable property for premium principles.

of the general premium principle in (2). There are also some other well-known premium principles used in insurance practice when the premium generating function g_r is not monotone (might be strictly decreasing somewhere on $[0, 1]$). These include the Gini deviation, the mean-median deviation, the inter-quantile range with confidence level $\alpha \in [1/2, 1)$, and the inter-ES range with confidence level $\beta \in (0, 1)$, whose distortion functions are given by $g_r(t) = t - t^2$, $g_r(t) = t \wedge (1-t)$, $g_r(t) = \mathbf{1}_{\{1-\alpha \le t \le \alpha\}}$, and $g_r(t) = \frac{t}{1-\beta} \wedge 1 + \frac{\beta - t}{1-\beta} \wedge 0$, respectively. We refer interested readers to Table 1 in Wang et al. (2020) for more details. Besides, it is easy to observe that the premium must be greater than the expectation if we set the premium generating function as $\tilde{g}_r(t) := g_r(t) + t$, where $g_r(t)$ refers to one of the above mentioned non-monotone premium generating functions. This policy is very important and useful in many insurance applications.

If g_r is decreasing somewhere, the underlying pricing kernel may be negative somewhere. Note however that $\Pi(f(X)) \ge 0$ for all $f \in \mathcal{F}$, and if $g_r(t) > 0$ for all $t \in (0, 1]$ and $\mathbb{P}(f(X) > 0) > 0$ then $\Pi_{g_r}(f(X)) > 0$. Thus, it is not necessarily possible to construct an arbitrage opportunity[5] when g_r is non-monotone, which is a result of the constraints in $\mathcal{F}$. Also, from (2), we get that if $f_1'(z) \ge f_2'(z)$ for all $z \in [0, M]$, then $\Pi_{g_r}(f_1(X)) \ge \Pi_{g_r}(f_2(X))$.

Finally, we discuss the preferences of the insurer. The insurer is assumed to minimize a *distortion risk measure*. A distortion risk measure ρ_g of a non-negative random variable Z is given by

$$\rho_g(Z) = \int_0^\infty g(\overline{F}_Z(z))\mathrm{d}z, \tag{3}$$

whenever the integral exists, where $g : [0, 1] \mapsto [0, 1]$ is a non-decreasing and left-continuous function such that $g(0) = 0$ and $g(1) = 1$. The set of all such distortion functions g is denoted by $\mathcal{G}_d$, i.e.,

$$\mathcal{G}_d = \{g \in \mathcal{G} : g(0) = 0, g(1) = 1, g \text{ is non-decreasing and left-continuous}\}.$$

2.2 Bowley reinsurance solutions with asymmetric information

The distortion risk measure used by the insurer is hidden information for the reinsurer, and the reinsurer only knows that there are just finitely many possible distortion risk measures of the insurer. For brevity of our result, we proceed under the case that there are only two possible distortion risk measures of the insurer. To be precise, the reinsurer holds the opinion that insurer minimizes $\rho_{g_{i1}}$ or $\rho_{g_{i2}}$ with probability p and $1-p$, respectively, where $p \in [0, 1]$ and $\{g_{i1}, g_{i2}\} \subset \mathcal{G}_d$.

For the reinsurance indemnity function $f \in \mathcal{F}$, the total retained loss for the insurer is equal to $X - f(X) + \Pi_{g_r}(f(X))$, where $g_r \in \mathcal{G}$ and Π_{g_r} is the corresponding reinsurance premium principle given by (2). The two-step game played by the insurer and the reinsurer is formalized as follows:

- (Decision problem faced by the insurer) For any given $g_r \in \mathcal{G}$ provided by the

[5] An arbitrage opportunity is here understood as the existence of a reinsurance indemnity that is positive somewhere with positive probability and that has a non-positive premium.

reinsurer, the insurer chooses the optimal ceded loss function $f \in \mathcal{F}$ by solving

$$\min_{f\in\mathcal{F}} \rho_{g_i}(X - f(X) + \Pi_{g_r}(f(X))), \tag{4}$$

where $g_i = g_{i1}$ or $g_i = g_{i2}$, depending the identity of the insurer.

- (Decision problem faced by the reinsurer) The reinsurer is uncertain about the identity of the insurer, but knows the distortion functions g_{i1} and g_{i2} and probability p. The reinsurer selects the optimal reinsurance premium generating function g_r^* by maximizing the expected net profit. Then, the optimization problem of interest is

$$\begin{aligned}\max_{g_r\in\mathcal{G}} W(g_r) := \max_{g_r\in\mathcal{G}} \{ & p\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i1}\}}(X)) - f_{\{g_r;g_{i1}\}}(X)] - C(f_{\{g_r;g_{i1}\}})\} \\ & + (1-p)\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i2}\}}(X)) - f_{\{g_r;g_{i2}\}}(X)] - C(f_{\{g_r;g_{i2}\}})\}\},\end{aligned} \tag{5}$$

where $C(f_{\{g_r;g_{ij}\}})$ denotes the aggregate administrative cost paid by the reinsurer if the insurer purchases the policy $f_{\{g_r;g_{ij}\}}$, for $j = 1, 2$.

Only after the insurer selects the indemnity function that is optimal for him/her, the identity of the insurer is revealed to the reinsurer via this indemnity selection. In fact, the reinsurer can distinguish insurers according to different responses from different types of the insurer. The problem of this paper is summarized as follows:

$$\begin{aligned}&\max_{g_r\in\mathcal{G}} W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}}) \\ &\text{s.t. } f_{\{g_r;g_{ij}\}} \in \operatorname*{argmin}_{f\in\mathcal{F}} \rho_{g_{ij}}(X - f(X) + \Pi_{g_r}(f(X))), \quad j = 1, 2,\end{aligned}$$

where $W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}})$ is the expected net profit of the reinsurer in (5) for given indemnity functions $f_{\{g_r;g_{i1}\}}$ and $f_{\{g_r;g_{i2}\}}$. Solutions are called *first-best Bowley solutions.*

Problem (4) has been solved by Cheung et al. (2019), which is stated in the following lemma.[6]

Lemma 2.1 (Cheung et al. (2019)) *For any $g_r \in \mathcal{G}$, the optimal ceded loss function $f^*_{\{g_r,g_{ij}\}}$ that solves problem (4) is given by*

$$\begin{aligned} f^*_{\{g_r,g_{ij}\}}(x) \;=\; & \mu(\{z \in [0, x] \mid \psi_j(F_X(z)) > 0\}) \\ & + \int_0^x h_j(z)\mathbf{1}_{\{\psi_j(F_X(z))=0\}}\mu(\mathrm{d}z), \quad x \geq 0, \end{aligned} \tag{6}$$

where the function ψ_j is defined as

$$\psi_j(t) := g_{ij}(1-t) - g_r(1-t), \quad t \in [0, 1],$$

and h_j could be any measurable function with $0 \leq h_j(z)\mathbf{1}_{\{\psi_j(F_X(z))=0\}} \leq 1$, for $j = 1, 2$.

[6] Cheung et al. (2019) require that the set of admissible premium functions is given by $\mathcal{G}_d$ instead of $\mathcal{G}$, but this assumption is not needed in this proof of this result.

We assume that the administrative cost of offering the compensation is proportional to a distortion risk measure, i.e., let $C(f) =: \gamma\rho_{\hat{g}_R}(f(X))$ for any $f \in \mathcal{F}$, where $\gamma \geq 0$ is a fixed constant and $\hat{g}_R \in \mathcal{G}_d$.[7] Then, for $j \in \{1, 2\}$,

$$\begin{aligned}
\mathbb{E}[f_{\{g_r;g_{ij}\}}(X)] + \gamma\rho_{\hat{g}_R}(f_{\{g_r;g_{ij}\}}) &= \int_0^{f_{\{g_r;g_{ij}\}}(M)} \overline{F}_{f_{\{g_r;g_{ij}\}}(X)}(z)\mathrm{d}z \\
&\quad + \gamma \int_0^{f_{\{g_r;g_{ij}\}}(M)} \hat{g}_R(\overline{F}_{f_{\{g_r;g_{ij}\}}(X)}(z))\mathrm{d}z \\
&= \int_0^{f_{\{g_r;g_{ij}\}}(M)} [\overline{F}_{f_{\{g_r;g_{ij}\}}(X)}(z) + \gamma\hat{g}_R(\overline{F}_{f_{\{g_r;g_{ij}\}}(X)}(z))]\mathrm{d}z \\
&= \int_0^{f_{\{g_r;g_{ij}\}}(M)} g_R(\overline{F}_{f_{\{g_r;g_{ij}\}}(X)}(z))\mathrm{d}z \\
&= \Pi_{g_R}(f_{\{g_r;g_{ij}\}}(X)),
\end{aligned}$$

where $g_R(t) := t + \gamma\hat{g}_R(t), t \in [0, 1]$, is such that $g_R \in \mathcal{G}$, and where Π_{g_R} is defined in (2). Thus, we rewrite the objective in (5) as:

$$\begin{aligned}
W(g_r) &= p\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i1}\}}(X)) - f_{\{g_r;g_{i1}\}}(X)] - \gamma\rho_{\hat{g}_R}(f_{\{g_r;g_{i1}\}})\} \\
&\quad + (1-p)\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i2}\}}(X)) - f_{\{g_r;g_{i2}\}}(X)] - \gamma\rho_{\hat{g}_R}(f_{\{g_r;g_{i2}\}})\} \\
&= p\{\Pi_{g_r}(f_{\{g_r;g_{i1}\}}(X)) - \Pi_{g_R}(f_{\{g_r;g_{i1}\}}(X))\} \\
&\quad + (1-p)\{\Pi_{g_r}(f_{\{g_r;g_{i2}\}}(X)) - \Pi_{g_R}(f_{\{g_r;g_{i2}\}}(X))\}.
\end{aligned}$$

Note that the expectation is a special case of a distortion risk measure. When the costs are proportional to the expectation, Cheung et al. (2019) studied problem (5) when $p = 1$, i.e., the identity of the insurer is known by the reinsurer: there is symmetric information. Also for the case with costs that are proportional to the expectation, Boonen et al. (2021) studied the case where the premium generating function has to be non-decreasing and for a general value $p \in [0, 1]$.

According to Lemma 2.1, the insurer with distortion function g_{ij} is indifferent regarding the choice of $h_j(z)$ for $z \in [0, M]$ such that $\psi_j(F_X(z)) = 0$; however, the reinsurer may still make a profit by setting $h_j(z) = 1$. Note that if the premium rate $g_r^*(\overline{F}_X(z))$ is not profitable for the reinsurer, the reinsurer would prefer to select a higher premium rate. As studied in Laffont and Martimort (2009), we assume that the insurer is "willing to" improve the welfare of the reinsurer in case the insurer is faced with alternatives that are that the insurer is indifferent with. Therefore, we set $h_i(z) = 1$ in the sequel of this paper, which is also consistent with the setting in Cheung et al. (2019) and Boonen et al. (2021). From this and Lemma 2.1, it follows that problem (5) can be written as:

$$\max_{g_r \in \mathcal{G}} W(g_r) \quad = \quad \max_{g_r \in \mathcal{G}} \int_0^1 [g_r(t) - g_R(t)] \, [p\mathbf{1}_{\{g_r(t) \leq g_{i1}(t)\}} + (1-p)\mathbf{1}_{\{g_r(t) \leq g_{i2}(t)\}}]\nu_X(\mathrm{d}t)$$

[7]Modelling transaction costs proportional to a distortion risk measure is common in finance via bid-ask spreads (Bannor and Scherer, 2014; Eberlein et al., 2014). In optimal reinsurance, the administrative cost is also interpreted as a cost of holding risk capital that is measure by a distortion risk measure (Chi, 2012; Cheung and Lo, 2017).

$$
\begin{aligned}
&= \max_{g_r \in \mathcal{G}} \left\{ p \int_0^1 [g_r(t) - g_R(t)] \, \mathbf{1}_{\{g_r(t) \le g_{i1}(t)\}} \nu_X(\mathrm{d}t) \right. \\
&\qquad \left. + (1-p) \int_0^1 [g_r(t) - g_R(t)] \, \mathbf{1}_{\{g_r(t) \le g_{i2}(t)\}} \nu_X(\mathrm{d}t) \right\} \\
&=: \max_{g_r \in \mathcal{G}} \left\{ pW(g_r, g_{i1}) + (1-p) W(g_r, g_{i2}) \right\}, \qquad (7)
\end{aligned}
$$

where

$$
W(g_r, g_{ij}) = \int_0^1 [g_r(t) - g_R(t)] \, \mathbf{1}_{\{g_r(t) \le g_{ij}(t)\}} \nu_X(\mathrm{d}t), \; j = 1, 2,
$$

and where ν_X is the Radon measure on $[0,1)$ such that $\nu_X([a,b)) = (-F_X^{-1}(1-b)) - (-F_X^{-1}(1-a))$ for $0 \le a < b < 1$.

3 Main result

In this section, we provide our main result for problem (7). We assume $\{g_{i1}, g_{i2}\} \subset \mathcal{G}_d$. It will be helpful to define, for $t \in [0,1]$,

$$
\begin{aligned}
\phi(t) = &\mathbf{1}_{\{g_{i1}(t) > g_{i2}(t)\}} (g_{i2}(t) - (p \cdot g_{i1}(t) + (1-p) g_R(t))) \\
&+ \mathbf{1}_{\{g_{i1}(t) < g_{i2}(t)\}} ((1-p) \cdot g_{i2}(t) + p g_R(t) - g_{i1}(t)).
\end{aligned}
$$

Moreover, define

$$
\begin{aligned}
\mathcal{A} &= \{t \in [0,1] : \phi(t) < 0, g_{i1}(t) \ge g_R(t)\}, \\
\mathcal{B} &= \{t \in [0,1] : \phi(t) = 0, g_{i2}(t) \ge g_R(t)\}, \\
\mathcal{C} &= \{t \in [0,1] : \phi(t) > 0, g_{i2}(t) \ge g_R(t)\}.
\end{aligned}
$$

This allows us to state our main result in the following theorem, which provides the first-best Bowley solutions under asymmetric information.

Theorem 3.1 *The solution set to problem (7) contains those $g_r^* \in \mathcal{G}$ such that*

$$
g_r^*(t) = \begin{cases} g_{i1}(t), & \text{if } t \in \mathcal{A}, \\ \in \{g_{i1}(t), g_{i2}(t)\}, & \text{if } t \in \mathcal{B}, \\ g_{i2}(t), & \text{if } t \in \mathcal{C}, \end{cases}
$$

and

$$
\nu_X \{t \in [0,1] \backslash (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) : g_r^*(t) \le \max\{g_{i1}(t), g_{i2}(t)\}\} = 0.
$$

Moreover, for any of these g_r^*, we have, for $x \in [0, M]$,

$$
\begin{aligned}
f^*_{\{g_r^*, g_{i1}\}}(x) &= \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{A}\}) \\
&\quad + \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{B}, g_r^*(\overline{F}_X(z)) \le g_{i1}(\overline{F}_X(z))\}) \\
&\quad + \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{C}, g_{i1}(\overline{F}_X(z)) > g_{i2}(\overline{F}_X(z))\}),
\end{aligned}
$$

and

$$
\begin{aligned}
f^*_{\{g_r^*, g_{i2}\}}(x) &= \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{A}, g_{i1}(\overline{F}_X(z)) < g_{i2}(\overline{F}_X(z))\}) \\
&\quad + \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{B}, g_r^*(\overline{F}_X(z)) \le g_{i2}(\overline{F}_X(z))\}) \\
&\quad + \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{C}\}).
\end{aligned}
$$

Proof. The proof uses the technique of path-wise optimization. Equation (7) writes as

$$\begin{aligned}
&\max_{g_r\in\mathcal{G}}\int_0^1 [g_r(t)-g_R(t)]\,[p\mathbf{1}_{\{g_r(t)\le g_{i1}(t)\}}+(1-p)\mathbf{1}_{\{g_r(t)\le g_{i2}(t)\}}]\nu_X(\mathrm{d}t)\\
=\;&\max_{g_r\in\mathcal{G}}\int_0^1 [g_r(t)-g_R(t)]\,[p\mathbf{1}_{\{g_{i2}(t)<g_r(t)\le g_{i1}(t)\}}+(1-p)\mathbf{1}_{\{g_{i1}(t)<g_r(t)\le g_{i2}(t)\}}\\
&\qquad\qquad +\mathbf{1}_{\{g_r(t)\le\min\{g_{i1}(t),g_{i2}(t)\}\}}]\nu_X(\mathrm{d}t)\\
=\;&\max_{g_r\in\mathcal{G}}\int_0^1 \psi(g_r(t),t)\nu_X(\mathrm{d}t)\\
\le\;&\int_0^1 \max_{g_r(t)\ge 0}\psi(g_r(t),t)\nu_X(\mathrm{d}t),
\end{aligned}$$

where

$$\psi(g_r(t),t):=\begin{cases}[g_r(t)-g_R(t)]\,[p\mathbf{1}_{\{g_{i2}(t)<g_r(t)\le g_{i1}(t)\}}+\mathbf{1}_{\{g_r(t)\le g_{i2}(t)\}}] & \text{if } g_{i1}(t)>g_{i2}(t),\\ [g_r(t)-g_R(t)]\,\mathbf{1}_{\{g_r(t)\le g_{i1}(t)\}} & \text{if } g_{i1}(t)=g_{i2}(t),\\ [g_r(t)-g_R(t)]\,[(1-p)\mathbf{1}_{\{g_{i1}(t)<g_r(t)\le g_{i2}(t)\}}+\mathbf{1}_{\{g_r(t)\le g_{i1}(t)\}}] & \text{if } g_{i1}(t)<g_{i2}(t).\end{cases}$$

Now, we solve the maximization problem path-wise, and therefore we first fix $t\in[0,1]$. We are next constructing solutions of $\max_{g_r(t)\ge 0}\psi(g_r(t),t)$. We separate three different cases.

(i) $g_{i1}(t)>g_{i2}(t)$: Then, $\psi(g_r(t),t):=[g_r(t)-g_R(t)]\,[p\mathbf{1}_{\{g_{i2}(t)<g_r(t)\le g_{i1}(t)\}}+\mathbf{1}_{\{g_r(t)\le g_{i2}(t)\}}]$. For all $t\in[0,1]$, it holds that $\psi(\cdot,t)$ is strictly increasing on $[0,g_{i2}(t)]$ and on $(g_{i2}(t),g_{i1}(t)]$, and $\psi(\cdot,t)=0$ on $(g_{i1}(t),\infty)$. Thus, the maximum value of $\psi(\cdot,t)$ is either located at the possible discontinuities, $g_{i2}(t)$ and $g_{i1}(t)$, or it is zero. Hence,

$$\begin{aligned}
\max_{g_r(t)\ge 0}\psi(g_r(t),t) &= \max\{p(g_{i1}(t)-g_R(t)),g_{i2}(t)-g_R(t),0\}\\
&= \max\{p(g_{i1}(t)-g_R(t))\\
&\qquad +\max\{0,g_{i2}(t)-(p\cdot g_{i1}(t)+(1-p)g_R(t))\},0\}\\
&= \max\{p(g_{i1}(t)-g_R(t))+\max\{0,\phi(t)\},0\}.
\end{aligned}$$

Hence, if $\phi(t)<0$ and $g_{i1}(t)\ge g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)$ is solved by $g_r(t)=g_{i1}(t)$. Likewise, if $\phi(t)>0$ and $g_{i2}(t)\ge g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)$ is solved by $g_r(t)=g_{i2}(t)$. If $\phi(t)=0$ and $g_{i2}(t)\ge g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)$ is solved by either $g_r(t)=g_{i1}(t)$ or $g_r(t)=g_{i2}(t)$. Finally, if $g_{i1}(t)<g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)=0$, and it is thus solved by any $g_r(t)>g_{i1}(t)$.

(ii) $g_{i1}(t)=g_{i2}(t)$: Then, $\psi(g_r(t),t):=[g_r(t)-g_R(t)]\,\mathbf{1}_{\{g_r(t)\le g_{i1}(t)\}}$. For all $t\in[0,1]$, it holds that $\psi(\cdot,t)$ is strictly increasing on $[0,g_{i1}(t)]$, and $\psi(\cdot,t)=0$ on $(g_{i1}(t),\infty)$. Thus, the maximum value of $\psi(\cdot,t)$ is either located at $g_{i1}(t)$, or it is zero. Hence,

$$\max_{g_r(t)\ge 0}\psi(g_r(t),t) = \max\{g_{i1}(t)-g_R(t),0\}.$$

Recall that $\phi(t)=0$. If $g_{i1}(t)\ge g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)$ is solved by $g_r(t)=g_{i1}(t)=g_{i2}(t)$. If $g_{i1}(t)<g_R(t)$, then $\max_{g_r(t)\ge 0}\psi(g_r(t),t)=0$, and is thus solved by any $g_r(t)>g_{i1}(t)$.

(iii) $g_{i1}(t) < g_{i2}(t)$: Then, $\psi(g_r(t), t) := [g_r(t) - g_R(t)]\,[(1-p)\mathbf{1}_{\{g_{i1}(t)<g_r(t)\leq g_{i2}(t)\}} + \mathbf{1}_{\{g_r(t)\leq g_{i1}(t)\}}]$. For all $t \in [0,1]$, it holds that $\psi(\cdot, t)$ is strictly increasing on $[0, g_{i1}(t)]$ and on $(g_{i1}(t), g_{i2}(t)]$, and $\psi(\cdot, t) = 0$ on $(g_{i2}(t), \infty)$. Thus, the maximum value of $\psi(\cdot, t)$ is either located at the possible discontinuities, $g_{i1}(t)$ and $g_{i2}(t)$, or it is zero. Hence,

$$\begin{aligned}\max_{g_r(t)\geq 0} \psi(g_r(t), t) &= \max\{(1-p)(g_{i2}(t) - g_R(t)), g_{i1}(t) - g_R(t), 0\} \\ &= \max\{(1-p)(g_{i2}(t) - g_R(t)) \\ &\qquad + \max\{0, g_{i1}(t) - ((1-p)\cdot g_{i2}(t) + p g_R(t))\}, 0\} \\ &= \max\{(1-p)(g_{i2}(t) - g_R(t)) + \max\{0, -\phi(t)\}, 0\}.\end{aligned}$$

Hence, if $\phi(t) < 0$ and $g_{i1}(t) \geq g_R(t)$, then $\max_{g_r(t)\geq 0} \psi(g_r(t), t)$ is solved by $g_r(t) = g_{i1}(t)$. Likewise, if $\phi(t) > 0$ and $g_{i2}(t) \geq g_R(t)$, then $\max_{g_r(t)\geq 0} \psi(g_r(t), t)$ is solved by $g_r(t) = g_{i2}(t)$. If $\phi(t) = 0$ and $g_{i2}(t) \geq g_R(t)$, then $\max_{g_r(t)\geq 0} \psi(g_r(t), t)$ is solved by either $g_r(t) = g_{i1}(t)$ or $g_r^*(t) = g_{i2}(t)$. Finally, if $g_{i1}(t) < g_R(t)$, then $\max_{g_r(t)\geq 0} \psi(g_r(t), t) = 0$, and is thus solved by any $g_r(t) > g_{i1}(t)$.

Now we constructed the solutions of $\max_{g_r(t)\geq 0} \psi(g_r(t), t)$ for all $t \in [0,1]$. Let $g_r^* \in \mathcal{G}$ such that it solves $\max_{g_r(t)\geq 0} \psi(g_r(t), t)$ for all $t \in (0,1]$. Note that $g_r^*(0) = 0$ is a solution to $\max_{g_r(0)\geq 0} \psi(g_r(0), 0) = 0$. Thus,

$$\begin{aligned}\int_0^1 \max_{g_r(t)\geq 0} \psi(g_r(t), t)\nu_X(\mathrm{d}t) &= \int_0^1 \psi(g_r^*(t), t)\nu_X(\mathrm{d}t) \\ &\leq \max_{g_r\in\mathcal{G}} \int_0^1 \psi(g_r(t), t)\nu_X(\mathrm{d}t) \\ &\leq \int_0^1 \max_{g_r(t)\geq 0} \psi(g_r(t), t)\nu_X(\mathrm{d}t).\end{aligned}$$

Thus, the inequalities can be replaced by equalities, which concluded the proof of the premium generating functions g_r^* in first-best Bowley solutions.

For a fixed premium generating function g_r^*, the optimal indemnity functions $f^*_{\{g_r^*, g_{i1}\}}$ and $f^*_{\{g_r^*, g_{i2}\}}$ follow from Lemma 2.1. This concludes the proof. ■

While the function ϕ is merely used as ancillary function to construct the first-best Bowley solutions, it has an interpretation. At a given value $t \in [0,1]$, $\phi(t)$ is the *marginal profit* that the reinsurer makes by choosing $g_r^*(t) = g_{i2}(t)$ instead of $g_r^*(t) = g_{i1}(t)$. So, if $\phi(t)$ is positive (negative), then it is profitable for the reinsurer to select the premium generating function $g^*(t)$ that makes the type 2 (1) insurer indifferent between buying or not buying marginal reinsurance. While, for the marginal profit, reinsurance prices often make one type of insurer "indifferent", this does not imply that the insurer will be indifferent between insuring or not insuring. In fact, since it may hold that $g_r^*(t) < g_{ij}(t)$ for some $t \in [0,1]$, the insurer can strictly profit from buying reinsurance.

Note that $(f^*_{\{g_r^*, g_{ij}\}})'(z) > 0$ for some $j \in \{1,2\}$ implies for $z \in [0, M]$ a.e. that $\overline{F}_X(z) \in \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$. Then, the solution in Theorem 3.1 is such that $g_r^*(\overline{F}_X(z)) \geq g_R(\overline{F}_X(z)) \geq \overline{F}_X(z)$ for $z \in [0, M]$ a.e. As a direct consequence of this chain of inequality, it must hold that $\Pi_{g_r^*}(f^*_{\{g_r^*, g_{ij}\}}(X)) \geq \mathbb{E}[f^*_{\{g_r^*, g_{ij}\}}(X)]$. In fact, a solution in Theorem 3.1 can always be chosen such that $g_r^*(t) \geq t$ for all $t \in [0,1]$. This observation implies that an optimal premium principle is such that the premium always exceeds the expected value of the insurable loss.

We have assumed that the insurer uses two possible distortion risk measures that the insurer is endowed with: g_{i1} and g_{i2}. It is important to note that our results can be generalized straightforwardly for finitely many possible distortion risk measures of the insurer. We provide optimal solutions for such case in Appendix A, but prefer to further omit this case in the main text to avoid cumbersome expressions. Note that in Boonen et al. (2021), it is not trivial to extend the optimal reinsurance contracts beyond the case of two possible distortion risk measures of the insurer unless the distortion risk measures are all equal to a VaR.

4 An example with ordered distortion functions of the insurer

In this section, we provide an example for the special case where $g_{i1}(t) \geq g_{i2}(t)$ for all $t \in [0,1]$. For ease of implementing the calculation and comparisons on the net gains between the result of Theorem 3.1 and that in Theorem 5.1 of Boonen et al. (2021), we set $\hat{g}_R(t) = t$ so that $g_R(t) = (1+\gamma)t$ and $\Pi_{\hat{g}_R}(f_{\{g_r;g_{ij}\}}(X)) = (1+\gamma)\mathbb{E}[f_{\{g_r;g_{ij}\}}(X)]$, $j = 1,2$. In this case, the function ϕ simplifies as

$$\phi(t) = g_{i2}(t) - (p \cdot g_{i1}(t) + (1-p)(1+\gamma)t), \quad t \in [0,1].$$

From Theorem 3.1, we get that, for any optimal g_r^*, we have, for $x \in [0, M]$,

$$\begin{aligned} f^*_{\{g_r^*,g_{i1}\}}(x) &= \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{A}\}) \\ &\quad + \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{B}, g_r^*(\overline{F}_X(z)) = g_{i1}(\overline{F}_X(z))\}), \end{aligned}$$

and

$$f^*_{\{g_r^*,g_{i2}\}}(x) = \mu(\{z \in [0,x] \mid \overline{F}_X(z) \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\}).$$

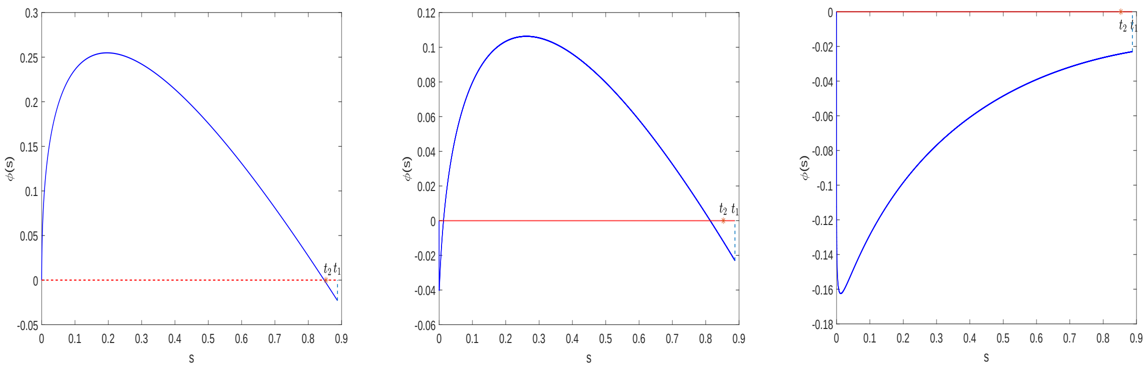
As studied in Boonen et al. (2021), we define the *second-best* Bowley solutions as the solutions of problem (7), but now we restrict the set of feasible premium generating functions to $g_r \in \mathcal{G}_{nd}$, where

$$\mathcal{G}_{nd} = \{g : [0,1] \mapsto \mathbb{R}_+ | g(0) = 0, g \text{ is bounded, left-continuous and non-decreasing}\}.$$

Note that in Theorem 3.1, we allow for the situation where g_r^* might be decreasing on some interval. We will highlight this in the numerical examples in the next two sections. Because $\mathcal{G}_{nd} \subset \mathcal{G}$, it holds that first-best Bowley solutions, as shown in Theorem 3.1, yield a menu of reinsurance contracts from which the reinsurer may benefit more compared with the second-best Bowley solutions. In other words, $W(g_r^*)$ is larger in the for the first-best Bowley solutions than for the second-best Bowley solutions. The insurer, on the other hand, may however be better off in a second-best Bowley solution. Also note that if a premium generating function $g_r^* \in \mathcal{G}_{nd}$ constitutes a first-best Bowley solution, then g_r^* also constitutes a second-best Bowley solution, and the net profit for the reinsurer $W(g_r^*)$ is the same in both solutions.

The next example serves as an illustration of the result in Theorem 3.1. As will be observed, the value of probability p plays a key role in determining the optimal premium generating function and the corresponding ceded loss functions.

Example 4.1 *Suppose that the distortion functions of the insurer are given by $g_{i1}(t) = t^{\beta_1}$ and $g_{i2}(t) = t^{\beta_2}$, for $t \in [0,1]$, where $\beta_1 = 0.2$ and $\beta_2 = 0.4$. Clearly, $g_{i1}(t) \geq g_{i2}(t)$, for all $t \in [0,1]$. Let $\gamma = 0.1$. Then, the solutions of the equations $g_{i1}(t) = g_R(t)$ and $g_{i2}(t) = g_R(t)$ on $t \in (0,1)$ can be calculated as $t_1 = 0.8877$ and $t_2 = 0.8531$, respectively. Assume that the risk X has an exponential distribution with expectation 1. According to definitions of the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we need to determine first the signs of the function $\phi(t)$ for $t \in [0, t_1]$, and then get the explicit expressions of these three sets. Consider the following three examples of the probability p:*



(a) $p = 0.1$ (b) $p = 0.4$ (c) $p = 0.8$

Figure 1: Plot of the function $\phi(t)$ on $t \in [0, t_1]$ for three different values of p. Corresponding to Example 4.1.

(i) $p = 0.1$. Figure 1(a) plots the function $\phi(t)$ on $t \in [0, t_1]$. From this figure it can be observed that $\phi(t) = 0$ has a unique solution on $[0, t_1]$, which is given by $t_3 = 0.8478$. Moreover, $\phi(t) > 0$ for $t \in (0, t_3)$ and $\phi(t) < 0$ for $t \in (t_3, t_1)$. Hence, we have $\mathcal{A} = (t_3, t_1]$, $\mathcal{B} = \{0, t_3\}$, and $\mathcal{C} = (0, t_3)$. Premium generating functions g_r^* in first-best Bowley solutions are then given by

$$g_r^*(t) = \begin{cases} t^{0.4}, & \text{if } t \in [0, 0.8478], \\ t^{0.2}, & \text{if } t \in (0.8478, 0.8877], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.8877, 1] : \tilde{g}_r(t) \leq t^{0.2}\} = 0, \end{cases}$$

which can be chosen such that $g_r^* \in \mathcal{G}_{nd}$. These premium generating functions $g_r^* \in \mathcal{G}_{nd}$ constitute also second-best Bowley solutions. Furthermore, the optimal ceded loss functions are given by

$$f^*_{\{g_r^*, g_{i1}\}}(x) \quad = \quad \mu(\{z \in [0, x] \mid e^{-z} \in (t_3, t_1]\})$$

$$
\begin{aligned}
&= \mu(\{z \in [0,x] \mid z \in [-\ln(t_1), -\ln(t_3))\}) \\
&= \min\{(x+\ln(t_1))_+, \ln(t_1/t_3)\} \\
&= \min\{(x-0.1191)_+, 0.0460\}, \quad x \in [0,M].
\end{aligned}
$$

where $y_+ := \max\{y, 0\}$, *and*

$$f^*_{\{g_r^*, g_{i2}\}}(x) = (x-0.1191)_+, \quad x \in [0,M].$$

This means that traditional stop-loss policy is provided for the type 2 insurer, while a limited stop-loss reinsurance contract is designed for the type 1 insurer. Moreover, the expected net profit acquired by the insurer can be calculated as $W(g_r^*) = 1.4077$.

(ii) $p = 0.4$. *For this case, the plot of* $\phi(t)$ *for* $t \in [0, t_1]$ *is shown in Figure 1(b). The equation* $\phi(t) = 0$ *has two solutions on* $[0, t_1]$, *which are* $t_3 = 0.0132$ *and* $t_4 = 0.8136$. *Moreover,* $\phi(t) > 0$ *for* $t \in (t_3, t_4)$ *and* $\phi(t) < 0$ *for* $t \in (0, t_3) \cup (t_4, t_1)$, *which lead to* $\mathcal{A} = (0, t_3) \cup (t_4, t_1]$, $\mathcal{B} = \{0, t_3, t_4\}$, *and* $\mathcal{C} = (t_3, t_4)$. *Then, premium generating functions* g_r^* *in first-best Bowley solutions are given by*

$$g_r^*(t) = \begin{cases} t^{0.2}, & \text{if } t \in [0, 0.0132] \cup (0.8136, 0.8877], \\ t^{0.4}, & \text{if } t \in (0.0132, 0.8136], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.8877, 1] : \tilde{g}_r(t) \le t^{0.2}\} = 0. \end{cases}$$

Obviously, any g_r^* *as specified above is not non-decreasing, which means that the functions* g_r^* *do not constitute second-best Bowley solutions. Moreover, the corresponding ceded loss function for both types of the insurer is given by*

$$
\begin{aligned}
f^*_{\{g_r^*, g_{i1}\}}(x) &= \mu(\{z \in [0,x] \mid e^{-z} \in [0, t_3] \cup [t_4, t_1]\}) \\
&= \min\{(x-0.1191)_+, 0.0871\} + (x-4.3275)_+, \quad x \in [0,M],
\end{aligned}
$$

and

$$f^*_{\{g_r^*, g_{i2}\}}(x) = (x-0.1191)_+, \quad x \in [0,M].$$

Hence, the stop-loss contract is signed between the reinsurer and the type 2 insurer, while a two layer stop-loss policy is provided for the type 1 insurer. Besides, the expected net profit for the reinsurer is $W(g_r^*) = 1.8158$.

(iii) $p = 0.8$. *The plot of* $\phi(t)$ *for* $t \in [0, t_1]$ *is shown in Figure 1(c). It can be seen that* $\mathcal{A} = (0, t_1]$, $\mathcal{B} = \{0\}$, *and* $\mathcal{C} = \emptyset$. *Premium generating functions* g_r^* *in first-best Bowley solutions are then given by*

$$g_r^*(t) = \begin{cases} t^{0.2}, & \text{if } t \in [0, 0.8877], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.8877, 1] : \tilde{g}_r(t) \le t^{0.2}\} = 0. \end{cases}$$

Thus, a pooling reinsurance contract is provided for both types of the insurer

$$f^*_{\{g_r^*, g_{i1}\}}(x) = f^*_{\{g_r^*, g_{i2}\}}(x) = (x-0.1191)_+, \quad x \in [0,M].$$

The profit acquired by the reinsurer can be computed as $W(g_r^*) = 3.1247$.

5 Examples with general distortion functions of the insurer

In this section, we present another two numerical examples to validate the finding in Theorem 3.1. We first consider the situation where the type 1 insurer adopts a VaR measure and the type 2 insurer uses a convex distortion risk measure. Setting $\hat{g}_R(t) = t$, we have $g_R(t) = (1+\gamma)t$ and $\Pi_{g_R}(f_{\{g_r;g_{ij}\}}(X)) = (1+\gamma)\mathbb{E}[f_{\{g_r;g_{ij}\}}(X)]$, $j = 1, 2$.

Example 5.1 *Suppose that the distortion functions of the insurer may take the expressions of* $g_{i1}(t) = \mathbf{1}_{\{1-\alpha<t\leq 1\}}$ *and* $g_{i2}(t) = t^\beta$, *for* $t \in [0,1]$, *where* $\beta = 0.2$. *Thus, it can be seen that* $g_{i1}(t) \geq g_{i2}(t)$, *for all* $t \in (1-\alpha, 1]$, *and* $g_{i1}(t) \leq g_{i2}(t)$, *for all* $t \in [0, 1-\alpha]$. *The solution of the equation* $g_{i2}(t) = g_R(t)$ *on* $t \in (0,1)$ *can be calculated as* $t_1 = (1+\gamma)^{1/(\beta-1)}$. *Assume that the risk* X *has an exponential distribution with expectation 1. Consider the following two cases of the confidence level* α *and cost coefficient* γ*:*

(i) Suppose $\alpha = 0.7$ *and* $\gamma = 3$. *From Theorem 3.1, it can be checked that, for any* $p \in [0,1]$, $\mathcal{A} = \emptyset$, $\mathcal{B} = \emptyset$, *and* $\mathcal{C} = [0, t_1]$, *where* $t_1 = 0.1768$. *Premium generating functions* g_r^* *in first-best Bowley solutions are given by* $g_r^*(t) = t^{0.2}$ *for* $t \in [0, 0.1768]$; *otherwise,* $\nu_X\{t \in (0.1768, 1] : g_r^*(t) \leq \max\{\mathbf{1}_{\{0.3<t\leq 1\}}, t^{0.2}\}\} = 0$. *Then,* $f^*_{\{g_r^*, g_{i1}\}}(X) = 0$ *and* $f^*_{\{g_r^*, g_{i2}\}}(x) = (x - 1.7327)_+$, *for* $x \in [0, M]$, *which means that a shutdown policy is provided for the type 1 insurer, and a stop-loss contract is ceded to the type 2 insurer. The expected net profit for the reinsurer is* $W(g_r^*) = 2.8284(1-p)$, *which depends on the probability that the insurer is of type 2. Since there exist first-best Bowley solutions that coincide with the second-best Bowley solutions, the expected net profits for the reinsurer in the second-best Bowley solutions coincide with the first-best Bowley solutions.*

(ii) Suppose $\alpha = 0.9$ *and* $\gamma = 0.1$. *Define* $t_2 = 1/(1+\gamma) = 0.9091$, *which is a solution of* $g_{i1}(t) = g_R(t)$. *First, it can be seen that* $\phi(t) > 0$ *always holds on* $t \in [0, 1-\alpha]$. *Also,* $(t_2, 1] \subset [0,1] \backslash (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, *and thus* $\nu_X\{t \in (t_2, 1] : g_r^*(t) \leq 1\} = 0$. *Now, for* $t \in [1-\alpha, t_2]$, *we consider the following three cases of the probability p:* $p = 0.3$, $p = 0.7$, *and* $p = 0.9$.

(a) $p = 0.3$. *For this case, it can found that the solution of* $\phi(t) = 0$ *on* $t \in (0.1, 0.9091]$ *is* $t_3 = 0.8748$ *(see Figure 2(a)). Hence,* $\mathcal{A} = (0.8748, 0.9091]$, $\mathcal{B} = \{0.8748\}$, *and* $\mathcal{C} = [0, 0.8748)$. *Then, according to Theorem 3.1, premium generating functions* g_r^* *in first-best Bowley solutions are given by*

$$g_r^*(t) = \begin{cases} t^{0.2}, & \textit{if } t \in [0, 0.8748], \\ 1, & \textit{if } t \in (0.8748, 0.9091], \\ \tilde{g}_r(t), & \textit{such that } \nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \leq 1\} = 0. \end{cases}$$

Further, the corresponding ceded loss functions for the two types of the insurer can be calculated as

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 2.2073\}, \quad x \in [0, M],$$

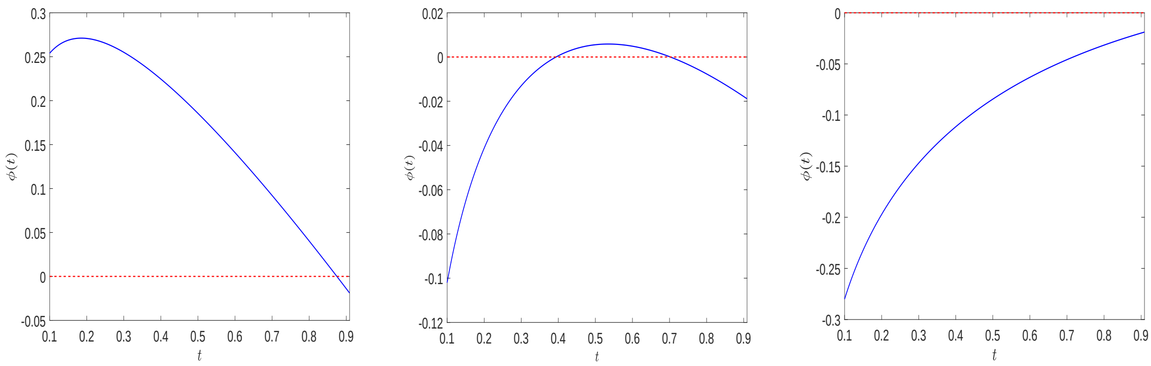


Figure 2: Plot of the function $\phi(t)$ on $t \in [1-\alpha, t_2]$ for three different values of p. Corresponding to Example 5.1.

and

$$f^*_{\{g^*_r, g_{i2}\}}(x) = (x - 0.0953)_+, \quad x \in [0, M].$$

The net gain obtained by the reinsurer can be calculated as $W(g^*_r) = 2.9925$. *It can be verified that the optimal ceded loss functions in this case coincide with the second-best Bowley solutions, as there exists a first-best Bowley solution with* $g^*_r \in \mathcal{G}_{nd}$.

(b) $p = 0.7$. *As displayed in Figure 2(b),* $\phi(t) = 0$ *has two solutions on* $t \in (0.1, 0.9091]$, *which are given by 0.3938 and 0.7024. Then* $\mathcal{A} = (0.1, 0.3938) \cup (0.7024, 0.9091]$, $\mathcal{B} = \{0.3938, 0.7024\}$, *and* $\mathcal{C} = [0, 0.1) \cup (0.3938, 0.7024)$. *Thus, premium generating functions* g^*_r *in first-best Bowley solutions are given by*

$$g^*_r(t) = \begin{cases} 1, & \text{if } t \in (0.1, 0.3938] \cup (0.7024, 0.9091], \\ t^{0.2}, & \text{if } t \in [0, 0.1] \cup (0.3938, 0.7024], \\ \tilde{g}_r(t), & \text{if } t \in (0.9091, 1], \end{cases}$$

where $\tilde{g}_r(t)$ *is such that* $\nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \le 1, \text{ for } t \in (0.9091, 1]\} = 0$. *Obviously, the function* g^*_r *is not monotone on* $[0, 1]$. *As a result, the optimal ceded loss function for the type 1 insurer admits the limited stop-loss reinsurance contract*

$$f^*_{\{g^*_r, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 2.2073\}, \quad x \in [0, M],$$

and for the type 2 insurer admits the two-layer stop-loss reinsurance contract

$$f^*_{\{g^*_r, g_{i2}\}}(x) = \min\{(x - 0.3534)_+, 0.5787\} + (x - 2.3026)_+, \quad x \in [0, M].$$

Then the expected net profit for the reinsurer under this reinsurance contract is $W(g^*_r) = 1.8378$. *For the same example, ceded loss functions in the second-best Bowley solutions are given by*

$$\tilde{f}^*_{\{g_r, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 2.2073\}, \quad x \in [0, M],$$

and $\tilde{f}^*_{\{g_r,g_{i2}\}}(x) = (x-0.3534)_+$, *for* $x \in [0, M]$, *where we use Case 3 of Theorem 4.1(i) of Boonen et al. (2021). Then, the corresponding expected net profit for the reinsurer can be computed as 1.7764. Thus, the expected net profit for the reinsurer under the first-best strategy is larger than that under the second-best policy, which is mainly caused by the assumption that the premium generating function might be decreasing under some interval.*

(c) $p = 0.9$. *Under this setting, it can be checked from Figure 2(c) that* $\phi(t) < 0$ *for all* $t \in (0.1, 0.9091]$. *Thus,* $\mathcal{A} = (0.1, 0.9091]$, $\mathcal{B} = \emptyset$, *and* $\mathcal{C} = [0, 0.1]$. *Then we have*

$$g_r^*(t) = \begin{cases} 1, & \text{if } t \in (0.1, 0.9091], \\ t^{0.2}, & \text{if } t \in [0, 0.1], \\ \tilde{g}_r(t), & \text{if } t \in (0.9091, 1], \end{cases}$$

where $\tilde{g}_r(t)$ *is such that* $\nu_X\{t \in [0, 0.1] : \tilde{g}_r(t) \le t^{0.2},\ \text{for } t \in [0, 0.1]\} = 0$ *and* $\nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \le 1,\ \text{for } t \in (0.9091, 1]\} = 0$. *The corresponding reinsurance indemnity functions are given by*

$$f^*_{\{g_r^*,g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 2.2073\}, \quad x \in [0, M],$$

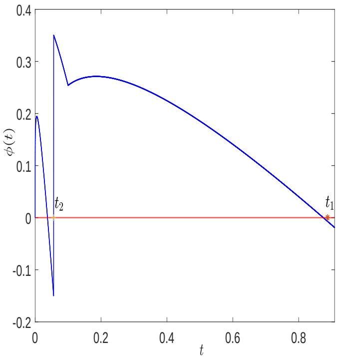
and $f^*_{\{g_r^*,g_{i2}\}}(x) = (x - 0.3534)_+$, *for* $x \in [0, M]$. *The expected net profit for the reinsurer is* $W(g_r^*) = 1.49$. *Since there exist first-best Bowley solutions with* $g_r^* \in \mathcal{G}_{nd}$, *the second-best Bowley solutions yield the same expected net profit for the reinsurer.*

As seen in Example 5.1, it is worth addressing that Theorem 3.1 not only improves the welfare gains of the reinsurer in the reinsurance contract, but also solves the Bowley solution in much more general cases than Boonen et al. (2021) (cf. Theorem 5.1 therein).

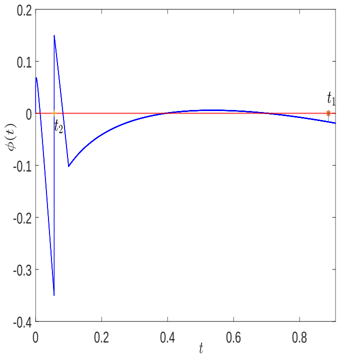
In the next example, we consider the case when the insurer adopts a convex distortion risk measure or a TVaR measure, which was not studied by Boonen et al. (2021).

Example 5.2 *Suppose the insurer adopts either* $g_{i1}(t) = \min\{t/(1-\alpha), 1\}$ *or* $g_{i2}(t) = t^\beta$, *for* $t \in [0, 1]$, *where* $\beta = 0.2$. *Thus, it can be seen that* $g_{i1}(t) \ge g_{i2}(t)$, *for all* $t \in (t_2, 1]$, *and* $g_{i1}(t) \le g_{i2}(t)$, *for all* $t \in [0, t_2]$, *where* $t_2 = (1-\alpha)^{1/(1-\beta)}$ *is the intersection point of* $g_{i1}(t)$ *and* $g_{i2}(t)$ *on* $t \in (0, 1)$. *The solution of* $g_{i2}(t) = g_R(t)$ *on* $t \in (0, 1)$ *can be calculated as* $t_1 = (1+\gamma)^{1/(\beta-1)}$. *Let* $t_3 = 1/(1+\gamma)$. *Assume that the risk* X *has an exponential distribution with expectation 1.*

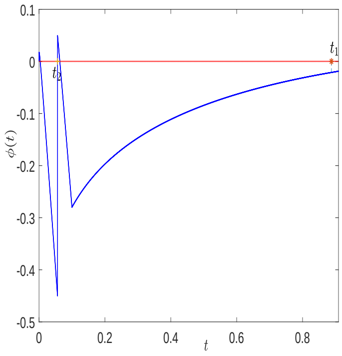
(i) *Let* $\alpha = 0.7$ *and* $\gamma = 3$. *It holds that, for any* $p \in [0, 1]$, $\mathcal{A} = \emptyset$, $\mathcal{B} = \emptyset$, *and* $\mathcal{C} = [0, t_1]$, *where* $t_1 = 0.1768$. *This case is very similar to case (i) in Example 5.1. Premium generating functions in first-best Bowley solutions are then given by* $g_r^*(t) = t^{0.2}$ *for* $t \in [0, 0.1768]$; *otherwise,* $\nu_X\{t \in (0.1768, 1] : g_r^*(t) \le \max\{\min\{t/0.3, 1\}, t^{0.2}\}\} = 0$. *Then,* $f^*_{\{g_r^*,g_{i1}\}}(X) = 0$ *and* $f^*_{\{g_r^*,g_{i2}\}}(x) = (x - 1.7327)_+$, *for* $x \in [0, M]$. *Hence, a "shutdown policy" is provided for the type 1 insurer, and a stop-loss contract is ceded to the type 2 insurer. Then, the expected net profit for the reinsurer is given by* $W(g_r^*) = 2.8284(1-p)$, *which is linear in the probability* $1-p$ *since the reinsurer shuts down the contract for the type 1 insurer.*



(a) $p = 0.3$



(b) $p = 0.7$



(c) $p = 0.9$

Figure 3: Plot of the function $\phi(t)$ on $t \in [0, t_3]$ for three different values of p. Corresponding to Example 5.2.

(ii) Let $\alpha = 0.9$ and $\gamma = 0.1$. Consider the following three cases of the probability p: $p = 0.3$, $p = 0.7$, and $p = 0.9$.

(a) $p = 0.3$. For this setting, the plot of $\phi(t)$ on $t \in [0, t_3]$ is shown in Figure 3(a), from which we see that $\phi(t) = 0$ has two solutions on $t \in (0, t_3)$, namely 0.0375 and 0.8748. Then, we can obtain that $\mathcal{A} = (0.0375, 0.0562] \cup (0.8748, 0.9091]$, $\mathcal{B} = \{0, 0.0375, 0.8748\}$, and $\mathcal{C} = (0, 0.0375) \cup (0.0562, 0.8748)$. According to Theorem 3.1, premium generating functions g_r^* in first-best Bowley solutions are given by

$$g_r^*(t) = \begin{cases} t^{0.2}, & \text{if } t \in [0, 0.0375] \cup (0.0562, 0.8748], \\ \min\{10t, 1\}, & \text{if } t \in (0.0375, 0.0562] \cup (0.8748, 0.9091], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \leq 1\} = 0. \end{cases}$$

Then, the corresponding ceded loss functions for the two types of the insurer can be derived as

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 3.1881\}, \quad x \in [0, M],$$

and

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \min\{(x - 0.0953)_+, 2.7835\} + (x - 3.2834)_+, \quad x \in [0, M].$$

This means that a layer stop-loss contract is provided for the type 1 insurer, while a two-layer stop-loss contract is ceded to the type 2 insurer. Moreover, the expected net profit for the reinsurer is given by $W(g_r^*) = 3.1089$.

(b) $p = 0.7$. For this case, Figure 3(b) presents the plot of $\phi(t)$ on $t \in [0, t_3]$, from which we see that $\phi(t) = 0$ has four solutions on $t \in (0, t_3)$, that are 0.0138, 0.0829, 0.3938 and 0.7023. Thus, we have $\mathcal{A} = (0.0138, 0.0562] \cup (0.0829, 0.3938] \cup (0.7023, 0.9091]$, $\mathcal{B} = \{0, 0.0138, 0.0829, 0.3938, 0.7023\}$, and

$\mathcal{C} = (0, 0.0138) \cup (0.0562, 0.0829) \cup (0.3938, 0.7023)$. *Further, premium generating functions g_r^* in first-best Bowley solutions are given by*

$$g_r^*(t) = \begin{cases} t^{0.2}, & \text{if } t \in [0, 0.0138] \cup (0.0562, 0.0829] \cup (0.3938, 0.7023], \\ \min\{10t, 1\}, & \text{if } t \in (0.0138, 0.0562] \cup (0.0829, 0.3938] \cup (0.7023, 0.9091], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \leq 1\} = 0. \end{cases}$$

Therefore, the corresponding ceded loss functions for the two types of the insurer are

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 4.1878\}, \quad x \in [0, M],$$

and

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \min\{(x - 0.0953)_+, 2.7835\} + (x - 4.2831)_+, \quad x \in [0, M].$$

This set of contracts is similar with the one in case (a). Furthermore, the net gain obtained by the insurer is $W(g_r^*) = 2.2385$.

(c) $p = 0.9$. *For this case, the solution of* $\phi(t) = 0$ *on* $t \in (0, t_3)$ *can be calculated as 0.0036 and 0.0632 (see Figure 3(c)). Thus, we have* $\mathcal{A} = (0.0036, 0.0562] \cup (0.0632, 0.9091]$, $\mathcal{B} = \{0, 0.0036, 0.0632\}$, *and* $\mathcal{C} = (0, 0.0036) \cup (0.0562, 0.0632)$. *Premium generating functions g_r^* in first-best Bowley solutions are then given by*

$$g_r^*(t) = \begin{cases} t^{0.2}, & \text{if } t \in [0, 0.0036] \cup (0.0562, 0.0632], \\ \min\{10t, 1\}, & \text{if } t \in (0.0036, 0.0562] \cup (0.0632, 0.9091], \\ \tilde{g}_r(t), & \text{such that } \nu_X\{t \in (0.9091, 1] : \tilde{g}_r(t) \leq 1\} = 0. \end{cases}$$

Hence, the corresponding ceded loss functions for the two types of the insurer have the expressions

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 5.5315\}, \quad x \in [0, M],$$

and

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \min\{(x - 0.0953)_+, 2.7835\} + (x - 5.6268)_+, \quad x \in [0, M].$$

This menu of reinsurance contracts is also consistent with the findings both in cases (a) and (b). However, the shape of the optimal premium generating function g_r^* for cases (a) and (c) is very different from that in case (b). For this case (c), the expected net profit for the reinsurer is $W(g_r^*) = 2.1694$.

6 Extensions to heterogeneous beliefs

In this section, we study the first-best Bowley solutions under heterogeneous beliefs of both parties on the probability distributions of X. Belief heterogeneity in optimal reinsurance has been motivated by several papers, including Boonen (2016), Ghossoub (2017), Chi (2019), and Boonen and Ghossoub (2020).

Let the reinsurer use probability measure $\mathbb{P}_R$ on the measurable space $(\Omega, \mathcal{F})$, and the type-i insurer uses $\mathbb{P}_i$ on $(\Omega, \mathcal{F})$, $i \in \{1, 2\}$. For ease of presentation, we denote $\overline{F}_X^R(z) = \mathbb{P}_R(X > z)$ and $\overline{F}_X^i(z) = \mathbb{P}_i(X > z)$, $i \in \{1, 2\}$. Moreover, we assume that $\mathbb{P}_i \ll \mathbb{P}_R$, i.e., $\mathbb{P}_i$ is absolutely continuous with respect to $\mathbb{P}_R$. Similar to (2), we then write the premium charged by the reinsurer as follows:

$$\hat{\Pi}_{g_r}(f(X)) = \int_0^{f(M)} g_r(\overline{F}_{f(X)}^R(z))\mathrm{d}z = \int_0^M g_r(\overline{F}_X^R(z))h(z)\mathrm{d}z,$$

where h satisfies (1) and $g_r \in \mathcal{G}$ is the premium generating function adopted by the reinsurer. Then, the optimization problem of interest is

$$\begin{aligned}\max_{g_r\in\mathcal{G}} W(g_r) := \max_{g_r\in\mathcal{G}}\{p\{\mathbb{E}[\hat{\Pi}_{g_r}(f_{\{g_r;g_{i1}\}}(X)) - f_{\{g_r;g_{i1}\}}(X)] - C(f_{\{g_r;g_{i1}\}})\} \\ + (1-p)\{\mathbb{E}[\hat{\Pi}_{g_r}(f_{\{g_r;g_{i2}\}}(X)) - f_{\{g_r;g_{i2}\}}(X)] - C(f_{\{g_r;g_{i2}\}})\}\}, \quad (8)\end{aligned}$$

where the ceded loss functions $f_{\{g_r;g_{ij}\}}, j \in \{1, 2\}$, solve

$$\min_{f\in\mathcal{F}} \rho_{g_j}^{\mathbb{P}_j}(X - f(X) + \hat{\Pi}_{g_r}(f(X))),$$

where $\rho_{g_j}^{\mathbb{P}_j}$ is a distortion risk measure as in (3), but with the probability distribution $\mathbb{P}_j$.

It will be helpful to define, for $z \in [0, M]$,

$$\begin{aligned}\hat{\phi}(z) = \mathbf{1}_{\{g_{i1}(\overline{F}_X^1(z))>g_{i2}(\overline{F}_X^2(z))\}}(g_{i2}(\overline{F}_X^2(z)) - (p\cdot g_{i1}(\overline{F}_X^1(z)) + (1-p)g_R(\overline{F}_X^R(z)))) \\ + \mathbf{1}_{\{g_{i1}(\overline{F}_X^1(z))<g_{i2}(\overline{F}_X^2(z))\}}((1-p)\cdot g_{i2}(\overline{F}_X^2(z)) + pg_R(\overline{F}_X^R(z)) - g_{i1}(\overline{F}_X^1(z))).\end{aligned}$$

Moreover, define

$$\begin{aligned}\hat{\mathcal{A}} &= \{z \in [0, M] : \hat{\phi}(z) < 0, g_{i1}(\overline{F}_X^1(z)) \geq g_R(\overline{F}_X^R(z))\}, \\ \hat{\mathcal{B}} &= \{z \in [0, M] : \hat{\phi}(z) = 0, g_{i2}(\overline{F}_X^2(z)) \geq g_R(\overline{F}_X^R(z))\}, \\ \hat{\mathcal{C}} &= \{z \in [0, M] : \hat{\phi}(z) > 0, g_{i2}(\overline{F}_X^2(z)) \geq g_R(\overline{F}_X^R(z))\}.\end{aligned}$$

This allows us to state the main result in the following theorem, which provides the first-best Bowley solutions under asymmetric information and heterogeneous beliefs.

Theorem 6.1 *The solution set to problem (8) contains those $g_r^* \in \mathcal{G}$ such that*

$$g_r^*(\overline{F}_X^R(z)) = \begin{cases} g_{i1}(\overline{F}_X^1(z)), & \text{if } z \in \hat{\mathcal{A}}, \\ \in \{g_{i1}(\overline{F}_X^1(z)), g_{i2}(\overline{F}_X^2(z))\}, & \text{if } z \in \hat{\mathcal{B}}, \\ g_{i2}(\overline{F}_X^2(z)), & \text{if } z \in \hat{\mathcal{C}}, \end{cases}$$

and

$$\mathbb{P}_R(z \in [0, M]\backslash(\hat{\mathcal{A}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) : g_r^*(\overline{F}_X^R(z)) \leq \max\{g_{i1}(\overline{F}_X^1(z)), g_{i2}(\overline{F}_X^2(z))\}) = 0.$$

Moreover, for any of these g_r^**, we have, for* $x \in [0, M]$*,*

$$\begin{aligned} f^*_{\{g_r^*, g_{i1}\}}(x) &= \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{A}}\}) \\ &\quad + \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{B}}, g_r^*(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\}) \\ &\quad + \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{C}}, g_{i1}(\overline{F}_X^1(z)) > g_{i2}(\overline{F}_X^2(z))\}), \end{aligned}$$

and

$$\begin{aligned} f^*_{\{g_r^*, g_{i2}\}}(x) &= \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{A}}, g_{i1}(\overline{F}_X^1(z)) < g_{i2}(\overline{F}_X^2(z))\}) \\ &\quad + \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{B}}, g_r^*(\overline{F}_X^R(z)) \leq g_{i2}(\overline{F}_X^2(z))\}) \\ &\quad + \mu(\{z \in [0, x] \mid z \in \hat{\mathcal{C}}\}). \end{aligned}$$

Proof. Equation (8) writes as

$$\begin{aligned} &\max_{g_r \in \mathcal{G}} \int_0^M \left[g_r(\overline{F}_X^R(z)) - g_R(\overline{F}_X^R(z))\right] [p\mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\right\}} + (1-p)\mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq g_{i2}(\overline{F}_X^2(z))\right\}}]\mathrm{d}z \\ = &\max_{g_r \in \mathcal{G}} \int_0^M \left[g_r(\overline{F}_X^R(z)) - g_R(\overline{F}_X^R(z))\right] [p\mathbf{1}_{\left\{g_{i2}(\overline{F}_X^2(z)) < g_r(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\right\}} \\ &\qquad + (1-p)\mathbf{1}_{\left\{g_{i1}(\overline{F}_X^1(z)) < g_r(\overline{F}_X^R(z)) \leq g_{i2}(\overline{F}_X^2(z))\right\}} + \mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq \min\{g_{i1}(\overline{F}_X^1(z)), g_{i2}(\overline{F}_X^2(z))\}\right\}}]\mathrm{d}z \\ = &\max_{g_r \in \mathcal{G}} \int_0^M \hat{\psi}(g_r(\overline{F}_X^R(z)), z)\mathrm{d}z \\ \leq &\int_0^M \max_{g_r(\overline{F}_X^R(z)) \geq 0} \hat{\psi}(g_r(\overline{F}_X^R(z)), z)\mathrm{d}z, \end{aligned}$$

where $\hat{\psi}(g_r(\overline{F}_X^R(z)), z) :=$

$$\begin{cases} \left[g_r(\overline{F}_X^R(z)) - g_R(\overline{F}_X^R(z))\right] [p\mathbf{1}_{\left\{g_{i2}(\overline{F}_X^2(z)) < g_r(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\right\}} + \mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq g_{i2}(\overline{F}_X^2(z))\right\}}] \\ \qquad\qquad \text{if } g_{i1}(\overline{F}_X^1(z)) > g_{i2}(\overline{F}_X^2(z)), \\ \left[g_r(\overline{F}_X^R(z)) - g_R(\overline{F}_X^R(z))\right] \mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\right\}} \qquad \text{if } g_{i1}(\overline{F}_X^1(z)) = g_{i2}(\overline{F}_X^2(z)), \\ \left[g_r(\overline{F}_X^R(z)) - g_R(\overline{F}_X^R(z))\right] [(1-p)\mathbf{1}_{\left\{g_{i1}(\overline{F}_X^1(z)) < g_r(\overline{F}_X^R(z)) \leq g_{i2}(\overline{F}_X^2(z))\right\}} + \mathbf{1}_{\left\{g_r(\overline{F}_X^R(z)) \leq g_{i1}(\overline{F}_X^1(z))\right\}}] \\ \qquad\qquad \text{if } g_{i1}(\overline{F}_X^1(z)) < g_{i2}(\overline{F}_X^2(z)). \end{cases}$$

The remainder of the proof is similar to the proof of Theorem 3.1, and thus omitted. ■

As illustrated in the examples of Sections 4 and 5, the optimal ceded loss function derived in Theorem 6.1 not only depends on the sign of the function $\hat{\phi}$ (which further relies on the proportion value p), but also depends upon the exact distortion functions of both parties and their beliefs on the probability distributions of the underlying risk. Thus, it is not easy to figure out the exact shape of the optimal ceded loss functions. For the sake of brevity, we will not present numerical examples for validating the finding of Theorem 6.1 since it is very similar with those presented in Sections 4 and 5.

7 Conclusion

Under the framework of distortion risk measures, we have revisited the Bowley reinsurance problem when the identity of the insurer is unknown to the reinsurer. By assuming that the reinsurer adopts a general premium function such that the pricing kernel might be negative on some states and the reinsurer uses general distortion risk measures, the first-best Bowley solutions are derived in full generality with the help of marginal profit functions. The optimal ceded loss functions depend on the underlying risk distribution, the shape of distortion functions that are possibly used by insurer, the cost function, and the probabilities that the reinsurer assigns to the insurer of being a certain type. By implementing some numerical examples, we found that the shut-down policy, the pooling stop-loss policy, the layer or limited stop-loss contracts are possible candidates of the optimal indemnity functions for the insurer. Besides, the expected net profit under our first-best Bowley solution is naturally larger than the second-best solution studied in Boonen et al. (2021). Finally, we generalize our main result to the case when both of the reinsurer and the types of the insurer have heterogeneous beliefs regarding the distribution function of the underlying risk.

As a future study, we are interested in extending the current study to the case where the types of the insurer are in a continuum. Besides, since different types of insurers may also have different distributions of losses, we wish to design optimal Bowley reinsurance contracts with such kind of asymmetric information as well.

Acknowledgements

We are very grateful to the two anonymous referees whose constructive comments and suggestions have significantly improved the paper. Yiying Zhang thanks the financial support from the National Natural Science Foundation of China (No. 12101336).

References

Albrecher, H. and Dalit, D.-A. (2017). On effects of asymmetric information on non-life insurance prices under competition. *International Journal of Data Analysis Techniques and Strategies*, **9**, 287-299.

Anthropelos, M. and Boonen, T.J. (2020). Nash equilibria in optimal reinsurance bargaining. *Insurance: Mathematics and Economics*, **93**, 196-205

Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**, 203-228.

Assa, H. (2015). On optimal reinsurance policy with distortion risk measures and premiums. *Insurance: Mathematics and Economics*, **61**, 70-75.

Bannör, K.F. and Scherer, M. (2014). On the calibration of distortion risk measures to bid-ask prices. *Quantitative Finance*, **14**, 1217-1228.

Boonen, T.J. (2016). Optimal reinsurance with heterogeneous reference probabilities. *Risks*, **4**, 26.

Boonen, T.J., Cheung, K.C. and Zhang, Y. (2021). Bowley reinsurance with asymmetric information on the insurer's risk preferences. *Scandinavian Actuarial Journal*, **2021**, 623-644.

Boonen, T.J. and Ghossoub, M. (2020). On the existence of a representative reinsurer under heterogeneous beliefs. *Insurance: Mathematics and Economics*, **88**, 209-225.

Chan, F.Y. and Gerber, H.U. (1985). The reinsurer's monopoly and the Bowley solution. *ASTIN Bulletin*, **15**, 141-148.

Chen, L. and Y. Shen (2018). On a new paradigm of optimal reinsurance: A stochastic Stackelberg differential game between an insurer and a reinsurer. *ASTIN Bulletin*, **48**, 905-960.

Cheung, K.C. and Lo, A. (2017). Characterizations of optimal reinsurance treaties: a cost-benefit approach. *Scandinavian Actuarial Journal*, **2017**, 1-28.

Cheung, K.C., Yam, S.C.P., Yuen, F.L. and Zhang, Y. (2020). Concave distortion risk minimizing reinsurance design under adverse selection. *Insurance: Mathematics and Economics*, **91**, 155-165.

Cheung, K.C., Yam, S.C.P. and Zhang, Y. (2019). Risk-adjusted Bowley reinsurance under distorted probabilities. *Insurance: Mathematics and Economics*, **86**, 64-72.

Chi, Y. (2012). Reinsurance arrangements minimizing the risk-adjusted value of an insurer's liability. *ASTIN Bulletin*, **42**, 529–557.

Chi, Y. (2019). On the optimality of a straight deductible under belief heterogeneity. *ASTIN Bulletin*, **49**, 243-262.

Chi, Y., Tan, K.S. and Zhuang, S.C. (2020). A Bowley solution with limited ceded risk for a monopolistic reinsurer. *Insurance: Mathematics and Economics*, **91**, 188-201.

Cui, W., Yang, J. and Wu, L. (2013). Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles. *Insurance: Mathematics and Economics*, **53**, 74-85.

Denuit, M. and Vermandele, C. (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, **22**, 229-233.

Eberlein, E., Madan, D. B., Pistorius, M. and Yor, M. (2014). Bid and ask prices as non-linear continuous time G-expectations based on distortions. *Mathematics and Financial Economics*, **8**, 265-289.

Ghossoub, M. (2017). Arrow's theorem of the deductible with heterogeneous beliefs. *North American Actuarial Journal*, **21**, 15-35, Huberman, G., Mayers, D. and Smith Jr, C.W. (1983). Optimal insurance policy indemnity schedules. *Bell Journal of Economics*, **14**, 415-426.

Jarrow, R.A. and Madan, D.B. (1997). Is mean-variance analysis vacuous: Or was beta still born? *Review of Finance*, **1**, 15-30.

Laffont, J.J. and Martimort, D. (2009). The Theory of Incentives: The Principal-Agent Model. *Princeton University Press*, New Jersey, USA.

Rahi, R. and Zigrand, J.-P. (2014). Walrasian foundations for equilibria in segmented markets. *Mathematics and Financial Economics*, **8**, 249-264.

Wang, S.S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin*, **26**, 71-92.

Wang, Q., Wang, R. and Wei, Y. (2020). Distortion riskmetrics on general spaces. *ASTIN Bulletin*, **50**, 827-851.

Yaari, M.E. (1987). The dual theory of choice under risk. *Econometrica*, **55**, 95-115.

A Multiple types of the insurer

In this appendix, we extend our main result to the case of finitely many types of the insurer. Let there be $n > 1$ types, and every type $j \in \{1, \ldots, n\}$ is identified by a distortion function g_{ij}, i.e., $\{g_{i1}, \ldots, g_{in}\} \subset \mathcal{G}_d$. The reinsurer believes that these types occur with probability $p_j \geq 0$ such that $\sum_{j=1}^{n} p_j = 1$. It will be helpful to define, for $t \in [0,1]$, $\sigma^t : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ as the bijective order of $\{g_{i1}, \ldots, g_{in}\}$: $g_{i\sigma^t(1)}(t) \leq g_{i\sigma^t(2)}(t) \leq \cdots \leq g_{i\sigma^t(n)}(t)$. Moreover, define

$$\begin{aligned}\mathcal{A}^k = \{t \in [0,1] :& [g_{i\sigma^t(k)}(t) - g_R(t)] \sum_{\ell=k}^{n} p_{\sigma(\ell)} \geq [g_{i\sigma^t(m)}(t) - g_R(t)] \sum_{\ell=m}^{n} p_{\sigma(\ell)} \\ & \text{for all } m = 1, 2, \ldots, n, g_{i\sigma^t(k)}(t) \geq g_R(t)\},\end{aligned}$$

for all $k \in \{1, \ldots, n\}$. Note that the sets $\mathcal{A}^k$ are not mutually exclusive, and $\bigcup_{k=1}^{n} \mathcal{A}^k \subset [0,1]$. Here, k is understood as a fixed rank in the bijective ordering σ^t for $t \in [0,1]$.

Theorem A.1 *The solution set to problem (7) contains those $g_r^* \in \mathcal{G}$ such that*

$$\nu_X\{t \in [0,1] \backslash (\bigcup_{k=1}^{n} \mathcal{A}^k) : g_r^*(t) \leq g_{i\sigma^t(n)}(t)\} = 0,$$

and

$$g_r^*(t) \in \left\{g_{i\sigma^t(k)}(t) | k \in \{1, \ldots, n\} \text{ such that } t \in \mathcal{A}^k\right\}, \text{ for all } t \in [0,1].$$

The proof of Theorem A.1 is similar to the proof of Theorem 3.1 and thus omitted. After obtaining an optimal premium generating function g_r^*, the indemnities follow directly from Lemma 2.1.