Static and dynamic risk capital allocations with the Euler rule

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Abstract

Risk capital allocations are of central importance in performance measurement. A popular solution concept in the academic literature is the Euler rule. This paper studies the volatility of the Euler rule for capital allocation in static and dynamic empirical applications with a simulated history. The Euler rule is not continuous in small changes of the underlying risk capital allocation problem, and we show that the Euler rule in combination with the Value-at-Risk is very sensitive to empirical measurement error. The use of a known distribution with estimated parameters helps to reduce this error. The Euler rule with an Expected Shortfall risk measure is less volatile, but still more volatile than the proportional rule.

Keywords: Dynamic capital allocation, Euler rule, proportional rule, simulation, Value-at-Risk.

1 Introduction

Financial institutions are required to hold a buffer, which is capital that is invested safely in order to mitigate the effects of adverse events. This buffer is referred to as risk capital, and is derived from the risk-based assessment of the aggregate risk of all its divisions. Withholding risk capital is costly, and therefore allocating risk capital to individual divisions is important for performance evaluation as well as for pricing. This problem is called the risk capital allocation problem. This is non-trivial because the sum of the risk evaluated in isolation is larger than the risk of the sum of the portfolios. Moreover, the firm may want to "reward" a division that has little or negative correlation with the firm's aggregate risk.

The Euler rule, also known as the Aumann-Shapley value, has received considerable attention in the academic literature on risk capital allocations. It is the gradient of an appropriately chosen function, indicating a marginal contribution of a division's portfolio to the aggregate

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risk of the firm. Contributions in the literature on the Euler rule are in the context of financial performance measurement (Tasche, 1999; Kromer and Overbeck, 2014b), cooperative game theory (Denault, 2001; Tsanakas and Barnett, 2003; Tsanakas, 2004; Kalkbrener, 2005), and insurance risk management (Myers and Read, 2001; Sherris, 2006).[1] Little attention has been spend to analyze the volatility of the Euler rule in dynamic applications. This paper studies that, and finds that the Euler rule is very volatile over time and is sensitive to modelling assumptions. This observation is also particularly important for practitioners working in enterprize risk management.

Van Gulick et al. (2012) are one of the first to explicitly acknowledge the lack of continuity of the Euler rule as a major drawback, and they propose an alternative method based on a game-theoretic concept. Lack of continuity implies that small changes in the underlying risk capital allocation problem may have a substantial impact on the Euler rule. In this paper, we aim to illustrate this drawback of the Euler method in simulation, and show that the Euler method may be very volatile in dynamic applications.

Tsanakas (2004) and Kromer and Overbeck (2014a) study dynamic capital allocations with coherent distortion or convex risk measures. In particular, Tsanakas (2004) proposes a Bayesian updating mechanism to determine the determine distorted probabilities at any point in time, and then uses the Euler method to determine capital allocations. We differ in two ways. First, we study the Value-at-Risk, which is not a coherent risk measure, but is line with Solvency II regulations. Second, we use the empirical distribution to calculate or approximate the underlying multivariate distribution, whereas Tsanakas (2004) takes the underlying prior distribution as given. Tsanakas (2004) shows that capital allocations are rather smooth for the Euler rule, but this is due to the focus on distortion risk measures generated by proportional hazard transforms. Kromer and Overbeck (2014a) study a representation based on backward stochastic differential equations. Moreover, recently, Boonen et al. (2018) study the forecasting of dynamic capital allocations based on compositional, empirical data. They focus on forecasting in a time-series model, where the allocation rule is given.

Our advice for risk managers is as follows. The use of the Euler rule in combination with the Value-at-Risk leads to a very high exposure to empirical measurement error, both for a static and a dynamic horizon. The use of a known distribution with estimated parameters helps to reduce this error, but it requires a careful test to determine the underlying distribution. Capital allocation with an Expected Shortfall risk measure is less volatile. This paper does not aim to advertise the proportional rule as alternative, but the proportional rule is used as benchmark to compare the Euler rule with.

This paper is set out as follows. In Section 2, we define the static model for risk capital allocation. In Section 3, we derive the theoretical and empirical results for the Euler rule in a static setting. In Section 4, we apply the static model to a dynamic setting, and show that the Euler rule is very volatile over time. In Section 5, we conclude. In the appendix A, we show the results for Expected Shortfall instead of the Value-at-Risk, and in Appendix B, we study risk capital allocation with Student's t-distributions.

[1] Guo et al. (2018) provide an excellent overview of different approaches leading to the Euler rule.

2 Static risk capital allocation models

To understand the concept of capital allocation, we first conceptualize the risk measure and allocation methods for the case of a static horizon. A risk measure is used to calculate the risk capital, which is understood as a buffer that the firm needs to hold. Let $(\Omega, A, \mathbb{P})$ be a probability space, and denote $\mathcal{X}$ as the class of random variables on it. We interpret random variables as future losses, that are realized in one year from now.

In line with Solvency II regulations, we study the risk measure Value-at-Risk, given by

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} \big| \mathbb{P}(X \leq x) \geq \alpha\}, \qquad \alpha \in (0,1), \tag{1}$$

for all $X \in \mathcal{X}$. If X is continuously distributed with cumulative distribution function F_X, then $VaR_\alpha(X) = F_X^{-1}(\alpha)$. In this section, we fix the parameter $\alpha = 99.5\%$, in line with Solvency II regulation.

The Value-at-Risk is popular in insurance regulation, because it is easy to interpret and has a clear connection to the probability of insolvency. Moreover, the Value-at-Risk satisfies a property called elicitability (see, e.g., Ziegel, 2016). This property is important to be able to verify and compare competing estimation procedures. Also, Yamai and Yoshiba (2005) and Kellner and Rösch (2016) show that the Value-at-Risk is less sensitive to model risk compared to the popular alternative Expected Shortfall, which is for instance used in Basel regulation for banks. Therefore, the Value-at-Risk typically needs less data or simulations to get reasonable estimates. We ignore the Expected Shortfall for now, but we will redo our analysis in this paper for the case with Expected Shortfall in Appendix A.

Consider a firm that consists of multiple divisions facing risk. The finite set of all divisions within a firm is denoted by N. Each division $i \in N$ holds a stochastic loss $X(e_i) \in \mathcal{X}$, where e_i is the i-th unit vector in $\mathbb{R}^N$. We define

$$X(\boldsymbol{\lambda}) = \sum_{i \in N} \lambda_i X_i, \qquad \boldsymbol{\lambda} \in \mathbb{R}_+^N,$$

so that the total loss of the firm is given by $X(e_N) = \sum_{i \in N} X_i$,[2] where e_N is the vector of ones in $\mathbb{R}^N$. The firm needs to hold risk capital given by $VaR_\alpha(X(e_N))$. The problem is to allocate this risk capital among all divisions. A vector $\boldsymbol{K} \in \mathbb{R}^N$ is called a risk capital allocation when

$$\sum_{i \in N} K_i = VaR_\alpha(X(e_N)).$$

The Euler rule, denoted by $\boldsymbol{K}^E$, is given by

$$K_i^E = \frac{\partial VaR_\alpha(X(\boldsymbol{\lambda}))}{\partial \lambda_i}\Big|_{\boldsymbol{\lambda}=e_N}, \qquad i \in N, \tag{2}$$

[2] Alternative non-linear risk aggregation has been studied recently by Boonen et al. (2017) and Major (2018). They still show under some model assumptions that the Euler rule of a homogeneous risk aggregation function is useful in applications.

whenever the partial derivatives exist. We focus in this paper on cases where the Euler rule exists, which holds under mild conditions (Tasche, 1999)[3]. Due to the positive homogeneity of the Value-at-Risk[4], the Euler rule is a risk capital allocation, whenever it exists (Tasche, 1999). Due to Tasche (2008), the Euler rule can be written as:

$$K_i^E = \mathbb{E}[X(e_i)|X(e_N) = VaR_\alpha(X(e_N))], \qquad i \in N. \tag{3}$$

In this paper, we compare the Euler rule with a "simple" allocation rule. For instance, Dhaene et al. (2012) provide an overview of some basic allocation rules, and we choose to compare the Euler rule with the proportional rule. The proportional rule, denoted by $\boldsymbol{K}^P$, is given by

$$K_i^P = \frac{VaR_\alpha(X(e_N))}{\sum_{j\in N} VaR_\alpha(X(e_j))} VaR_\alpha(X(e_i)), \qquad i \in N. \tag{4}$$

In the following example, we illustrate the motivation of the Euler rule, as well as a major drawback of the Euler rule compared to the proportional rule. The latter drawback is a central topic of this paper.

Example 2.1 Let $\alpha = 99.5\%$, $N = \{1, 2, 3\}$, and moreover, $X(e_1) = X(e_2) = -X(e_3) = Z$, where $Z \sim N(0, 1)$. Then, $VaR_\alpha(X(e_i)) = VaR_\alpha(X(e_N)) = VaR_\alpha(Z) \approx 2.58$. We derive that $K_1^E = K_2^E \approx 2.58, K_3^E \approx -2.58$, and $K_i^P \approx 0.89$ for $i = 1, 2, 3$. Hence, the Euler rule considers the dependence structure by "rewarding" Division 3 with a low capital allocation, due to the fact that its risk is negatively correlated with the total risk of the firm. The proportional rule does not consider the use of Division 3 as a good hedge for the firm, and only evaluated the risk of the division in isolation.

The Euler rule is however sensitive to small changes in the underlying problem, as we next illustrate. Let $\alpha = 99.5\%$, $N = \{1, 2\}$, $X(e_1) = Z, X(e_2) = -\delta Z$, where $Z \sim N(0, 1)$ and $\delta > 0$. Then, $VaR_\alpha(X(e_1)) = VaR_\alpha(Z) \approx 2.58, VaR_\alpha(X(e_2)) = VaR_\alpha(\delta Z) \approx 2.58\delta$ and

$$VaR_\alpha(X(e_N)) = VaR_\alpha((1-\delta)Z) = \begin{cases} (1-\delta)VaR_\alpha(Z) \approx 2.58(1-\delta) & \text{if } \delta \in (0, 1], \\ (\delta - 1)VaR_\alpha(Z) \approx 2.58(\delta - 1) & \text{if } \delta > 1. \end{cases}$$

We derive that

$$K_1^E \approx \begin{cases} 2.58 & \text{if } \delta \in (0, 1), \\ -2.58 & \text{if } \delta > 1, \end{cases} \qquad K_2^E \approx \begin{cases} -2.58\delta & \text{if } \delta \in (0, 1), \\ 2.58\delta & \text{if } \delta > 1, \end{cases}$$

and

$$K_1^P \approx \begin{cases} \frac{2.58(1-\delta)}{(1+\delta)} & \text{if } \delta \in (0, 1], \\ \frac{2.58(\delta-1)}{(1+\delta)} & \text{if } \delta > 1, \end{cases} \qquad K_2^P \approx \begin{cases} \frac{2.58\delta(1-\delta)}{(1+\delta)} & \text{if } \delta \in (0, 1], \\ \frac{2.58\delta(\delta-1)}{(1+\delta)} & \text{if } \delta > 1. \end{cases}$$

[3]Non-existence of the Euler rule in the context of coherent or convex risk measures is studied by Boonen et al. (2012) and Centrone and Rosazza Gianin (2018).

[4]A risk measure ρ is positive homogeneous if for any $\hat{X} \in \mathcal{X}$, we have $\rho(\alpha\hat{X}) = \alpha\rho(\hat{X})$ for all $\alpha > 0$.

The Euler rule is not defined when $\delta = 1$, and moreover a small perturbation of δ around 1 may have a substantial impact on the allocated capital. Note that the risk capital itself, given by $VaR_\alpha(X(e_N))$, and the proportional rule are both continuous in δ. The discontinuity of the Euler rule may lead to considerable volatility in the risk capital allocations, as we will illustrate in the sequel of this paper. $\triangledown$

3 Gaussian distributions in a static model

3.1 Theory

Suppose the vector of losses of the divisions $\{X(e_i)\}_{i \in N}$ is multivariate Gaussian distributed with parameters $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \Sigma)$, where $\boldsymbol{\mu}$ is the vector of expectations, $\boldsymbol{\sigma}$ is the vector of standard deviations, and Σ is the correlation matrix. Denote Φ as the Cumulative Distribution Function (CDF) of a standard Gaussian distributed random variable $Z \sim N(0, 1)$. It is well-known that $VaR_\alpha(X(e_i)) = \mu_i + \Phi^{-1}(\alpha)\sigma_i$ for all $i \in N$. In general, $X(\boldsymbol{\lambda})$ has a Gaussian distribution as well, and it holds that

$$VaR_\alpha(X(\boldsymbol{\lambda})) = \sum_{i \in N} \lambda_i \mu_i + \Phi^{-1}(\alpha) \sqrt{\sum_{i \in N} \lambda_i^2 \sigma_i^2 + 2 \sum_{i,j \in N, i<j} \lambda_i \lambda_j \sigma_{ij}}, \tag{5}$$

where σ_{ij} is the covariance of $(X(e_i), X(e_j))$.

Taking the partial derivative of (5) and substituting $\boldsymbol{\lambda} = e_N$ yield directly the following expressing of the Euler rule:

$$K_i^E = \mu_i + \Phi^{-1}(\alpha) \sum_{j \in N} \sigma_{ij} \Big/ \sqrt{\sum_{j \in N} \sigma_j^2 + 2 \sum_{j,k \in N, j<k} \sigma_{jk}} \tag{6}$$

$$= \mu_i + \Phi^{-1}(\alpha) \frac{cov(X(e_N), X(e_i))}{\sqrt{var(X(e_N))}}, \qquad i \in N. \tag{7}$$

3.2 Numerical approach

In practice, the precise distribution of losses is unknown, and it is approximated empirically. Therefore, also the Value-at-Risk needs to be approximated. For instance, in Solvency II regulation in insurance, CEIOPS (2010) calibrates the Solvency Capital Requirement for, e.g., equity risk using daily data of yearly returns. This leads to an overlapping window of returns, and it creates "the greatest possible quantity of relevant data". However, the data is not stationary over time. So, an empirical quantile is not an accurate approximation of the Value-at-Risk, even though the size of the data is artificially large.

We use an alternative approach in order to construct a stationary sample of losses. We assume that the daily losses for division $i \in N$ are generated by the process

$$\ell_{i,t}^d := L_{i,t} - L_{i,t-1} = \mu_i + \sigma_i \varepsilon_{i,t}, \tag{8}$$

where $L_{i,t}$ is the value of the liabilities of division i at day t, and $\varepsilon_{\cdot,t} \overset{i.i.d.}{\sim} N(\boldsymbol{\mu}, \Sigma)$.[5] Gaussian

[5]Remark that $L_{i,t}$ is interpreted as a loss.

distributions are popular in the calibration of Solvency II (see, e.g., CEIOPS, 2010), but the use of Gaussian distributions has often been criticized (see, e.g., Sandström, 2007). More advanced models could be used to generate our data. Unless some other time-series model is found to be substantially better, we use (8) because of its simplicity and straightforward interpretation. Note that Gaussian returns typically do not have fat tails compared to empirical returns for any type of financial or insurance risk (e.g., Cont, 2001). In this paper, we will claim that the volatility of the Euler rule is not solely a result of fat tails in the underlying multivariate distribution. So, to be prudent in our conclusions, we chose the Gaussian distribution to simulate our data.

We assume that every year has 250 working days. Since the data is serially independent and has a Gaussian distribution, the daily losses are annualized as follows:

$$\ell^y_{i,t} := L_{i,t} - L_{i,t-250} = \sum_{t'=1}^{250} \ell^d_{i,t-250+t'} \stackrel{d}{=} 250\mu_i + \sqrt{250}\sigma_i\varepsilon_{i,t} = 250\mu_i + \sqrt{250}(\ell^d_{i,t} - \mu_i). \quad (9)$$

The daily losses are generated by (8), and we simulate the daily losses in the past 40 year. This yields a total of 10,000 daily losses. Let $\alpha = 99.5\%$, $N = \{1, 2, 3\}$, $\boldsymbol{\mu} = (0, 0, 0)$, $\boldsymbol{\sigma} = (\frac{16\%}{\sqrt{250}}, \frac{16\%}{\sqrt{250}}, \frac{16\%}{\sqrt{250}})$, and the correlation matrix is given by the following positive semi-definite matrix:

$$\Sigma = \begin{bmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}.$$

The losses are expressed in a common currency; for instance, the divisions may have the same present value, and a unit is equal to this value of a division. Note that the standard deviation of the yearly losses is equal to 16% for every division. The example is constructed such that Division 3 is relatively attractive for the firm due to the negative correlation between the losses of Division 3 and Divisions 1 and 2. Also, in the underlying data generating process, Divisions 1 and 2 are symmetric.

We use (9) to construct yearly losses in the past, i.e., we multiply the daily losses by $\sqrt{250}$. We get 10,000 past realizations of the random variable $(X(e_i))_{i\in N}$, given by: $(\sqrt{250}\ell^d_{i,t})_{i\in N}, t = -9,999; \ldots; 0$, where the present is indexed by day $t = 0$. These past realization are used to determine the empirical distribution function of $(X(e_i))_{i\in N}$. Based on the empirical distribution function, we determine the 99.5%-Value-at-Risk of $X(e_N)$. This is given by the 9,950th order statistic, when we order the simulated past losses $\sum_{i\in N}\sqrt{250}\ell^d_{i,t}, t = -9,999; \ldots; 0$, from small to large.

From (5) and (7), we readily get that, theoretically, the risk capital and risk capital allocations are given by:

$$\begin{aligned} VaR_\alpha(X(e_N)) &= \Phi^{-1}(99.5\%) \cdot 16\% \cdot \sqrt{2} \approx 58.3\%, \\ \boldsymbol{K}^E &\approx (29.1\%, 29.1\%, 0), \\ \boldsymbol{K}^P &\approx (19.4\%, 19.4\%, 19.4\%). \end{aligned}$$

In Figure 1, we display the Euler rule and the proportional rule, determined via the simulated past, for Divisions 1 and 3. We run 10,000 simulations of the past 10,000 days. Since Divisions 1 and 2 are symmetric in the underlying simulation model, the allocations of both divisions are similar. Compared to the proportional rule, the Euler rule is very sensitive to the simulation index. This implies that the Euler rule is more prone to empirical measurement error than the proportional rule. Because the Euler rule selects only one specific past realization, as seen in (3), this risk is not expected to be mitigated by increasing the number of past losses. In other words, the error cannot be reduced substantially when we increase the past data horizon beyond 40 years. Moreover, this would be very difficult to do in practice, as such past losses may not exist for certain divisions, or the distribution of losses beyond 40 years ago is different from the distribution of future losses in the coming year.

In practice, insurance companies should be wary about this uncertainty that the Euler rule yields empirically. The Gaussian distributions and serial independence yield smooth Euler allocations in theory; but this does not hold empirically, even when we simulate the underlying risk under the assumption that it is Gaussian and serially independent.

In Appendix A, we redo the analysis of this section, but then with the risk measure Expected Shortfall. For the Expected Shortfall, it is shown by Yamai and Yoshiba (2005) that the total risk capital is much more prone to simulation error. This implies that much more past data is needed to find a good estimate of the risk capital. For capital allocation with the Euler rule and Expected Shortfall, we find however that the simulation error is much smaller than when the Value-at-Risk is used. This is due to the fact that, empirically, the Euler rule based on Value-at-Risk selects one specific order statistic of the aggregate risk, and allocates risk capital according this realization (see (3)). The Euler rule with Expected Shortfall yields an average of realizations, and this averaging mitigates the measurement error of the Euler rule.

In Appendix B, we redo the analysis of this section, but then for risk that is simulated with the multivariate Student's t-distribution. For the underlying distribution, we keep the first two moments equal to the two moments in our study above in Figure 1 with Gaussian distribution. Moreover, the dependency between the risks is given by the same correlation matrix Σ. The Student's t-distribution has fatter tails than the Gaussian distribution. We find that our conclusions amplify substantially compared to the Guassian distribution, and so we consider our conclusions with the Guassian distribution to be prudent.

3.3 Fitting Gaussian distributions

To decrease the volatility of the Euler rule in empirical applications, a solution is to first fit empirically the first two moments to a Gaussian distribution. So, from the last 10,000 days data, we estimate the parameters $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \Sigma)$. From the estimated parameters $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}, \hat{\Sigma})$, we obtain the (Gaussian) distribution. Based on this distribution, we then theoretically determine the Value-at-Risk (risk capital), the Euler rule, and the proportional rule from Section 3.1.

Since the underlying data is generated by a Gaussian distribution, this approach leads to a smaller measurement error in the underlying distribution. This is confirmed by Figure 2. In this figure, we show the Euler rule and the proportional rule based on the fitted

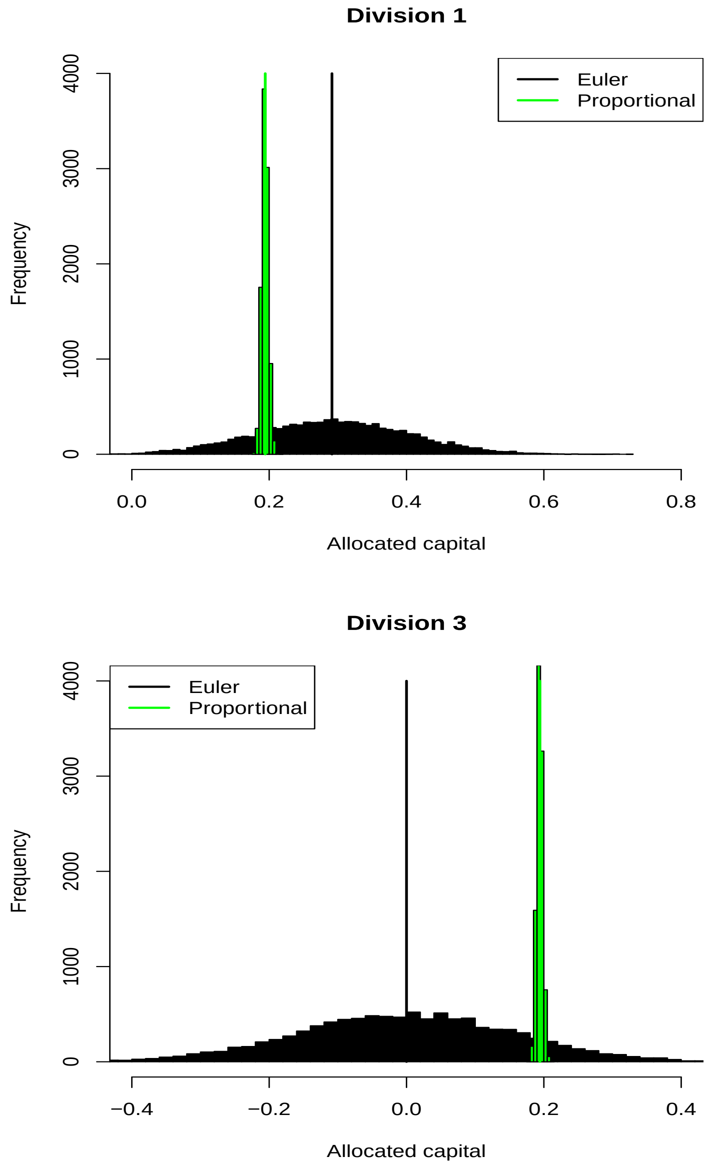


Figure 1: The Euler rule and proportional rule for Division 1 (top) and Division 3 (bottom). The past 10,000 days is simulated 10,000 times. The vertical lines show the theoretical allocations with the given parameters, corresponding to Section 3.2.

Gaussian distribution. We simulate the past 10,000 times. We still find that the Euler rule is considerably more volatile than the proportional rule.

Remark that this approach may lead to substantial underestimations of the risk capital when the losses are not generated by a Gaussian distribution but with a distribution that has a heavy tail. Therefore, we recommend that a careful analysis of the distribution is performed in practice. Our empirical approach in Section 3.2 does not require such a pre-test, as it only uses the empirical distribution.

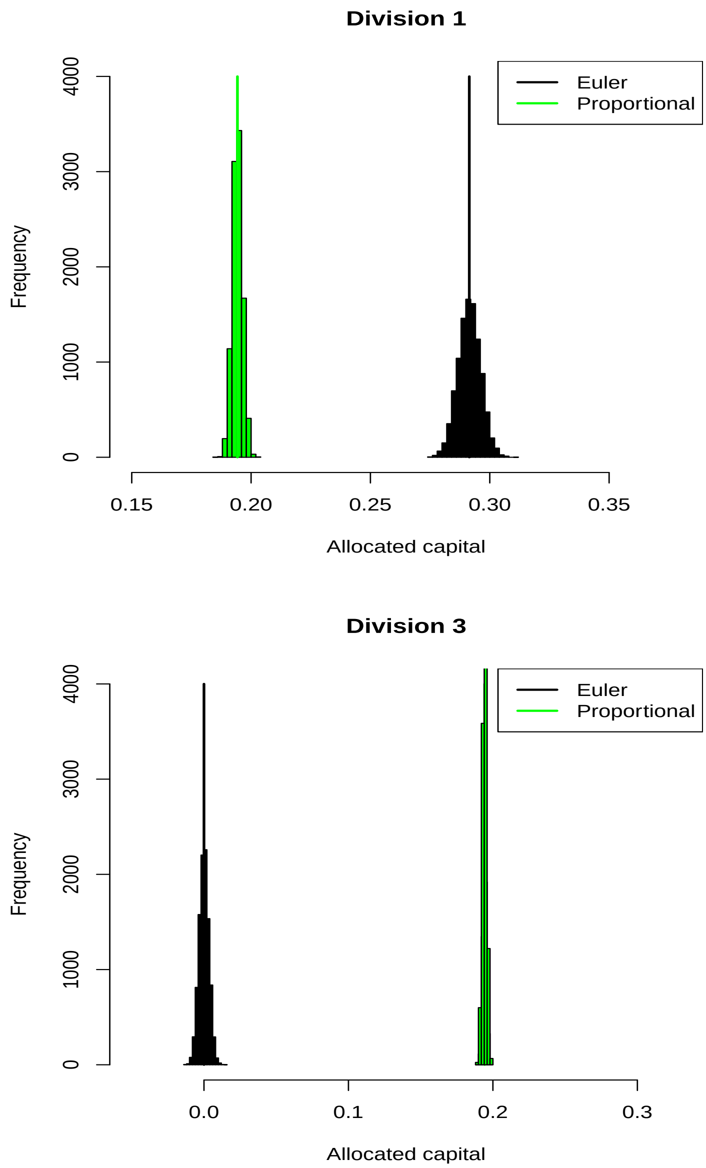


Figure 2: The Euler rule and proportional rule for Division 1 (top) and Division 3 (bottom). We first fit the Gaussian distribution using a simulated past of 10,000 simulations, and then determine the allocation rules theoretically, corresponding to Section 3.3. The vertical lines show the theoretical allocations with the given parameters.

4 Dynamic approach to risk capital allocations

In this section, we apply the Euler rule in a dynamic setting. If the risks of the firm are stationary over time, the Euler rule and the proportional rule are stable in theory. If the risk is determined empirically using past data, the distribution is not stationary anymore. The firm still wants the capital allocations to be stable.

We assume that the Value-at-Risk is again determined as in Section 3.2: using the past 10,000 daily losses. We use rolling window of 40 years past data. Our approach to dynamic capital allocation with the Euler rule is given as follows:

- We simulate the past 40 years of data using (8).
- At time $t = 0$, we determine the Euler rule using the last 40 years of daily losses as in Section 3.2. So, we use the time window $-9,999; \ldots; 0$, and determine the Euler rule empirically.
- We simulate an additional year data using (8).
- At time $t = 250$, we use the daily losses in the time window $-9,724; \ldots; 250$, and compute the Euler rule at day $t = 250$.
- Repeat this for ten years in the future.

In this way, we generate the Euler rule at times $t = 0; 250; \ldots; 2,500$. We repeat this dynamic approach for the proportional rule.

In Figure 3, we display trajectories of the Euler rule and the proportional rule. We show two different scenarios to illustrate whether the allocated capital is smooth over time. We find that the Euler rule is very irregular, while the proportional rule has a smooth pattern over time. The Euler rule is therefore not practical for practical purposes, as one year additional data has a substantial impact on the allocated capital.

We proceed with studying the confidence intervals of future capital allocations. We simulate the past and future jointly. The average capital allocations and corresponding confidence intervals are expected to be flat, as the data is generated by a stationary process. We display this in Figure 4, where we run 10,000 simulations. We find that the confidence intervals of the Euler rule are wide compared to the proportional rule. This implies that the Euler rule is very sensitive to empirical measurement error. This confirms our findings of capital allocation with a static horizon in Section 3.2.

If the firm believes that the data is stationary, the firm aims to keep the capital allocations stable over time. The data generated by (8) is stationary. The trajectories of the future capital allocations are shown in Figure 5. We simulate the past once, and then simulate the future 10,000 times. Our findings are consistent with Figure 4 in that the confidence intervals for the Euler are rather wide - already after one year in the future.

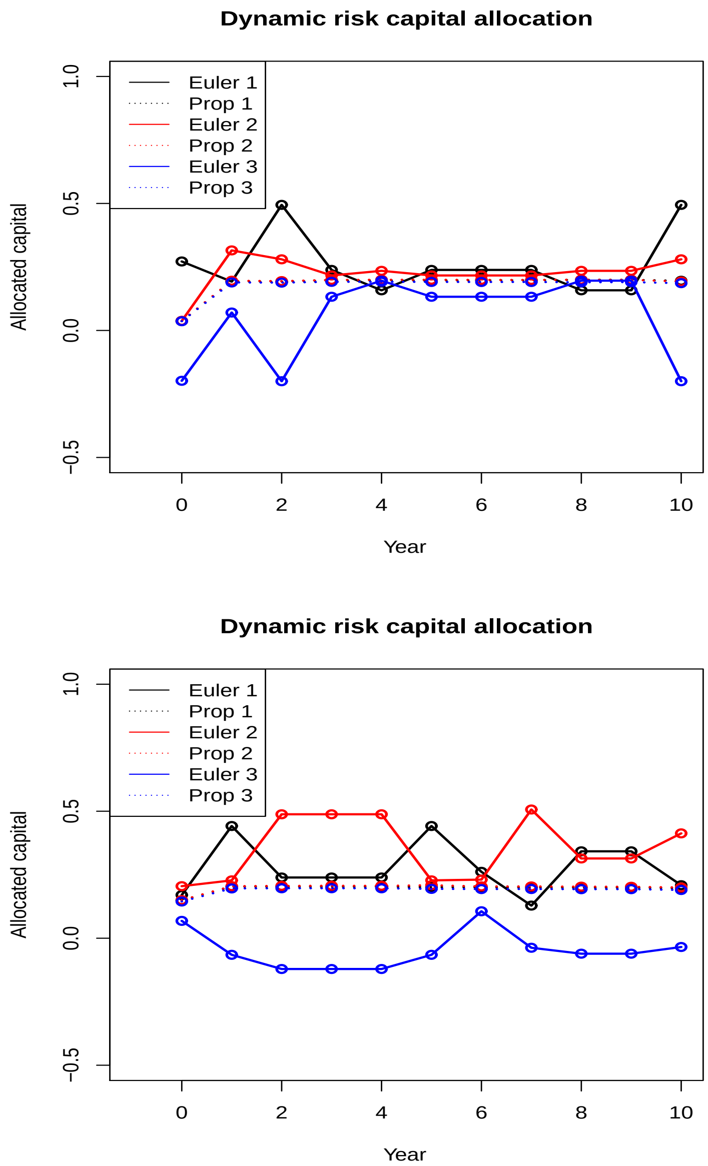
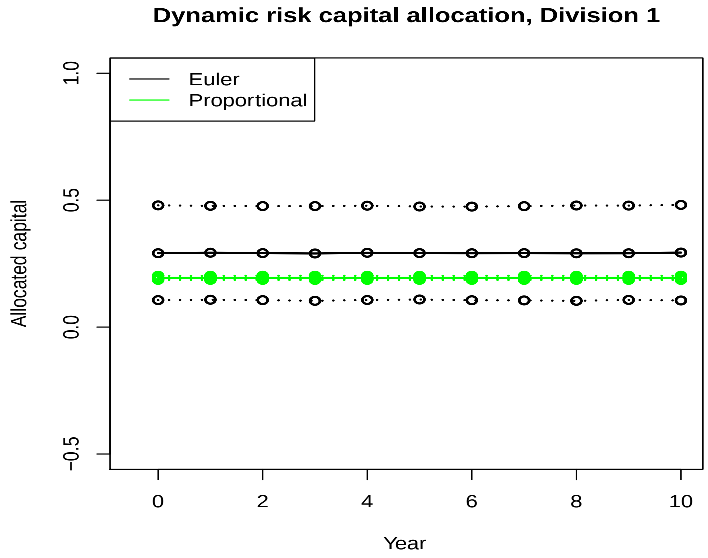


Figure 3: The Euler rule and proportional rule in dynamic setting for all three divisions in N. We display here two different simulated scenarios.



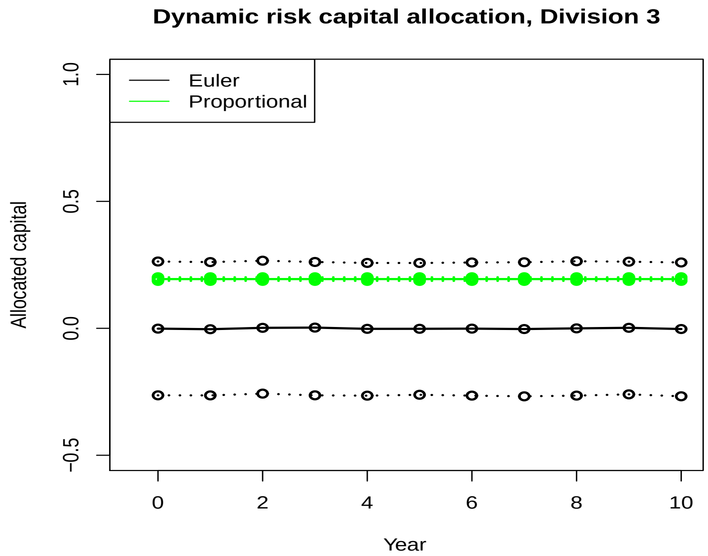
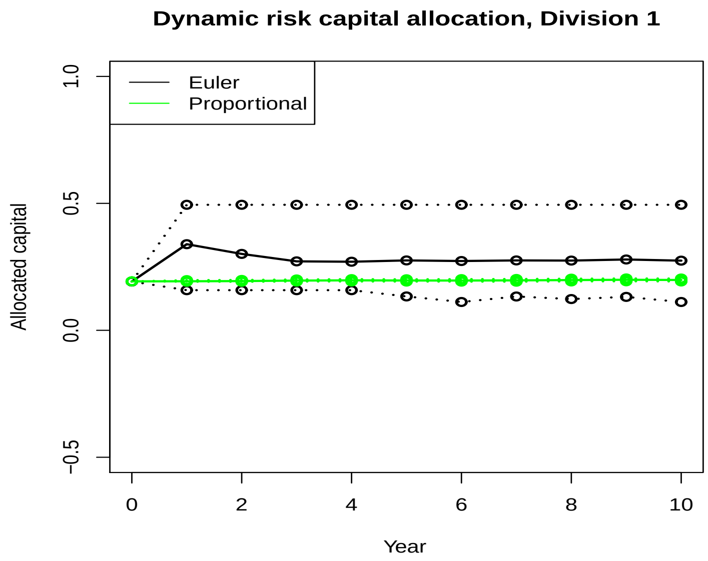


Figure 4: The Euler rule and proportional rule in dynamic setting for Division 1 (top) and Division 3 (bottom). We simulate the past and future jointly, with 10,000 loss trajectories from $t = -9,999; \ldots ; 2,500$. The solid lines are the average allocations, and the dotted lines are the 95%-confidence intervals.



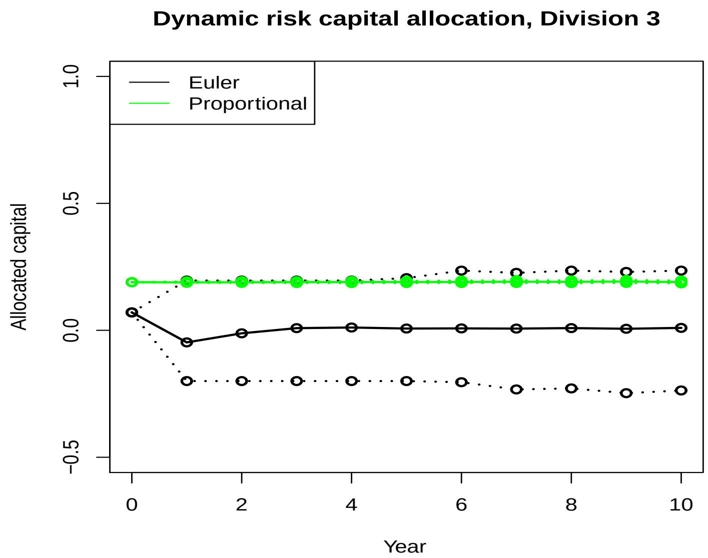


Figure 5: The Euler rule and proportional rule in dynamic setting for Division 1 (top) and Division 3 (bottom). We simulate the past once, and then determine the allocation rules for 10,000 simulated future loss trajectories. The solid lines are the average allocations, and the dotted lines are the 95%-confidence intervals.

5 Conclusion

This paper provides a careful warning for the use of the Euler rule in empirical applications for risk capital allocation. In particular, we show that the Euler rule is very sensitive to empirical measurement errors if the Value-at-Risk is used. In dynamic applications, the Euler rule can vary substantially over time. The use of a known distribution with estimated parameters helps to reduce this error. Capital allocation with an Expected Shortfall risk measure is less volatile, but there it is still considerably more volatile than the proportional rule. Note that it is well-known that determining the risk capital itself with the Expected Shortfall is difficult in simulation-based approaches (see, e.g., Ziegel, 2016).

The past data for the risk of the firm is simulated using a Gaussian distribution, independent and identically distributed over time. This time series process is often too simple in practice. If there is serial dependence, it is typically inaccurate to use past losses to estimate the distribution of future losses. Also, if there are fat tails in the underlying distribution, fitting a Gaussian distribution can substantially underestimate the corresponding Value-at-Risk. If more advanced models are used to generate the data, the volatility of the Euler rule is expected to be larger than under the smooth assumptions of this paper.

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A Expected Shortfall for capital allocations

The α-Expected Shortfall (ES) is given by

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\tau(X)d\tau, \qquad \alpha \in [0,1), \tag{10}$$

for all $X \in \mathcal{X}$, whenever the integral converges. If the random variable X is continuously distributed, the ES may be even more intuitively expressed as the conditional tail expectation (CTE):

$$ES_\alpha(X) = \mathbb{E}[X \mid X \geq VaR_\alpha(X)]. \tag{11}$$

If $X \sim N(\mu, \sigma^2)$, then $ES_\alpha(X) = \mu + \frac{\Phi'(\Phi^{-1}(\alpha))}{1-\alpha}\sigma$ (Yamai and Yoshiba, 2005). Theoretically, the Euler rule for the Gaussian distributions is therefore given by (7), where we replace the factor $\Phi^{-1}(\alpha)$ by $\frac{\Phi'(\Phi^{-1}(\alpha))}{1-\alpha}$. If $\alpha = 99\%$, then $\frac{\Phi'(\Phi^{-1}(\alpha))}{1-\alpha} \approx 2.67$. Therefore, if we redo the analysis of fitting a distribution, and then determining the allocation rules, the outcomes are in line with Section 3.3. The only difference is that the factor $\Phi^{-1}(\alpha)$ (a constant) is replaced by $\frac{\Phi'(\Phi^{-1}(\alpha))}{1-\alpha}$.

We redo the analysis of Section 3.2, but now with the 99%-Expected Shortfall. This risk measure is used in the Swiss Solvency Test (SST) for insurers. In Figure 6 and 7, we display the Euler rule and proportional rule. In Figure 6 we redo the analysis done for Figure 1 in Section 3.2, but now with the 99%-Expected Shortfall. In Figure 7, we redo the analysis done for Figure 5 in Section 4 with 99%-Expected Shortfall, where we have a dynamic horizon, the past is simulated once, and the future is simulated 10,000 times. We find that for the Expected Shortfall, the empirical measurement error in the Euler rule is considerably smaller than for the Value-at-Risk. This is due to the fact that, empirically, the Expected Shortfall takes an average of order statistics, while the Value-at-Risk considers only one specific order statistic. As a result, the Euler rule uses an averaging of specific realizations as well. This averaging mitigates the measurement error of the Euler rule. Interestingly, while the Expected Shortfall needs more data to get reasonable estimates for the risk capital than the Value-at-Risk, the Euler rule for capital allocation with the Expected Shortfall is substantially less volatile than with the Value-at-Risk.

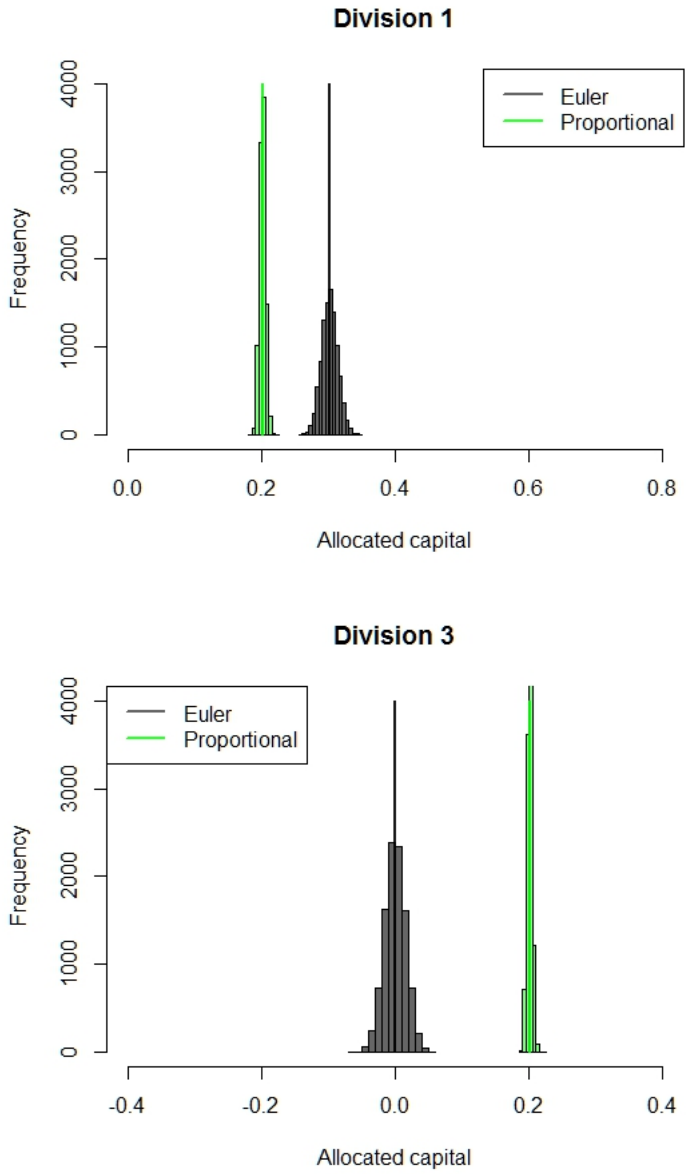


Figure 6: The Euler rule and proportional rule for Division 1 (top) and Division 3 (bottom), based on 99%-Expected Shortfall. The past 10,000 days is simulated 10,000 times. The vertical lines show the theoretical allocations with the given parameters.

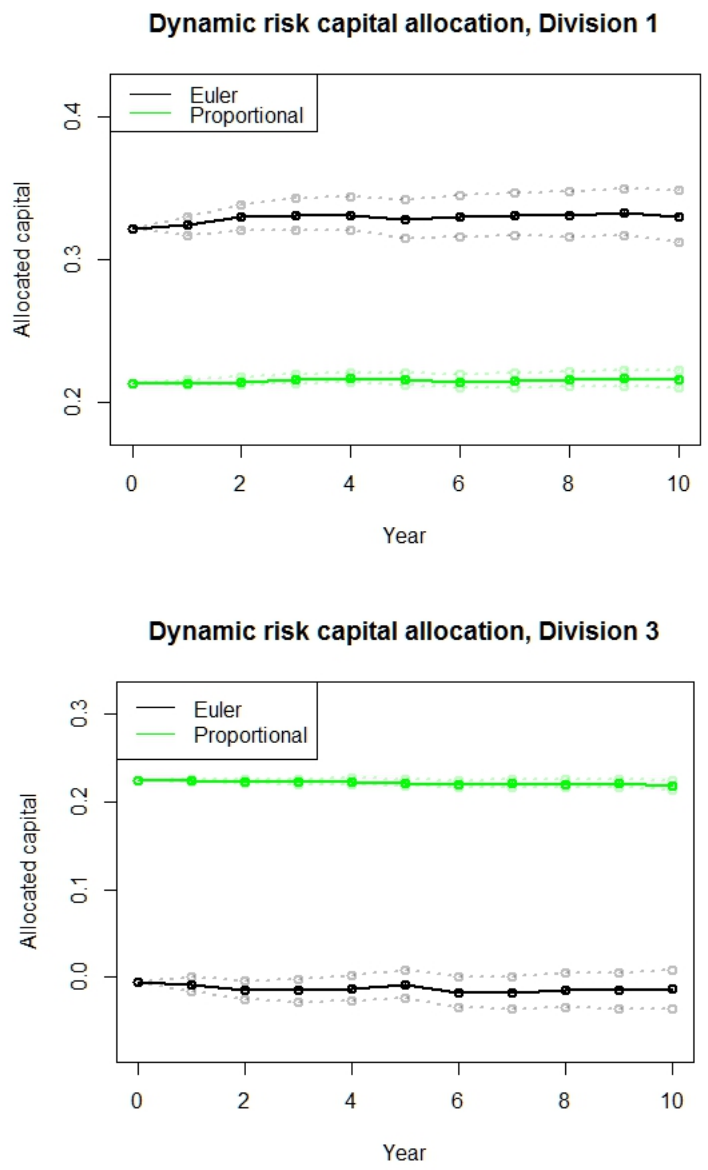


Figure 7: The Euler rule and proportional rule in dynamic setting for Division 1 (top) and Division 3 (bottom), based on 99%-Expected Shortfall. We simulate the past once, and then determine the allocation rules for 10,000 simulated future loss trajectories. The solid lines are the average allocations, and the dotted lines are the 95%-confidence intervals.

B Student's t-distributions for capital allocations

In this appendix, we simulate the multivariate Student's t-distribution with $\nu > 2$ degrees of freedom. We normalize the annualized losses by multiplying it with a constant, so that the standard deviation of yearly losses is identical to 16% for all three divisions. Thus, we simulate

$$X(e_i) \sim 16\% \cdot Z_{\nu,i} \Big/ \sqrt{\frac{\nu}{\nu - 2}}, i \in N, \tag{12}$$

for 10,000 times, with $Z_{\nu,i}$ is Student's t-distributed with parameter ν. The correlation matrix of $(X(e_1), X(e_2), X(e_3))$ is again given by Σ, as defined in Section 3.2. If ν gets larger, the tails are getting less heavy. We redo the analysis of Section 3.2 for the multivariate Student's t-distribution with parameters $\nu = 3$ (Figure 8) and $\nu = 5$ (Figure 9).

We find that our results of Section 3.2 amplify; the Euler rule is substantially more volatile with Student's t-distributions compared to the Gaussian distributions. The proportional rule provides capital allocations that are substantially less volatile than the Euler rule. When the tails are more heavy (so when ν is smaller), we find that this effect is stronger.

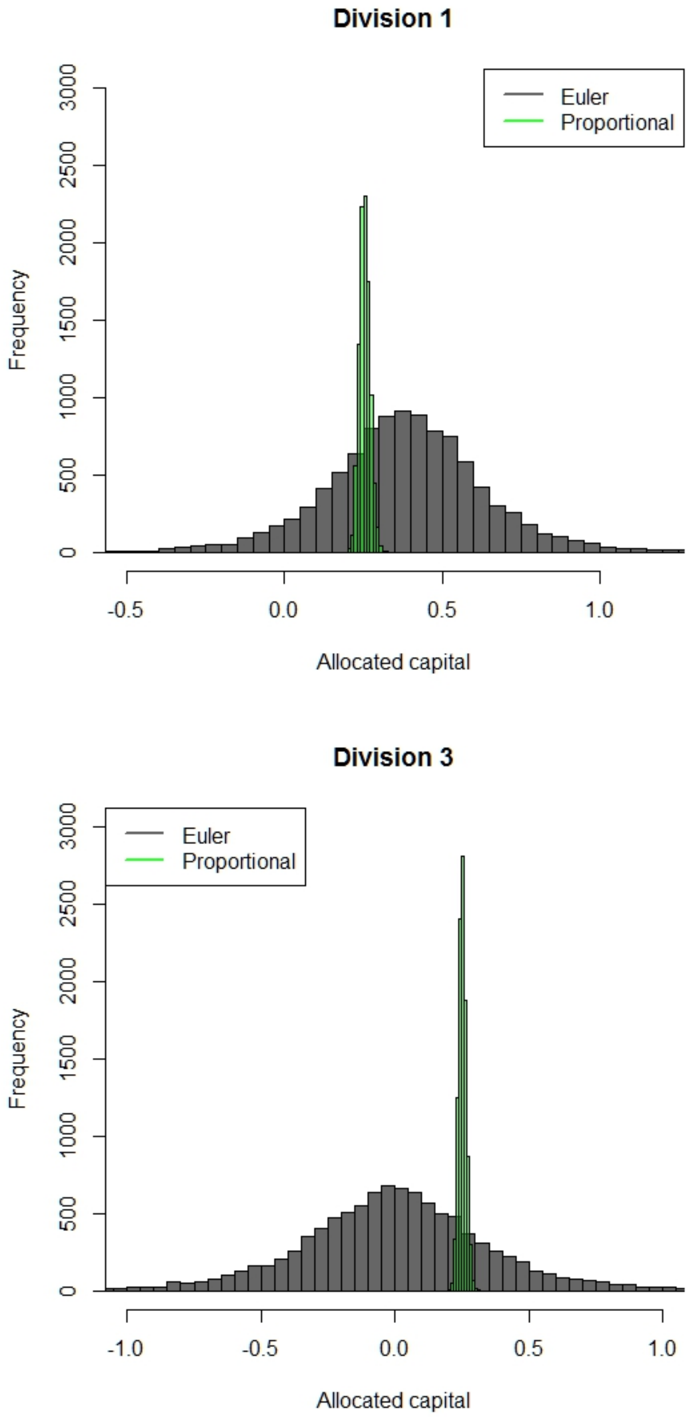
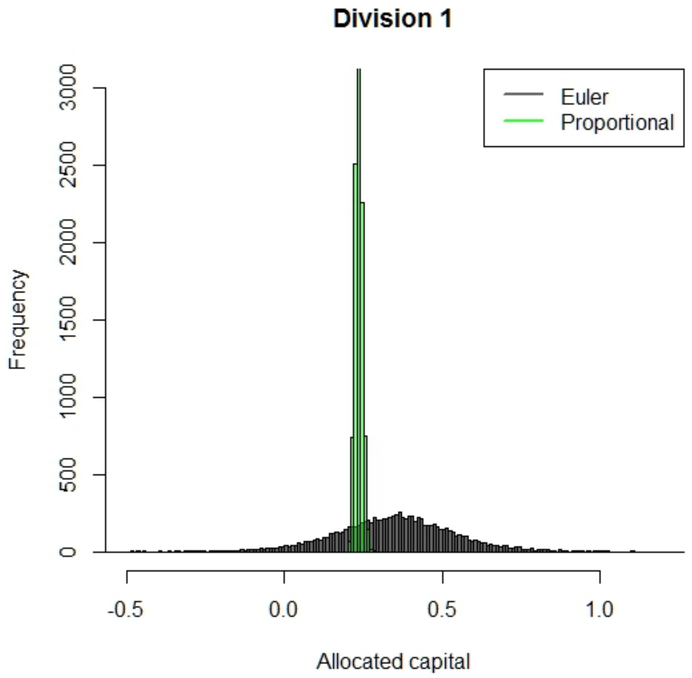


Figure 8: The Euler rule and proportional rule for Division 1 (top) and Division 3 (bottom). The past 10,000 days is simulated 10,000 times from a Student's t-distribution with $\nu = 3$, corresponding to Appendix B.



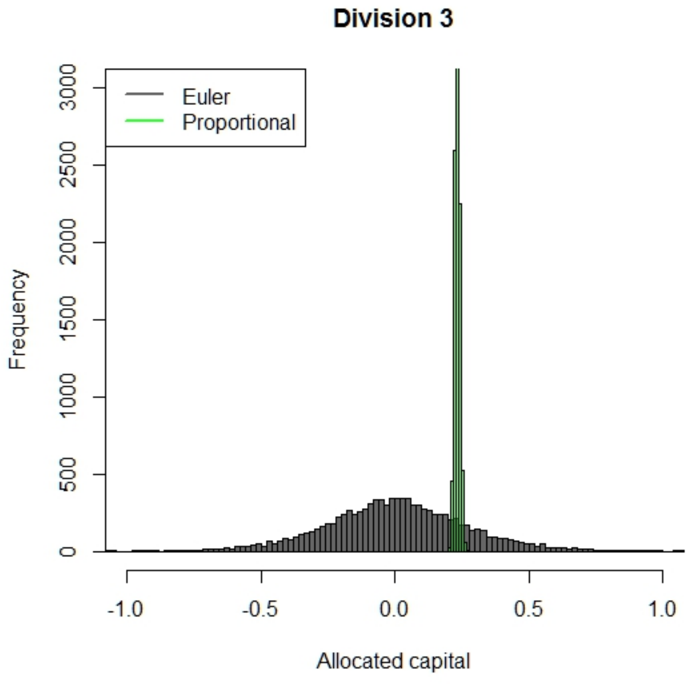


Figure 9: The Euler rule and proportional rule for Division 1 (top) and Division 3 (bottom). The past 10,000 days is simulated 10,000 times from a Student's t-distribution with $\nu = 5$, corresponding to Appendix B.