Optimal ratcheting of dividends with irreversible reinsurance

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Abstract

This paper considers an insurance company that faces two key constraints: a ratcheting dividend constraint and an irreversible reinsurance constraint. The company allocates part of its reserve to pay dividends to its shareholders while strategically purchasing reinsurance for its claims. The ratcheting dividend constraint ensures that dividend cuts are prohibited at any time. The irreversible reinsurance constraint ensures that reinsurance contracts cannot be prematurely terminated or sold to external entities. The dividend rate and reinsurance level are modeled as nondecreasing processes, thereby satisfying the constraints. Claims are modeled using a Brownian risk model. The main objective is to maximize the cumulative expected discounted dividend payouts until the time of ruin. The reinsurance and dividend levels are restricted to a finite set. The optimal value function is shown to be the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation. A threshold strategy is constructed and shown to be optimal. Finally, numerical examples are presented to illustrate the optimality conditions and optimal strategies.

Keywords: optimal dividends, ratcheting dividends, irreversible reinsurance, viscosity solutions, HJB equations, stochastic optimal control.

1 Introduction

The optimal dividend payout problem is a fundamental topic in the field of actuarial science. The foundations of this topic were laid in a seminal paper by de Finetti [1957], which explored the optimal way to distribute dividends to the company's shareholders. The key insight is that the optimal dividend payout strategy is the one that maximizes the expected discounted sum of all dividends paid to shareholders until the company's time of bankruptcy, also called the ruin time.

This optimization problem is critical for financial institutions, including insurance companies and banks, as they must strike a balance between paying dividends to shareholders and holding reserves to cover potential future liabilities. Consequently, de Finetti [1957] has inspired numerous variations and extensions of the optimal dividend payout problem. An overview of techniques and strategies for solving optimal dividend payout problems is presented in Albrecher and Thonhauser [2009] and Avanzi [2009].

The use of control techniques, such as viscosity solutions and the Hamilton-Jacobi-Bellman (HJB) approach, in optimal dividend payout problems is discussed in Azcue and Muler [2014] and Schmidli [2008]. The case where the dividend rate can be unbounded is also discussed in Schmidli [2008]. The reserve process is commonly modeled via the classical risk model, sometimes referred to as the Cramér-Lundberg model, or the diffusion approximation model. Jump-diffusion processes have also been used to model the reserve process, for example in Belhaj [2010].

The term *ratcheting* has been used in the context of optimal dynamic consumption and investment problems, as discussed in Duesenberry [1949] and Dybvig [1995], where consumption is not allowed to fall over time and increases proportionately as wealth reaches a new threshold or maximum. The dividend ratcheting constraint is motivated by practitioners' observations and by the fact that shareholders are generally unhappy about reductions in dividend payments, as discussed in Avanzi et al. [2016] for the case of insurance companies. The positive and significant share-price response of insurers to dividend increases is discussed in Akhigbe et al. [1993].

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The dividend ratcheting constraint was first introduced in the context of optimal dividend payout problems in Albrecher et al. [2018], where the dividend rate can be increased only once and must not exceed the drift rate of the original reserve process under the Lévy risk model and the Brownian risk model. The general case of the ratcheting constraint, where the dividend rate is allowed to increase more than once during the lifetime of the process, is discussed in Albrecher et al. [2020] under the compound Poisson process and in Albrecher et al. [2022] under the Brownian risk model. There is more regularity present in the value function under the Brownian risk model; hence, an explicit differential equation was derived and used to identify the candidates for the optimal strategies in the latter work. Some extensions and variations include a habit-formation (or drawdown) constraint with ratcheting dividends as in Angoshtari et al. [2019], a periodic dividend payout with ratcheting as in Song and Sun [2023] and Sun and Song [2024], and ratcheting with capital injection as in Wang et al. [2024].

Chang et al. [1996] argue that reinsurance negotiations are expensive, time-consuming, and irreversible, unlike insurance futures and call spreads, which are standardized and exchange-traded. Reinsurance contracts are structured to provide long-term stability for the insurer, reflecting a mutual understanding between insurer and reinsurer regarding risk-sharing that cannot easily be modified without significant consequences for either party. With irreversible reinsurance, neither party can prematurely terminate the agreement, nor can the insurer lower the reinsurance coverage at any time. This arrangement benefits the insurer by allowing it to hedge an increasing proportion of its risks, thereby enhancing financial stability and the capacity to manage future uncertainties.

The modeling of irreversible reinsurance is motivated by problems such as the finite-fuel problem in Bank [2005], the irreversible investment problem in Bertola and Caballero [1994], and the irreversible liquidation problem in Ferrari and Koch [2021], all of which employ techniques for singular control problems. Yan et al. [2022] formulate the irreversible reinsurance problem using a singular control problem under a mean-reverting risk exposure process. Their objective function consists of a running cost function generated by the risk exposure and the cost of purchasing new reinsurance contracts. The work of Brachetta and Ceci [2021] proposes an optimal reinsurance problem, wherein a reinsurance contract with some fixed cost is purchased and simultaneously the risk retention level is decided at some random time before maturity. The work of Federico et al. [2024] studies the minimization of the flow of capital injections while purchasing perpetual reinsurance contracts at certain random times.

The optimal dividend payout with proportional reinsurance problem under the diffusion approximation, without the ratcheting dividend constraint and the irreversible reinsurance constraint, was first studied in Højgaard and Taksar [1999]. Excess-of-loss reinsurance policies have also been considered in optimal dividend payout problems, such as in Asmussen et al. [2000]. Impulse control has been used to solve optimal dividend payout problems with proportional reinsurance, such as in Wei et al. [2010], where regime-switching was also introduced.

In this work, we study a company that seeks to optimize dividend payouts subject to a ratcheting dividend constraint and an irreversible reinsurance constraint. This extends the work of Albrecher et al. [2022] by incorporating reinsurance and introducing the irreversible reinsurance constraint, resulting in a multi-dimensional stochastic control problem. The rate at which the company pays dividends is modeled as a nondecreasing process, while the level of risk retained by the company is modeled as a nonincreasing process. Consequently, the level of reinsurance is modeled as a nondecreasing process.

The discussion begins by modeling the reserve process via the classical Cramér-Lundberg model. We assume that the company's premium rate is computed using the expected value principle and that the company's incoming claims are reinsured using *proportional reinsurance*. Besides analytical tractability, proportional reinsurance offers a more straightforward mechanism for risk-sharing. It is also advantageous to the insurer in the sense that the reinsurer is required to participate in all types of claims.

We use the result of Grandell [1977] to obtain a diffusion approximation of the reserve process with proportional reinsurance. The value function exhibits greater regularity under the diffusion approximation of the Cramér-Lundberg model, which significantly enhances the analytical tractability of the problem. This allows for an analytical solution to the HJB equation, which serves as a basis for the construction of the optimal strategies. The diffusion approximation also mitigates the complexities associated with sudden claims, since one of its key assumptions is that the expected number of claims is sufficiently large. The dynamics of the controlled reserve process are then introduced; this process is essentially the diffusion approximation of the reserve process reduced by the dividend rate process. The goal is then to maximize the expected discounted dividends paid until ruin time.

The dynamic programming approach is then used to obtain the HJB equation. We start with the case of constant dividend and reinsurance levels, which yields a characteristic equation that motivates the form of the candidate value function. We then discuss the HJB equation for the case where the dividend and reinsurance levels belong to finite sets. We introduce viscosity solutions because the candidate value function is not necessarily twice continuously differentiable. The form of the value function is derived using two approaches: backward recursion and scale functions. The optimal strategy is constructed by determining the appropriate threshold levels for the dividend and reinsurance variables. The order in which these levels are changed must be carefully analyzed, which is one of the main differences from the work of Albrecher et al. [2022].

The main contributions of this paper are twofold. First, we obtain a recursive formula for the optimal value function when the reinsurance levels and dividend rates belong to finite sets. This restriction allows a clear construction of the optimal reinsurance and dividend strategies, simplifying insurers' decision-making. For instance, insurers can easily select from a finite set of retention levels that align well with their risk appetite when considering additional reinsurance coverage. Moreover, when presenting risk strategies to stakeholders, the use of finite choices creates a clearer narrative. This clarity can significantly improve discussions around risk management strategies, allowing stakeholders to better understand the implications of various decisions.

Second, the numerical illustrations in Section 5 provide new insights into the dynamics of optimal retention and dividend threshold levels in the context of ratcheting dividends and the irreversibility of reinsurance. We find that, under typical market regimes, it is almost always optimal to increase dividends before transferring additional risk to the reinsurer. We also observe that decreasing the retention level requires relatively high reserves. This result contrasts with the findings of Højgaard and Taksar [1999], who study an optimal dividend payout problem involving proportional reinsurance without imposing constraints on dividends and reinsurance. In their study, optimal retention levels decrease as the reserve level rises. In our framework, however, the irreversibility constraint requires insurers to increase their reserves before considering an increase in their reinsurance coverage.

A key factor influencing this dynamic is the ratcheting constraint on dividend rates. Increasing dividend rates results in a lower drift under the diffusion approximation model for the reserve process, effectively slowing the growth of the reserve level. This interplay between the two constraints highlights their impact on optimal strategies and offers a fresh and valuable perspective. It also provides actionable insights that may meaningfully influence insurers' strategic decisions in practice.

The paper is organized as follows. Section 2 discusses the model for the reserve process and the properties of the value function. Section 3 discusses the main results. The derivation of the optimal strategies is discussed in Section 4. Numerical illustrations are presented in Section 5. Section 6 concludes. Appendix A provides the intermediate lemmas, propositions, and their proofs, while Appendix B contains the proofs of the main results. An alternative derivation of the form of the optimal value function is provided in Appendix C.

2 Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We consider an insurance company that simultaneously uses part of its reserve to pay dividends to shareholders and strategically reinsures some of its claims. Suppose that the company's reserve process follows the classical Cramér-Lundberg model given by $R_t = x + pt - \sum_{i=1}^{N_t} Z_i$, where $x > 0$ is the initial reserve, $p > 0$ is the premium rate on the risk that the company is insuring, $\{N_t\}_{t\geq 0}$ is a Poisson process modeling claim frequency with intensity parameter $\lambda > 0$, and $\{Z_i\}_{i=1,2,\ldots}$ is a series of positive-valued, independent and identically distributed (i.i.d.) random variables with Z_i denoting the size of the ith loss. We assume that $\{N_t\}_{t\geq 0}$ and $\{Z_i\}_{i=1,2,\ldots}$ are independent under $\mathbb{P}$. We further suppose that the i.i.d. random variables $Z_1, Z_2, \ldots$ have a common distribution with finite mean μ_0 and finite variance σ_0^2.

We assume that the company's premium rate p is calculated using the expected value principle with relative safety loading $\gamma > 0$, that is, $p = (1+\gamma)\lambda\mu_0$. We further assume that the claims are reinsured by proportional reinsurance with retention level $A \in [0, 1]$. For a claim Z_i, the company pays AZ_i and the reinsurer pays $(1-A)Z_i$. Suppose that the reinsurance premium p^A is calculated using the expected value principle with relative safety loading $\theta > 0$, that is, $p^A = (1+\theta)(1-A)\lambda\mu_0$. The reserve process with proportional reinsurance, denoted by R^A, can then be written as

$$R_t^A = x + (p - p^A)t - A\sum_{i=1}^{N_t} Z_i = x + [A(1+\theta) - (\theta - \gamma)]\,\lambda\mu_0 t - A\sum_{i=1}^{N_t} Z_i.$$

The company can then choose a reinsurance strategy $A := \{A(t)\}_{t\geq 0}$, where $A(t)$ represents the proportion of the risk retained by the insurer, or simply the *retention level*, at time t. Following Grandell [1977], the diffusion approximation of the reserve process, denoted by $X = \{X_t\}_{t\geq 0}$ is given by

$$X_t = x + \int_0^t [\mu A(s) - b]\,ds + \int_0^t \sigma A(s) dW_s, \tag{1}$$

where $\mu := \theta\lambda\mu_0 > 0$, $b := (\theta - \gamma)\lambda\mu_0$, $\sigma := \sqrt{\lambda\sigma_0^2} > 0$, and $W := \{W_t\}_{t\geq 0}$ is a standard Brownian motion. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$, which is the $\mathbb{P}$-augmentation of the natural filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$ generated by W.

Remark 2.1. *The idea of the diffusion approximation in Grandell [1977] is to increase the number of claims while making their sizes smaller. Define $\overline{S}(t) := \sum_{i=1}^{N_t} Z_i$, the aggregate claims process. The approximation relies on a functional central limit theorem. Consider the rescaled and centered processes $\overline{S}_n(t) := [\overline{S}(nt) - \lambda\mu_0 nt](\lambda\sigma_0^2 n)^{-1/2}$. Then, as $n \to \infty$, we have $\overline{S}_n \xrightarrow{d} \tilde{W}$; that is, $\overline{S}_n$ converges in distribution to a standard Brownian motion $\tilde{W}$. More precisely, the convergence holds weakly in the Skorokhod space $D[0,\infty)$. The premium rates are also scaled accordingly so that the first and second moments of the reserve process remain unchanged. Taking $W := -\tilde{W}$ yields the result by the symmetry (scaling invariance) of Brownian motions.*

Moreover, the diffusion approximation of the reserve process (1) has a form that also works for the following premium principles:

- *Standard Deviation Premium Principle: $p = \lambda\mu_0 + \gamma\sqrt{\lambda\sigma_0^2}$ and $p^A = \lambda(1-A)\mu_0 + \theta(1-A)\sqrt{\lambda\sigma_0^2}$ implies $\mu = \theta\sqrt{\lambda\sigma_0^2}$, $b = (\theta-\gamma)\sqrt{\lambda\sigma_0^2}$, and $\sigma = \sqrt{\lambda\sigma_0^2}$.*
- *Modified Variance Principle: $p = \lambda\mu_0 + \gamma\frac{\sigma_0^2}{\mu_0}$ and $p^A = \lambda(1-A)\mu_0 + \theta(1-A)\frac{\sigma_0^2}{\mu_0}$ implies $\mu = \theta\frac{\sigma_0^2}{\mu_0}$, $b = (\theta-\gamma)\frac{\sigma_0^2}{\mu_0}$, and $\sigma = \sqrt{\lambda\sigma_0^2}$.*

Denote $C := \{C(t)\}_{t\geq 0}$ as the dividend strategy, where $C(t)$ is the rate at which the company pays dividends at time t. When applying the policy $\pi := (A, C)$, let $\{X_t^\pi\}_{t\geq 0}$ be the controlled reserve process. The dynamics for X_t^π can then be written as

$$\begin{cases} dX_t^\pi = [\mu A(t) - b]\,dt + \sigma A(t) dW_t - C(t)dt, \\ X_0^\pi = x. \end{cases}$$

Suppose that the retention level belongs to a set $\mathscr{A} \subseteq [\underline{a}, 1]$, where $\underline{a} \in [0,1]$ is the minimum retention level possible and satisfies $\underline{a} \in \mathscr{A}$, while the dividend rates belong to a set $\mathscr{C} \subseteq [0, \overline{c}]$, where $0 \leq \overline{c} \in \mathscr{C}$ is the maximum dividend rate possible. Given an initial reserve $X_0^\pi = x \geq 0$, a maximum initial retention level $a \in \mathscr{A}$, and a minimum initial dividend rate $c \in \mathscr{C}$, the policy π is said to be admissible if

(i) the processes $A = \{A(t)\}_{t\geq 0}$ and $C = \{C(t)\}_{t\geq 0}$ are adapted to $\mathbb{F}$,

(ii) A is nonincreasing, right-continuous, and $A(t) \in \mathscr{A}$ for all $t \geq 0$,

(iii) C is nondecreasing, right-continuous, and $C(t) \in \mathscr{C}$ for all $t \geq 0$.

Denote by $\Pi_{x,a,c}^{\mathscr{A},\mathscr{C}}$ the set of all admissible policies.

Remark 2.2. *A maximum initial retention level $a \in \mathscr{A}$ and a minimum initial dividend rate $c \in \mathscr{C}$ mean that $\underline{a} \leq A(0) \leq a$ and $c \leq C(0) \leq \overline{c}$. This further implies that given $a_1, a_2 \in \mathscr{A}$ with $a_1 \geq a_2$, we have $\Pi_{x,a_2,c}^{\mathscr{A},\mathscr{C}} \subseteq \Pi_{x,a_1,c}^{\mathscr{A},\mathscr{C}}$ for any $x \geq 0$ and $c \in \mathscr{C}$. In a similar way, given $c_1, c_2 \in \mathscr{C}$ with $c_1 \leq c_2$, we have $\Pi_{x,a,c_2}^{\mathscr{A},\mathscr{C}} \subseteq \Pi_{x,a,c_1}^{\mathscr{A},\mathscr{C}}$ for any $x \geq 0$ and $a \in \mathscr{A}$. It is important to note that this inclusion property will not hold if the definition of admissible policies requires $A(0) = a$ and $C(0) = c$. For clarity, we also mention that $a = \arg\max \mathscr{A}$ and $c = \arg\min \mathscr{C}$ do not necessarily hold.*

Remark 2.3. *The dividend rate $\bar{c}$ establishes an upper bound on the dividend rates. On the other hand, the minimum retention level $\underline{a}$ ensures that retention levels do not fall below this threshold. Consequently, if $\underline{a} > 0$, then "full reinsurance" is not possible regardless of the reserve level.*

Given $\pi = (A, C) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}$, the value function is given by

$$J(x; A, C) = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-qs} C(s)ds\right],$$

where $q > 0$ is the discount factor and $\tau_\pi := \inf\{t \geq 0 : X^\pi_t < 0\}$ is the ruin time. For any initial reserve $x \geq 0$, initial retention level $a \in \mathscr{A}$, and initial dividend rate $c \in \mathscr{C}$, the goal is to find the optimal value function defined as

$$V^{\mathscr{A},\mathscr{C}}(x, a, c) = \sup_{\pi \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}} J(x; A, C). \tag{2}$$

To simplify the notation, we write $V := V^{\mathscr{A},\mathscr{C}}$ for the general sets $\mathscr{A}$ and $\mathscr{C}$.

Remark 2.4. *The optimal dividend payout with proportional reinsurance problem under the diffusion approximation without the ratcheting dividend constraint and the irreversible reinsurance constraint is first studied in Højgaard and Taksar [1999]. Their model setup is similar to this work without the constraints on the dividends and the reinsurance and without the constant term in the drift coefficient of the reserve process. Their problem is considered to be one-dimensional without these constraints. The additional constant term in the drift is a trivial extension of their work, and the same results should still hold. Hence, if we denote by $V_{NC}(x)$ the optimal value function without constraints, then it is clear that $V_{NC}(x) \geq V(x, a, c)$ for all $x \geq 0$, $a \in \mathscr{A}$, and $c \in \mathscr{C}$. By Theorem 2.3 of Højgaard and Taksar [1999], V_{NC} is increasing, concave, twice continuously differentiable with $V_{NC}(0) = 0$ and $\lim_{x\to\infty} V_{NC}(x) = \frac{\bar{c}}{q}$. Thus, for any $0 \leq x_1 \leq x_2$, V_{NC} is Lipschitz, that is, it satisfies*

$$0 \leq V_{NC}(x_2) - V_{NC}(x_1) \leq L_0(x_2 - x_1), \tag{3}$$

where $L_0 = V'_{NC}(0)$.

The above remark on the relationship between the optimal value functions with constraints and no constraints is useful in proving some properties of the optimal value function V considered in this paper. In the remainder of this section, we state some properties of V. Moreover, for the rest of the paper, all proofs are provided in the appendices. The following proposition states that V satisfies boundedness and monotonicity.

Proposition 2.1. *The optimal value function $V(x, a, c)$ is bounded above by $\frac{\bar{c}}{q}$, nondecreasing in x and a, and nonincreasing in c.*

The optimal value function V never exceeds $\frac{\bar{c}}{q}$. Moreover, V increases with the reserve level and retention level, while it decreases with the dividend rate.

The next proposition states that V satisfies a Lipschitz property with respect to all of its variables. The proof requires the results in Claisse et al. [2016] for conditional expectations involving stopping times.

Proposition 2.2. *There exists a constant $L > 0$ such that*

$$0 \leq V(x_2, a_1, c_1) - V(x_1, a_2, c_2) \leq L\left[(x_2 - x_1) + (a_1 - a_2) + (c_2 - c_1)\right]$$

for all $0 \leq x_1 \leq x_2$, $a_1, a_2 \in \mathscr{A}$ with $a_2 \leq a_1$, and $c_1, c_2 \in \mathscr{C}$ with $c_1 \leq c_2$.

One way to interpret this property is that small or large perturbations in the reserve level, retention rate, or dividend rate do not lead to disproportionately large changes in the value of V. More precisely, the Lipschitz result implies that the change in V is bounded by a term that is proportional to the change in input values. Moreover, if V has bounded partial derivatives in x, a, and c, then the Lipschitz property is immediately satisfied. We will see in the next sections that V may not be smooth, that is, the partial derivatives may not exist.

The following lemma states that the optimal value function V satisfies the dynamic programming principle, which is essential to prove that V is a viscosity solution of the HJB equation, which is the subject of the next section. The proof is similar to that of Azcue and Muler [2014, Lemma 1.2].

Lemma 2.3. *Given any* $\mathbb{F}$*-stopping time* $\tilde{\tau}$*, we have*

$$V(x,a,c) = \sup_{(A,C)\in\Pi_{x,a,c}^{\mathscr{A},\mathscr{C}}} \mathbb{E}\left[\int_0^{\tau_\pi\wedge\tilde{\tau}} e^{-qs}C(s)ds + e^{-q(\tau_\pi\wedge\tilde{\tau})}V(X_{\tau_\pi\wedge\tilde{\tau}}^\pi, A(\tau_\pi\wedge\tilde{\tau}), C(\tau_\pi\wedge\tilde{\tau}))\right].$$

Throughout the paper, all stopping times are understood to be with respect to the filtration $\mathbb{F}$.

3 Main Results

In this section, we present the HJB equations for two different cases, each characterized by the form of the reinsurance set $\mathscr{A}$ and the dividend set $\mathscr{C}$: (1) **singleton sets**, where the retention level and the dividend rate are fixed, and (2) **finite sets**, where the retention level and the dividend rate can take on any value from a discrete set of choices. For the finite set case, we will prove that the optimal value function is the unique viscosity solution to its corresponding HJB equation.

3.1 Singleton Case

Consider the case $\mathscr{A} = \{a\}$ and $\mathscr{C} = \{c\}$ (i.e., the singleton set case). In this case, the unique admissible strategy consists of having a constant retention level a retained by the insurer and paying a constant dividend rate c up to the ruin time. The value function, denoted by $V^{\{a\},\{c\}}(x,a,c)$, is the unique solution of the second-order linear ordinary differential equation:

$$\mathcal{L}^{a,c}\left(V^{\{a\},\{c\}}\right) := \frac{1}{2}\sigma^2a^2V_{xx}^{\{a\},\{c\}} + (\mu a - b - c)V_x^{\{a\},\{c\}} - qV^{\{a\},\{c\}} + c = 0 \tag{4}$$

with boundary conditions $V^{\{a\},\{c\}}(0,a,c) = 0$ and $\lim_{x\to\infty} V^{\{a\},\{c\}}(x,a,c) = \frac{c}{q}$. The solutions of (4) are of the form $K_1e^{\theta_1(a,c)x} + K_2e^{\theta_2(a,c)x} + \frac{c}{q}$, where $K_1, K_2 \in \mathbb{R}$, and $\theta_1(a,c)$ and $\theta_2(a,c)$ are defined as $\theta_1(a,c) := \frac{b+c-\mu a+\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2}}{\sigma^2a^2}$ and $\theta_2(a,c) := \frac{b+c-\mu a-\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2}}{\sigma^2a^2}$. It is clear that $\theta_2(a,c) < 0 < \theta_1(a,c)$ since $\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2} > \sqrt{(b+c-\mu a)^2} \geq b+c-\mu a$. The other properties of the functions $\theta_1(a,c)$ and $\theta_2(a,c)$ and their partial derivatives are stated in Proposition A.1.

The solutions of (4) with boundary condition $V^{\{a\},\{c\}}(0) = 0$ are of the form

$$K\left[e^{\theta_1(a,c)x} - e^{\theta_2(a,c)x}\right] + \frac{c}{q}\left[1 - e^{\theta_2(a,c)x}\right], \tag{5}$$

where $K \in \mathbb{R}$. The unique solution of (4) satisfying the boundary conditions $V^{\{a\},\{c\}}(0,a,c) = 0$ and $\lim_{x\to\infty} V^{\{a\},\{c\}}(x,a,c) = \frac{c}{q}$ corresponds to $K = 0$. Hence, we have $V^{\{a\},\{c\}}(x,a,c) = \frac{c}{q}\left[1 - e^{\theta_2(a,c)x}\right]$. It is clear that $V^{\{a\},\{c\}}(x,a,c)$ is nondecreasing and concave in x.

Remark 3.1. *We have* $V(x,a,c) \geq V^{\{a\},\{c\}}(x,a,c) = \frac{c}{q}\left[1 - e^{\theta_2(a,c)x}\right]$*. Together with Proposition 2.1 and the fact that* $\theta_2(a,c) < 0$*, we obtain* $\lim_{x\to\infty} V(x,a,c) = \frac{c}{q}$*.*

3.2 Finite Set Case

We now consider the finite set case; that is, $\mathscr{A} := \{a_1, a_2, \ldots, a_m\}$ with $\underline{a} = a_m < a_{m-1} < \cdots < a_1 \leq 1$ and $\mathscr{C} := \{c_1, c_2, \ldots, c_n\}$ with $0 \leq c_1 < c_2 < \cdots < c_n = \bar{c}$. By definition, $V^{\mathscr{A},\mathscr{C}}(x,a_i,c_j) = V(x,a_i,c_j)$.

For $1 \leq i \leq m$ and $1 \leq j \leq n$, we simplify the notation as follows:

$$V^{a_i,c_j}(x) := V^{\mathscr{A},\mathscr{C}}(x,a_i,c_j). \tag{6}$$

Using Proposition 2.1, we have $V^{a_i,c_j}(x) \geq V^{a_{i+1},c_j}(x) \geq \cdots \geq V^{a_m,c_j}(x)$ for a fixed $1 \leq j \leq n-1$, and $V^{a_i,c_j}(x) \geq V^{a_i,c_{j+1}}(x) \geq \cdots \geq V^{a_i,c_n}(x)$ for a fixed $1 \leq i \leq m-1$. Moreover, to avoid separate boundary conditions, we adopt the convention that $V^{a_{i+1},c_j}(x) - V^{a_i,c_j}(x) = -\infty$ if $i = m$ and $V^{a_i,c_{j+1}}(x) - V^{a_i,c_j}(x) = -\infty$ if $j = n$. The HJB equation associated to (6) is given by

$$\max\{\mathcal{L}^{a_i,c_j}(V^{a_i,c_j})(x), V^{a_{i+1},c_j}(x) - V^{a_i,c_j}(x), V^{a_i,c_{j+1}}(x) - V^{a_i,c_j}(x)\} = 0, \tag{7}$$

for $x \geq 0$, $1 \leq i \leq m$, and $1 \leq j \leq n$.

We now define viscosity subsolutions and supersolutions for the HJB equation (7).

Definition 3.1. A (locally Lipschitz) function $u : [0,\infty) \to \mathbb{R}$ is a viscosity supersolution (subsolution, resp.) of (7) at $x \in (0,\infty)$ if any twice continuously differentiable function $\varphi : [0,\infty) \to \mathbb{R}$ with $\varphi(x) = u(x)$, such that $u - \varphi$ reaches a minimum (maximum, resp.) at x, satisfies

$$\max\{\mathcal{L}^{a_i,c_j}(\varphi)(x), V^{a_{i+1},c_j}(x) - \varphi(x), V^{a_i,c_{j+1}}(x) - \varphi(x)\} \leq 0 \quad (\geq 0,\ \text{resp.}). \tag{8}$$

Moreover, u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

We now state the first main result. The following theorem states that V^{a_i,c_j} is the unique viscosity solution of (7) and satisfies the boundary conditions.

Theorem 3.2. *The optimal value function $V^{a_i,c_j}(x)$ for $i = 1,\dots,m$ and $j = 1,\dots,n$ is the unique viscosity solution of the associated HJB equation (7) with the boundary conditions $V^{a_i,c_j}(0) = 0$ and $\lim_{x\to\infty} V^{a_i,c_j}(x) = \frac{\bar{c}}{q}$.*

Since $V^{a_i,c_j}(x)$ is the viscosity solution of (7) via Theorem 3.2, we can infer that there are values of x such that (i) $\mathcal{L}^{a_i,c_j}(V^{a_i,c_j}) = 0$, (ii) $V^{a_i,c_j}(x) = V^{a_{i+1},c_j}(x)$, and (iii) $V^{a_i,c_j}(x) = V^{a_i,c_{j+1}}(x)$. Hence, we can partition $(0,\infty)$ in two ways. The first way is fixing j (i.e., the dividend level is fixed). For $i < m$, $(0,\infty)$ can be partitioned in such a way that the optimal strategy is to retain a_i of the incoming claims when the current reserve is in the open set $\{x : V^{a_i,c_j}(x) > V^{a_{i+1},c_j}(x)\}$ and to decrease the risk exposure to a_{i+1} when the current reserve is in the closed set $\{x : V^{a_i,c_j}(x) = V^{a_{i+1},c_j}(x)\}$. The second way is fixing i (i.e., fixing the retention level). Similarly, for $j < n$, the optimal strategy is to pay dividends at rate c_j of the incoming claims when the current reserve is in the open set $\{x : V^{a_i,c_j}(x) > V^{a_i,c_{j+1}}(x)\}$ and to increase the dividend rate to c_{j+1} when the current reserve is in the closed set $\{x : V^{a_i,c_j}(x) = V^{a_i,c_{j+1}}(x)\}$.

Define the functions $y : \mathscr{A} \times \mathscr{C} \to [0,\infty]$ and $z : \mathscr{A} \times \mathscr{C} \to [0,\infty]$ such that $y(a_m, c_j) = +\infty$ for $1 \leq j \leq n$ and $z(a_i, c_n) = +\infty$ for $1 \leq i \leq m$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, we simplify the notation as follows

$$y_{i,j} := y(a_i, c_j) \quad \text{and} \quad z_{i,j} := z(a_i, c_j).$$

We call the values $y_{i,j}$ and $z_{i,j}$ the retention threshold and dividend threshold, respectively, at retention level a_i and dividend rate c_j. The functions y and z are called the retention threshold function and the dividend threshold function, respectively.

The next theorem states the form of the optimal value function.

Theorem 3.3. *Define the function $W^{y,z}(x, a_i, c_j)$ such that it satisfies the following recursive formula:*

$$W^{y,z}(x, a_m, c_n) = \frac{c_n}{q}\left[1 - e^{\theta_2(a_m,c_n)x}\right],$$

$$W^{y,z}(x, a_i, c_j) = \begin{cases} \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)x}\right] + k^{y,z}(a_i,c_j)\left[e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}\right] & \text{if } x < y_{i,j} \wedge z_{i,j}, \\ W^{y,z}(x, a_{i+1}, c_j)\mathbf{1}_{\{y_{i,j} < z_{i,j}\}} + W^{y,z}(x, a_i, c_{j+1})\mathbf{1}_{\{y_{i,j} > z_{i,j}\}} & \\ +W^{y,z}(x, a_{i+1}, c_{j+1})\mathbf{1}_{\{y_{i,j} = z_{i,j}\}} & \text{otherwise}, \end{cases}$$

for $i < m$ or $j < n$, where

$$k^{y,z}(a_i,c_j) := \begin{cases} \dfrac{W^{y,z}(y_{i,j}, a_{i+1}, c_j) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)y_{i,j}}\right]}{e^{\theta_1(a_i,c_j)y_{i,j}} - e^{\theta_2(a_i,c_j)y_{i,j}}} & \text{if } y_{i,j} < z_{i,j}, \\ \dfrac{W^{y,z}(z_{i,j}, a_i, c_{j+1}) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)z_{i,j}}\right]}{e^{\theta_1(a_i,c_j)z_{i,j}} - e^{\theta_2(a_i,c_j)z_{i,j}}} & \text{if } y_{i,j} > z_{i,j}, \\ \dfrac{W^{y,z}(y_{i,j}, a_{i+1}, c_{j+1}) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)y_{i,j}}\right]}{e^{\theta_1(a_i,c_j)y_{i,j}} - e^{\theta_2(a_i,c_j)y_{i,j}}} & \text{if } y_{i,j} = z_{i,j}. \end{cases}$$

Suppose y^* and z^* are the optimal threshold functions in the sense that $k^{y^*,z^*}(a_i,c_j)$ is the maximum for any $i < m$ or $j < n$. Then for $i = 1,\dots,m$ and $j = 1,\dots,n$, we have that $W^{y^*,z^*}(x, a_i, c_j)$ is the optimal value function $V^{a_i,c_j}(x)$ in (2).

Remark 3.2. *A derivation of the value function characterized in Theorem 3.3 based on scale functions in the theory of Lévy fluctuations is given in Appendix C. For more details on scale functions, we refer the interested readers to Kyprianou [2014].*

Remark 3.3. *If $i = m$ and $n = j$, then the corresponding problem is reduced to the singleton case. This is because we have reached the maximum possible dividend rate and the minimum possible retention level. Hence, the value function is concave in x. If $i < m$ or $n < j$, the concavity of the value function is not guaranteed. Figure 1b in Section 5 shows an example of a value function that is not necessarily concave.*

4 Derivation of the Optimal Strategy

In the previous section, we presented the form of the optimal value function. However, the optimal threshold functions y^* and z^* in Theorem 3.3 have not yet been specified. We can immediately observe from Theorem 3.3 that the optimal reinsurance and dividend strategies are of a stationary threshold type. This means that the strategy depends on the current levels of reserve, retention, and dividend. In this section, we will discuss how the form of the value function is derived and how the optimal threshold strategies y^* and z^* are constructed.

Consider the reinsurance-dividend strategies wherein the corresponding value function $v^{a_i,c_j}(x)$ satisfies the following:

(i) $v^{a_m,c_j}(x) = V^{a_m,c_j}(x)$ for $1 \le j \le n$ and $v^{a_i,c_n}(x) = V^{a_i,c_n}(x)$ for $1 \le i \le m$.

(ii) For $i < m$, $\mathcal{L}^{a_i,c_n}(v^{a_i,c_n}(x)) = 0$ if $x \in [0, y_{i,n})$ and $v^{a_i,c_n}(x) = v^{a_{i+1},c_n}(x)$ if $x \in [y_{i,n}, \infty)$.

(iii) For $j < n$, $\mathcal{L}^{a_m,c_j}(v^{a_m,c_j}(x)) = 0$ if $x \in [0, z_{m,j})$ and $v^{a_m,c_j}(x) = v^{a_m,c_{j+1}}(x)$ if $x \in [z_{m,j}, \infty)$.

(iv) For $i < m$ and $j < n$, $\mathcal{L}^{a_i,c_j}(v^{a_i,c_j}(x)) = 0$ if $x \in [0, y_{i,j} \wedge z_{i,j})$, $v^{a_i,c_j}(x) = v^{a_{i+1},c_j}(x)$ if $x \in [y_{i,j}, \infty)$, and $v^{a_i,c_j}(x) = v^{a_i,c_{j+1}}(x)$ if $x \in [z_{i,j}, \infty)$.

Denote the threshold strategy as

$$\pi^{y,z} := \left(\pi_{x,a_i,c_j}\right)_{(x,a_i,c_j)\in[0,\infty)\times\mathscr{A}\times\mathscr{C}}, \tag{9}$$

where $\pi_{x,a_i,c_j} := (A_{x,a_i,c_j}, C_{x,a_i,c_j}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_i,c_j}$ for $(x, a_i, c_j) \in [0,\infty) \times \mathscr{A} \times \mathscr{C}$. The threshold strategy $\pi^{y,z}$ is stationary (i.e. it depends on the current reserve level, retention level, and dividend rate) and is therefore represented as a family of admissible controls indexed by the initial state. That is, for each initial state (x, a_i, c_j), it assigns an admissible control $(A_{x,a_i,c_j}, C_{x,a_i,c_j}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_i,c_j}$. The threshold strategy $\pi^{y,z}$ is defined via backward recursion as follows:

(i) If $i = m$ and $j = n$, retain a_m of the incoming claims and pay dividends with rate c_n until ruin time, that is, $\pi_{x,a_m,c_n}(t) = (A_{x,a_m,c_n}, C_{x,a_m,c_n})(t) = (a_m, c_n)$.

(ii) If $i = m$, $j < n$, and $x \in [z_{m,j}, \infty)$, follow $\pi_{x,a_m,c_l} = (A_{x,a_m,c_l}, C_{x,a_m,c_l})$, where

$$l = \begin{cases} n & \text{if } x \ge z_{m,r},\ r = j+1,\dots,n-1, \\ \min\{r \in \{j+1,\dots,n-1\} : x < z_{m,r}\} & \text{otherwise.} \end{cases}$$

(iii) If $j = n$, $i < m$, and $x \in [y_{i,n}, \infty)$, follow $\pi_{x,a_k,c_n} = (A_{x,a_k,c_n}, C_{x,a_k,c_n})$, where

$$k = \begin{cases} m & \text{if } x \ge y_{s,n},\ s = i+1,\dots,m-1, \\ \min\{s \in \{i+1,\dots,m-1\} : x < y_{s,n}\} & \text{otherwise.} \end{cases}$$

(iv) If $i < m$, $j < n$, and $x \in [y_{i,j}, z_{i,j})$, follow $\pi_{x,a_k,c_j} = (A_{x,a_k,c_j}, C_{x,a_k,c_j})$, where

$$k = \begin{cases} m & \text{if } z_{s,j} > x \ge y_{s,j},\ s = i+1,\dots,m-1, \\ \min\{s \in \{i+1,\dots,m-1\} : z_{s,j} < y_{s,j}\} & \text{otherwise.} \end{cases}$$

(v) If $i < m$, $j < n$, and $x \in [z_{i,j}, y_{i,j})$, follow $\pi_{x,a_i,c_l} = (A_{x,a_i,c_l}, C_{x,a_i,c_l})$, where

$$l = \begin{cases} n & \text{if } y_{i,r} > x \ge z_{i,r},\ r = i+1,\dots,n-1, \\ \min\{r \in \{j+1,\dots,n-1\} : y_{i,r} < z_{i,r}\} & \text{otherwise.} \end{cases}$$

(vi) If $i < m$, $j < n$, and $x \in [y_{i,j} \vee z_{i,j}, \infty)$, follow $\pi_{x,a_k,c_l} = (A_{x,a_k,c_l}, C_{x,a_k,c_l})$, where

$$k = \begin{cases} m & \text{if } x \ge y_{s,j},\ s = i+1,\dots,m-1, \\ \min\{s \in \{i+1,\dots,m-1\} : x < y_{s,j}\} & \text{otherwise,} \end{cases}$$

and

$$l = \begin{cases} n & \text{if } x \ge z_{i,r},\ r = j+1,\dots,n-1, \\ \min\{r \in \{j+1,\dots,n-1\} : x < z_{i,r}\} & \text{otherwise.} \end{cases}$$

(vii) If $i < m$, $j < n$, and $x \in [0, y_{i,j} \wedge z_{i,j})$, retain a_i of the incoming claims and pay dividends with rate c_j until ruin time τ_π or until before the current reserve reaches $y_{i,j} \wedge z_{i,j}$, whichever comes first. If the current reserves reach $y_{i,j}$ before ruin time, follow $\pi_{x,a_{i+1},c_j} = (A_{x,a_{i+1},c_j}, C_{x,a_{i+1},c_j})$. If the reserves reach $z_{i,j}$ before ruin time, follow $\pi_{x,a_i,c_{j+1}} = (A_{x,a_i,c_{j+1}}, C_{x,a_i,c_{j+1}})$. If $y_{i,j} = z_{i,j}$ is reached before ruin time, follow $\pi_{x,a_{i+1},c_{j+1}} = (A_{x,a_{i+1},c_{j+1}}, C_{x,a_{i+1},c_{j+1}})$. That is,

$$\pi_{x,a_i,c_j}(t) = \begin{cases} (a_i, c_j) & \text{if } t < (\tau_a \wedge \tau_c) < \tau_\pi \text{ or } t < \tau_\pi < (\tau_a \wedge \tau_c), \\ \pi_{X_{\tau_a},a_{i+1},c_j}(t) & \text{if } \tau_a < t < (\tau_\pi \wedge \tau_c), \\ \pi_{X_{\tau_c},a_i,c_{j+1}}(t) & \text{if } \tau_c < t < (\tau_\pi \wedge \tau_a), \\ \pi_{X_{(\tau_a \vee \tau_c)},a_{i+1},c_{j+1}}(t) & \text{if } (\tau_a \vee \tau_c) < t < \tau_\pi, \end{cases}$$

where τ_a and τ_c are the first times the reserve level hits $y_{i,j}$ and $z_{i,j}$, respectively.

The expected payoff of the strategy $\pi^{y,z}$ is defined as $J(x; A_{x,a_i,c_j}, C_{x,a_i,c_j})$ and is equal to $W^{y,z}(x, a_i, c_j)$ defined in Theorem 3.3. Since the strategy is to retain a_i of the incoming claims and pay dividends with rate c_j whenever $x \in [0, y_{i,j} \wedge z_{i,j})$, we have $\mathcal{L}^{a_i,c_j}(W^{y,z})(x, a_i, c_j) = 0$ for $x \in [0, y_{i,j} \wedge z_{i,j})$. Moreover, $W^{y,z}(0, a_i, c_j) = 0$ since ruin is immediate as soon as the reserve level hits 0. Lastly, by definition, $W^{y,z}(x, a_i, c_j) = W^{y,z}(x, a_{i+1}, c_j)$ if $x \geq y_{i,j}$ and $W^{y,z}(x, a_i, c_j) = W^{y,z}(x, a_i, c_{j+1})$ if $x \geq z_{i,j}$. The formula for $W^{y,z}$ then follows from (5).

We now seek to maximize the expected payoff $W^{y,z}(x, a_i, c_j)$ over all possible threshold functions y and z. Recall from Theorem 3.3 that y^* and z^* are the optimal threshold functions in the sense that $k^{y^*,z^*}(a_i, c_j)$ attains the maximum for any $i < m$ and $j < n$. Since the function $W^{y,z}(x, a_m, c_n)$ is known, we can solve this optimization problem using backward recursion.

We look for the optimal threshold strategy, which can be viewed as a sequence of $(m-1) \times (n-1)$ one-dimensional optimization problems. These optimization problems consist in maximizing $k^{y,z}(a_i, c_j)$ for $i = m-1, \ldots, 1$ and $j = n-1, \ldots, 1$. If $W^{y^*,z^*}(x, a_k, c_l)$, $y^*_{k,l}$, and $z^*_{k,l}$ are known for $k = i+1, \ldots, m-1$ and $l = j+1, \ldots, n-1$, then using the recursive formula for $W^{y,z}$, we can solve for $W^{y^*,z^*}(x, a_{i+1}, c_j)$ and $W^{y^*,z^*}(x, a_i, c_{j+1})$. Consequently, we can solve for $W^{y^*,z^*}(x, a_i, c_j)$, $y^*_{i,j}$, and $z^*_{i,j}$.

Define the continuous functions $G^{\mathcal{A}}_{i,j} : [0, \infty) \to \mathbb{R}$, $G^{\mathcal{C}}_{i,j} : [0, \infty) \to \mathbb{R}$, and $G^{\mathcal{E}}_{i,j} : [0, \infty) \to \mathbb{R}$ as

$$G^{\mathcal{A}}_{i,j}(x) := \begin{cases} \frac{W^{y^*,z^*}(x,a_{i+1},c_j) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)x}\right]}{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}} & \text{if } x > 0, \\ \frac{\partial_x W^{y^*,z^*}(0,a_{i+1},c_j) + \frac{c_j}{q}\theta_2(a_i,c_j)}{\theta_1(a_i,c_j) - \theta_2(a_i,c_j)} & \text{if } x = 0, \end{cases} \tag{10}$$

$$G^{\mathcal{C}}_{i,j}(x) := \begin{cases} \frac{W^{y^*,z^*}(x,a_i,c_{j+1}) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)x}\right]}{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}} & \text{if } x > 0, \\ \frac{\partial_x W^{y^*,z^*}(0,a_i,c_{j+1}) + \frac{c_j}{q}\theta_2(a_i,c_j)}{\theta_1(a_i,c_j) - \theta_2(a_i,c_j)} & \text{if } x = 0, \end{cases}$$

and

$$G^{\mathcal{E}}_{i,j}(x) := \begin{cases} \frac{W^{y^*,z^*}(x,a_{i+1},c_{j+1}) - \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)x}\right]}{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}} & \text{if } x > 0, \\ \frac{\partial_x W^{y^*,z^*}(0,a_{i+1},c_{j+1}) + \frac{c_j}{q}\theta_2(a_i,c_j)}{\theta_1(a_i,c_j) - \theta_2(a_i,c_j)} & \text{if } x = 0. \end{cases}$$

For $k = \mathcal{A}, \mathcal{C}, \mathcal{E}$, we introduce the following notation:

$$g^k_{i,j} := \min\left[\operatorname*{arg\,max}_{x \in [0,\infty)} G^k_{i,j}(x)\right].$$

By Lemma A.3, the maximizer of $G^k_{i,j}$ exists in $[0, \infty)$ and $g^k_{i,j}$ also exists for $k = \mathcal{A}, \mathcal{C}, \mathcal{E}$. The optimal threshold functions $y^*_{i,j}$ and $z^*_{i,j}$ are then given by:

Case 1: If $\max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x) \vee \max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x)$, then

$$y^*_{i,j} = g^{\mathcal{A}}_{i,j} \quad \text{and} \quad z^*_{i,j} = +\infty.$$

Case 2: If $\max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) \vee \max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x)$, then

$$y^*_{i,j} = +\infty \quad \text{and} \quad z^*_{i,j} = g^{\mathcal{C}}_{i,j}.$$

Case 3: If $\max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) \vee \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x)$, then

$$y^*_{i,j} = z^*_{i,j} = g^{\mathcal{E}}_{i,j}.$$

Case 4: If $\max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) = \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x)$, then either

$$y^*_{i,j} = \begin{cases} g^{\mathcal{A}}_{i,j} & \text{if } g^{\mathcal{A}}_{i,j} \le g^{\mathcal{C}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{C}}_{i,j} & \text{if } g^{\mathcal{C}}_{i,j} < g^{\mathcal{A}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases}$$

or

$$y^*_{i,j} = \begin{cases} g^{\mathcal{A}}_{i,j} & \text{if } g^{\mathcal{A}}_{i,j} < g^{\mathcal{C}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{C}}_{i,j} & \text{if } g^{\mathcal{C}}_{i,j} \le g^{\mathcal{A}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases}$$

depending on whichever between decrease in retention rate and increase in dividend rate is prioritized.
Case 5: If $\max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x) = \max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x)$, then

$$y^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j}, \\ g^{\mathcal{A}}_{i,j} & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j}, \\ +\infty & \text{otherwise.} \end{cases}$$

Case 6: If $\max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x) = \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x) > \max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x)$, then

$$y^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{C}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{C}}_{i,j}, \\ g^{\mathcal{C}}_{i,j} & \text{otherwise.} \end{cases}$$

Case 7: If $\max_{x\in[0,\infty)} G^{\mathcal{E}}_{i,j}(x) = \max_{x\in[0,\infty)} G^{\mathcal{A}}_{i,j}(x) = \max_{x\in[0,\infty)} G^{\mathcal{C}}_{i,j}(x)$, then

$$y^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j} \wedge g^{\mathcal{C}}_{i,j}, \\ g^{\mathcal{A}}_{i,j} & \text{if } g^{\mathcal{A}}_{i,j} < g^{\mathcal{E}}_{i,j} \wedge g^{\mathcal{C}}_{i,j} \\ & \text{or } g^{\mathcal{A}}_{i,j} = g^{\mathcal{C}}_{i,j} < g^{\mathcal{E}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j} \wedge g^{\mathcal{C}}_{i,j}, \\ g^{\mathcal{C}}_{i,j} & \text{if } g^{\mathcal{C}}_{i,j} < g^{\mathcal{E}}_{i,j} \wedge g^{\mathcal{A}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases}$$

or

$$y^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j} \wedge g^{\mathcal{C}}_{i,j}, \\ g^{\mathcal{A}}_{i,j} & \text{if } g^{\mathcal{A}}_{i,j} < g^{\mathcal{E}}_{i,j} \wedge g^{\mathcal{C}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad z^*_{i,j} = \begin{cases} g^{\mathcal{E}}_{i,j} & \text{if } g^{\mathcal{E}}_{i,j} \le g^{\mathcal{A}}_{i,j} \wedge g^{\mathcal{C}}_{i,j}, \\ g^{\mathcal{C}}_{i,j} & \text{if } g^{\mathcal{C}}_{i,j} < g^{\mathcal{E}}_{i,j} \wedge g^{\mathcal{A}}_{i,j} \\ & \text{or } g^{\mathcal{C}}_{i,j} = g^{\mathcal{A}}_{i,j} < g^{\mathcal{E}}_{i,j}, \\ +\infty & \text{otherwise,} \end{cases}$$

depending on whichever between decrease in retention rate and increase in dividend rate is prioritized.

To establish continuity and differentiability of W^{y^*,z^*}, we consider three auxiliary problems of looking for the smallest solution $U^*_{\mathcal{A}}$, $U^*_{\mathcal{C}}$, and $U^*_{\mathcal{E}}$ of the equation $\mathcal{L}^{a_i,c_j}(U) = 0$ in $[0,\infty)$ with boundary condition $U(0) = 0$ above $W^{y^*,z^*}(\cdot, a_{i+1}, c_j)$, $W^{y^*,z^*}(\cdot, a_i, c_{j+1})$, and $W^{y^*,z^*}(\cdot, a_{i+1}, c_{j+1})$, respectively. Once we find $U^*_{\mathcal{A}}$, $U^*_{\mathcal{C}}$, and $U^*_{\mathcal{E}}$, we get, respectively,

$$y^*_{i,j} = \begin{cases} 0 & \text{if } U^*_{\mathcal{A}}(\cdot) > W^{y^*,z^*}(\cdot, a_{i+1}, c_j) \text{ in } (0,\infty), \\ \inf\{y > 0 : U^*_{\mathcal{A}}(\cdot) = W^{y^*,z^*}(\cdot, a_{i+1}, c_j)\} & \text{otherwise,} \end{cases}$$

$$z^*_{i,j} = \begin{cases} 0 & \text{if } U^*_{\mathcal{C}}(\cdot) > W^{y^*,z^*}(\cdot, a_i, c_{j+1}) \text{ in } (0,\infty), \\ \inf\{y > 0 : U^*_{\mathcal{C}}(\cdot) = W^{y^*,z^*}(\cdot, a_i, c_{j+1})\} & \text{otherwise,} \end{cases}$$

and

$$y^*_{i,j} = z^*_{i,j} = \begin{cases} 0 & \text{if } U^*_{\mathcal{E}}(\cdot) > W^{y^*,z^*}(\cdot, a_{i+1}, c_{j+1}) \text{ in } (0,\infty), \\ \inf\{y > 0 : U^*_{\mathcal{E}}(\cdot) = W^{y^*,z^*}(\cdot, a_{i+1}, c_{j+1})\} & \text{otherwise.} \end{cases}$$

We then have

$$\begin{cases} W^{y^*,z^*}(x, a_i, c_j) = U^*_{\mathcal{A}}(x) & \text{for } x < y^*_{i,j}, \\ W^{y^*,z^*}(x, a_i, c_j) = W^{y^*,z^*}(x, a_{i+1}, c_j) & \text{for } x \ge y^*_{i,j}, \end{cases}$$

$$\begin{cases} W^{y^*,z^*}(x,a_i,c_j) = U^*_{\mathcal{C}}(x) & \text{for } x < z^*_{i,j}, \\ W^{y^*,z^*}(x,a_i,c_j) = W^{y^*,z^*}(x,a_i,c_{j+1}) & \text{for } x \geq z^*_{i,j}, \end{cases}$$

and

$$\begin{cases} W^{y^*,z^*}(x,a_i,c_j) = U^*_{\mathcal{E}}(x) & \text{for } x < y^*_{i,j} = z^*_{i,j}, \\ W^{y^*,z^*}(x,a_i,c_j) = W^{y^*,z^*}(x,a_{i+1},c_{j+1}) & \text{for } x \geq y^*_{i,j} = z^*_{i,j}. \end{cases}$$

By Lemma A.4, $U^*_{\mathcal{A}}$, $U^*_{\mathcal{C}}$, and $U^*_{\mathcal{E}}$ exist.

Remark 4.1. *The construction of the optimal threshold functions ensures that $y^*_{i,j} \wedge z^*_{i,j} < \infty$; that is, the optimal threshold strategy always exists. Moreover, the strategy depends on which of the functions $G^{\mathcal{A}}_{i,j}$, $G^{\mathcal{C}}_{i,j}$, and $G^{\mathcal{E}}_{i,j}$ maximizes the expected payoff $W^{y,z}$ at the current levels of reserve, retention, and dividend. For instance, if the maximum value of $G^{\mathcal{A}}_{i,j}$ is the higher than the other two functions, the next threshold to consider would be the reinsurance level. In the case of a tie among these functions, the insurer must decide whether to increase the retention level or the dividend level, based on their strategic priorities.*

Remark 4.2. *Given $y_{i,j}, z_{i,j} : \mathscr{A} \times \mathscr{C} \to [0,\infty]$, we have defined in (9) a threshold strategy $\pi^{y,z}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. This strategy can be extended to*

$$\tilde{\pi}^{y,z} := (\tilde{\pi}_{x,a,c})_{(x,a,c)\in[0,\infty)\times[a_m,a_1]\times[c_1,c_n]},$$

where $\tilde{\pi}_{x,a,c} := (A_{x,a,c}, C_{x,a,c}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}$, as follows:

(i) If $a \in (a_{i+1}, a_i)$, $c \in (c_j, c_{j+1})$, and $x < y_{i,j} \wedge z_{i,j}$, retain proportion a of the incoming claims and pay dividends at rate c until the time of ruin. If the current reserve reaches $y_{i,j}$ before ruin time, follow $\left(A_{y_{i,j},a_{i+1},c}, C_{y_{i,j},a_{i+1},c}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c}$. If the current reserve reaches $z_{i,j}$ first before ruin time, follow $\left(A_{z_{i,j},a,c_{j+1}}, C_{z_{i,j},a,c_{j+1}}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c_{j+1}}$. If the current reserve reaches $y_{i,j}$ and $z_{i,j}$ simultaneously before ruin time, follow $\left(A_{y_{i,j},a_{i+1},c_{j+1}}, C_{y_{i,j},a_{i+1},c_{j+1}}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c_{j+1}}$.

(ii) If $a \in (a_{i+1}, a_i)$, $c \in (c_j, c_{j+1})$, and $x \geq y_{i,j}$, follow $\left(A_{x,a_{i+1},c_j}, C_{x,a_{i+1},c_j}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c_j}$. More precisely, if $(x,a,c) \in [0, y_{i,j}) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$ with $y_{i,j} < z_{i,j}$, then $(A_{x,a}, C_{x,c}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}$ is defined as $(A_{x,a,c}, C_{x,a,c})(t) = (a,c)$ and so $X^{(A_{x,a,c},C_{x,a,c})}_t = X_t - ct$ for

$$t < \tau_\pi \wedge \tau^{\mathcal{A}}_i \quad \textit{where } \tau^{\mathcal{A}}_i := \min\{s : X^{(A_{x,a,c},C_{x,a,c})}_s = y_{i,j}\},$$

and $(A_{x,a,c}, C_{x,a,c})(t) = (A_{y_{i,j},a_{i+1},c_j}, C_{y_{i,j},a_{i+1},c_j})(t - \tau^{\mathcal{A}}_i) \in \Pi^{\mathscr{A},\mathscr{C}}_{y_{i,j},a_{i+1},c_j}$ for $t \geq \tau^{\mathcal{A}}_i$. Finally, it holds that $(A_{x,a,c}, C_{x,a,c}) = (A_{x,a_{i+1},c_j}, C_{x,a_{i+1},c_j}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c_j}$ for $(x,a,c) \in [y_{i,j}, \infty) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$.

(iii) If $a \in (a_{i+1}, a_i)$, $c \in (c_j, c_{j+1})$ and $x \geq z_{i,j}$, follow $\left(A_{x,a_i,c_{j+1}}, C_{x,a_i,c_{j+1}}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_i,c_{j+1}}$. More precisely, if $(x,a,c) \in [0, z_{i,j}) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$ with $z_{i,j} < y_{i,j}$, then $(A_{x,a,c}, C_{x,a,c}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}$ is defined as $(A_{x,a,c}, C_{x,a,c})(t) = (a,c)$ and so $X^{(A_{x,a,c},C_{x,a,c})}_t = X_t - ct$ for

$$t < \tau_\pi \wedge \tau^{\mathcal{C}}_j \quad \textit{where } \tau^{\mathcal{C}}_j := \min\{s : X^{(A_{x,a,c},C_{x,a,c})}_s = z_{i,j}\},$$

and $(A_{x,a,c}, C_{x,a,c})(t) = (A_{z_{i,j},a_i,c_{j+1}}, C_{z_{i,j},a_i,c_{j+1}})(t - \tau^{\mathcal{C}}_j) \in \Pi^{\mathscr{A},\mathscr{C}}_{y_{i,j},a_i,c_{j+1}}$ for $t \geq \tau^{\mathcal{C}}_j$. Finally, it holds that $(A_{x,a,c}, C_{x,a,c}) = (A_{x,a_i,c_{j+1}}, C_{x,a_i,c_{j+1}}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_i,c_{j+1}}$ for $(x,a,c) \in [z_{i,j}, \infty) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$.

(iv) If $a \in (a_{i+1}, a_i)$, $c \in (c_j, c_{j+1})$ and $x \geq y_{i,j} = z_{i,j}$, follow $\left(A_{x,a_{i+1},c_{j+1}}, C_{x,a_{i+1},c_{j+1}}\right) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c_{j+1}}$. More precisely, if $(x,a,c) \in [0, z_{i,j}) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$ with $z_{i,j} = y_{i,j}$, then $(A_{x,a,c}, C_{x,a,c}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a,c}$ is defined as $(A_{x,a,c}, C_{x,a,c})(t) = (a,c)$ and so $X^{(A_{x,a,c},C_{x,a,c})}(t) = X_t - ct$ for

$$t < \tau_\pi \wedge \tau^{\mathcal{E}}_{i,j} \quad \textit{where } \tau^{\mathcal{E}}_{i,j} := \min\{s : X^{(A_{x,a,c},C_{x,a,c})}_s = z_{i,j}\},$$

and $(A_{x,a,c}, C_{x,a,c})(t) = (A_{z_{i,j},a_{i+1},c_{j+1}}, C_{z_{i,j},a_{i+1},c_{j+1}})(t - \tau^{\mathcal{E}}_{i,j}) \in \Pi^{\mathscr{A},\mathscr{C}}_{y_{i,j},a_{i+1},c_{j+1}}$ for $t \geq \tau^{\mathcal{E}}_{i,j}$. Finally, $(A_{x,a,c}, C_{x,a,c}) = (A_{x,a_{i+1},c_{j+1}}, C_{x,a_{i+1},c_{j+1}}) \in \Pi^{\mathscr{A},\mathscr{C}}_{x,a_{i+1},c_{j+1}}$ for $(x,a,c) \in [z_{i,j}, \infty) \times (a_{i+1}, a_i) \times (c_j, c_{j+1})$.

The value function of the extended stationary strategy $\tilde{\pi}^{y,z}$ is defined as

$$J^{\tilde{\pi}^{y,z}}(x,a,c) := J(x; A_{x,a,c}, C_{x,a,c}) : [0,\infty) \times [a_m, a_1] \times [c_1, c_n] \to \mathbb{R}.$$

5 Numerical Illustrations

In this section, we present several numerical examples to gain further insight into the optimal dividend and reinsurance strategies, as well as the optimal value function. These examples are intended to provide a clearer understanding of the optimal retention and dividend threshold levels and how they affect the optimal value function.

Figure 1 presents three examples, each illustrating the optimal value function. The vertical dotted lines represent the optimal threshold levels. We fix the following parameters across all three examples: $\sigma = 1.5$, $b = 2$, and $q = 0.1$.

Consider the case $\mu = 6$, $\mathscr{A} = \{0.80, 0.90\}$, and $\mathscr{C} = \{2, 4\}$. We obtain the following optimal threshold levels: $z^*_{1,1} = 13.04$ and $y^*_{1,2} = 348.5$. The sequence of retention-dividend rate changes is as follows: $(0.9, 2) \xrightarrow{x=13.04} (0.9, 4) \xrightarrow{x=348.5} (0.8, 4)$, and this is illustrated in Figures 1a and 1b. The optimal strategy is to increase the dividend level first and then decrease the retained proportion of incoming claims if ruin time does not occur before or within these level changes. In Figure 1b, we can see that $W^{y^*,z^*}(x, a_1, c_1)$ (in red) is greater than $W^{y^*,z^*}(x, a_1, c_2)$ (in blue) when $x < z^*_{1,1}$. We can also see here how concavity is not guaranteed when both the minimum retention level and the maximum dividend rate have not been attained.

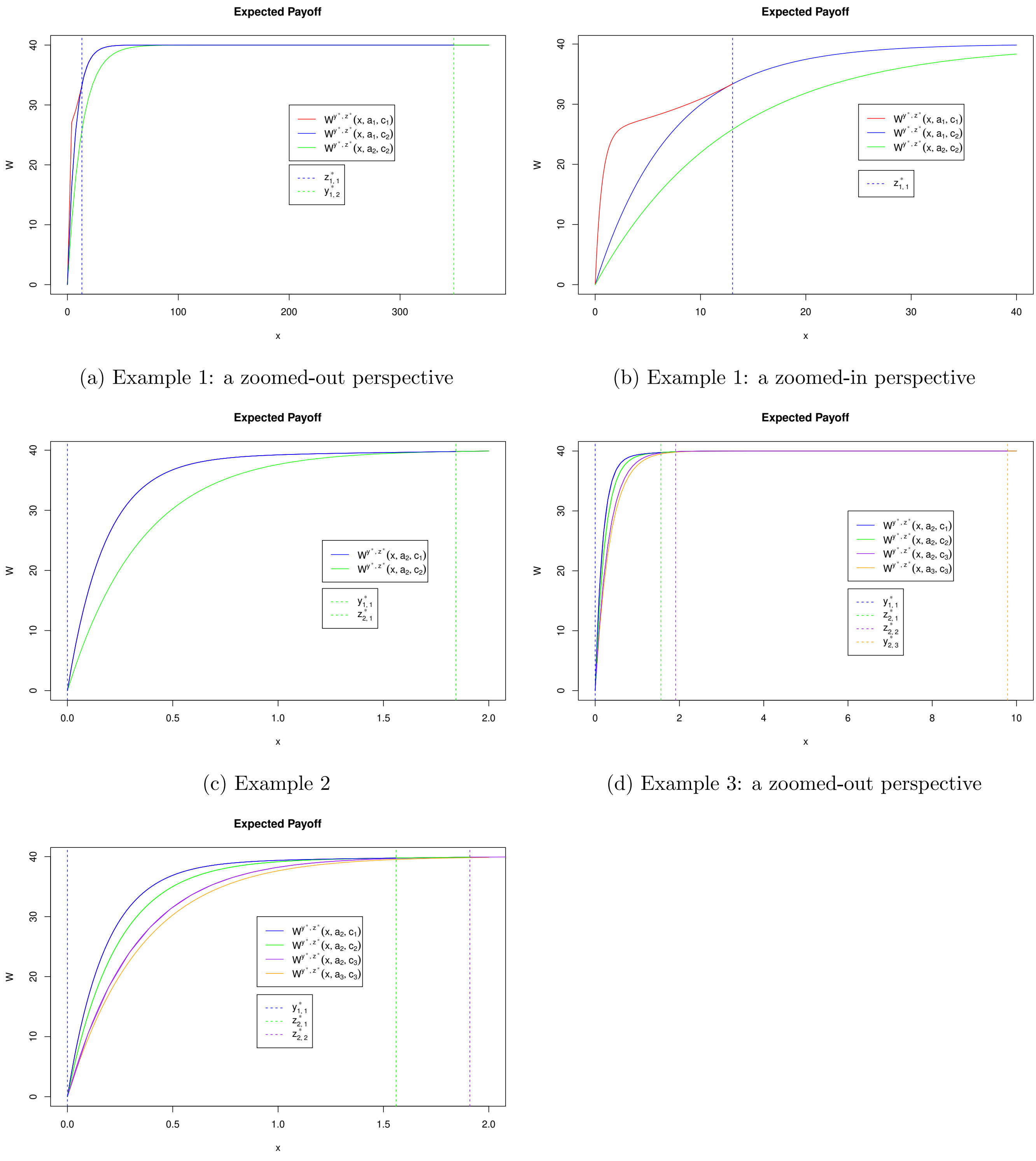
Next, we consider a similar setup with a higher adjusted mean $\mu = 10$. We obtain the following optimal threshold levels: $y^*_{1,1} = 0$ and $z^*_{2,1} = 1.92$. The sequence of retention-dividend level changes is as follows: $(0.90, 2) \xrightarrow{x=0} (0.80, 2) \xrightarrow{x=1.92} (0.80, 4)$, and this is illustrated in Figure 1c. The curves $W^{y^*,z^*}(x, a_1, c_1)$ and $W^{y^*,z^*}(x, a_2, c_1)$ coincide since it is optimal to increase the reinsurance level right away assuming that the initial reserve level is positive. The dividend level is then increased as soon as the reserve reaches the threshold level $z^*_{2,1} = 1.92$. It is also interesting to note that in the second setup, we have $\mu a_1 - 2(b + c_1) < 0$ and $\sigma < \sqrt{\frac{-\mu(\mu a_1 - 2(b+c_1))}{2qa_1}}$, which implies that $\frac{\partial}{\partial a}\theta_2(a, c) < 0$. In the first setup, we have $\mu a_1 - 2(b + c_1) > 0$, which implies that $\frac{\partial}{\partial a}\theta_2(a, c) > 0$.

Finally, consider the case where $\mu = 10$, $\mathscr{A} = \{0.80, 0.85, 0.90\}$, and $\mathscr{C} = \{2, 3, 4\}$. As in the previous examples, the maximum dividend rate and the minimum retention level remain unchanged. The main difference is the presence of an intermediate level between the extremes. We obtain the following optimal threshold levels: $y^*_{1,1} = 0$, $z^*_{2,1} = 1.56$, $z^*_{2,2} = 1.91$, and $y^*_{2,3} = 9.79$. The sequence of retention-dividend level changes is as follows: $(0.90, 2) \xrightarrow{x=0} (0.85, 2) \xrightarrow{x=1.56} (0.85, 3) \xrightarrow{x=1.91} (0.85, 4) \xrightarrow{x=9.79} (0.80, 4)$, and this is illustrated in Figures 1d and 1e. As in the second example, the optimal strategy is to increase the reinsurance level immediately. The dividend level is then increased to its maximum before the reinsurance level is maximized, provided that ruin does not occur during these level changes.

Following the numerical examples, we now investigate the impact of the model parameters on the optimal retention and dividend threshold strategies through a sensitivity analysis. For our analysis, we use the model parameters from Example 1 of Figure 1a as our base case: $\mu = 6$, $\sigma = 1.5$, $b = 2$, $q = 0.1$, $a_1 = 0.9$, $a_2 = 0.8$, $c_1 = 2$, and $c_2 = 4$. The sensitivity analysis varies the following parameters: the adjusted mean μ, the adjusted volatility σ, the nonhomogeneous term b, the discount rate q, the retention levels a_1 and a_2, and the dividend rates c_1 and c_2.

In each table, the columns for $y^*_{1,1}$ and $z^*_{1,1}$ represent the optimal retention and dividend threshold levels given the initial retention level a_1 and the initial dividend rate c_1. If $y^*_{1,1} < \infty$ and $z^*_{1,1} = \infty$, then the retention level must be decreased first. Otherwise, the dividend rate must be increased first. The column for $y^*_{1,2}$ represents the optimal retention threshold level following an increase in the dividend rate, while the column for $z^*_{2,1}$ represents the optimal dividend threshold level after a decrease in the retention level. A value of "N/A" for $y^*_{1,2}$ means that the retention level has already been decreased, whereas an "N/A" for $z^*_{2,1}$ means that the dividend rate has already been increased. The rightmost column represents the optimal strategy: "D-R" signifies that the dividend rate should be increased first, followed by a decrease in the retention level, whereas "R-D" signifies that the retention level must be decreased first, followed by an increase in the dividend rate.

It can be observed that there is an inverse relationship between the optimal reinsurance threshold level $y^*_{1,2}$ and the parameters μ, q, a_2, and (a_1, a_2) (refer to Tables 1a, 1d, 1f, and 1i). A higher μ implies a higher drift for the reserve process, to which the downward trend of $y^*_{1,2}$ can be attributed. As q rises, the incentive to hold more reserves decreases because the future "costs" associated with these reserves are also less significant. When a_2 is higher, the change in retention level from $a_1 = 0.90$ becomes larger. The same holds when the pair (a_1, a_2) has higher values.



(a) Example 1: a zoomed-out perspective

(b) Example 1: a zoomed-in perspective

(c) Example 2

(d) Example 3: a zoomed-out perspective

(e) Example 3: a zoomed-in perspective

Figure 1: Examples 1, 2, and 3

In contrast, a direct relationship can be observed between $y_{1,2}^*$ and the parameters σ, b, c_2, and (c_1, c_2) (refer to Tables 1b, 1c, 1h, and 1j). A higher σ implies a higher diffusion for the reserve process, which corresponds to higher volatility. As either b or c_2 increases, the drift for the reserve process decreases, which makes lowering the retention level less advantageous. This trend also holds if both c_1 and c_2 increase simultaneously.

For the optimal dividend threshold level $z_{1,1}^*$, there is an inverse relationship with the parameters q, a_1, and (a_1, a_2) (see Tables 1d, 1e, and 1i). As q increases, the valuation of future dividend flows decreases, leading to an optimal strategy of increasing the dividend rate at lower threshold levels. As a_1 increases, the *expected* reserve level $\mathbb{E}[X_t]$ also increases, making it more attractive to increase the dividend rate at a

μ	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
2	∞	0	1209.44	N/A	D-R
3	∞	10.67	1000.02	N/A	D-R
4	∞	16.98	782.9	N/A	D-R
6	∞	13.04	348.49	N/A	D-R
8	∞	3.24	37.33	N/A	D-R
10	0	∞	N/A	1.92	R-D

(a) Sensitivity analysis for μ

σ	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0.375	∞	6.15	333.66	N/A	D-R
0.75	∞	11.86	336.73	N/A	D-R
1.5	∞	13.04	348.49	N/A	D-R
3	∞	16.24	389.44	N/A	D-R
6	∞	20.4	507.03	N/A	D-R
16	0	∞	N/A	6.72	R-D

(b) Sensitivity analysis for σ

b	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0	0	∞	N/A	3.48	R-D
1	∞	5.62	107.1	N/A	D-R
2	∞	13.04	348.49	N/A	D-R
4	∞	16.48	893.2	N/A	D-R
8	∞	0	1992.24	N/A	D-R

(c) Sensitivity analysis for b

q	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0.025	∞	47.43	1414.25	N/A	D-R
0.05	∞	24.56	698.51	N/A	D-R
0.1	∞	13.04	348.49	N/A	D-R
0.5	∞	3.40	75.64	N/A	D-R
1	∞	1.93	41.66	N/A	D-R

(d) Sensitivity analysis for q

a_1	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0.81	∞	15.26	344.36	N/A	D-R
0.85	∞	15.26	348.49	N/A	D-R
0.9	∞	13.04	348.49	N/A	D-R
0.95	∞	11.03	348.49	N/A	D-R
1	∞	8.91	348.49	N/A	D-R

(e) Sensitivity analysis for a_1

a_2	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0.1	∞	13.04	1496.84	N/A	D-R
0.5	∞	13.04	834.14	N/A	D-R
0.8	∞	13.04	348.49	N/A	D-R
0.85	∞	13.04	272.39	N/A	D-R
0.89	∞	13.04	213.53	N/A	D-R

(f) Sensitivity analysis for a_2

c_1	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
0	∞	12.66	348.49	N/A	D-R
1	∞	12.77	348.49	N/A	D-R
2	∞	13.04	348.49	N/A	D-R
3	∞	14.21	348.49	N/A	D-R
3.5	∞	16.47	348.49	N/A	D-R

(g) Sensitivity analysis for c_1

c_2	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
2.5	∞	3.89	42.9	N/A	D-R
3	∞	5.60	105.98	N/A	D-R
4	∞	13.04	348.49	N/A	D-R
6	∞	22.31	906.18	N/A	D-R
8	∞	25.99	1479.92	N/A	D-R

(h) Sensitivity analysis for c_2

(a_1, a_2)	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
$(0.5, 0.4)$	∞	14.28	999.09	N/A	D-R
$(0.7, 0.6)$	∞	15.93	669.88	N/A	D-R
$(0.9, 0.8)$	∞	13.04	348.49	N/A	D-R
$(0.95, 0.85)$	∞	11.03	272.39	N/A	D-R
$(1, 0.9)$	∞	8.91	201.12	N/A	D-R

(i) Sensitivity analysis for (a_1, a_2)

(c_1, c_2)	$y^*_{1,1}$	$z^*_{1,1}$	$y^*_{1,2}$	$z^*_{2,1}$	Strategy
$(0, 2)$	0	∞	N/A	2.92	R-D
$(1, 3)$	∞	5.16	105.98	N/A	D-R
$(2, 4)$	∞	13.04	348.49	N/A	D-R
$(3, 5)$	∞	20	623.72	N/A	D-R
$(4, 6)$	∞	29.07	906.18	N/A	D-R

(j) Sensitivity analysis for (c_1, c_2)

Table 1: Comparison of optimal threshold levels with base case: $\mu = 6$, $\sigma = 1.5$, $b = 2$, $q = 0.1$, $a_1 = 0.9$, $a_2 = 0.8$, $c_1 = 2$, and $c_2 = 4$

lower threshold level. The same can be said when both a_1 and a_2 increase simultaneously.

On the other hand, $z^*_{1,1}$ exhibits a direct relationship with the parameters σ, c_1, c_2, and (c_1, c_2) (see

Tables 1b, 1g, 1h, and 1j). Higher dividend rates make increasing the dividend rate less attractive. A smaller σ implies lower volatility, allowing a lower threshold level for a dividend increase.

There is no noticeable trend for $z_{1,1}^*$ in Tables 1a and 1c. It remains constant in Table 1f since $z_{1,1}^*$ does not depend on the next retention level a_2. This is also similar to the stable behavior of $y_{1,2}^*$ with respect to c_1 in Table 1g, which can be attributed to the fact that $y_{1,2}^*$ depends on c_2, and not on c_1.

The optimal strategy is predominantly to increase the dividends first and then decrease the retention level (i.e., D-R). However, a shift to the R-D strategy occurs in "extreme" cases, as illustrated in Tables 1a, 1b, 1c, and 1j. If the adjusted mean μ is sufficiently large or if b is relatively small, the drift of the reserve process becomes large, making it more advantageous to decrease the retention level (or, equivalently, to increase the reinsurance coverage). Similarly, if σ is excessively high, the resulting volatility in reserve levels necessitates an immediate decrease in the retention level. The same behavior is observed if both c_1 and c_2 are sufficiently small. It is also noteworthy that when the optimal strategy is R–D, the retention level is decreased immediately.

6 Conclusion

In this paper, we have extended the literature on optimal dividend payout problems with ratcheting by incorporating an irreversible reinsurance constraint, under which the reinsurance and dividend levels are restricted to a finite set. Through a dynamic programming approach, we have derived the HJB equation. We have presented a method to determine the optimal threshold levels for the dividend and reinsurance variables. The sensitivity analysis suggests that it is almost always optimal to increase the dividends before increasing the reinsurance coverage. A possible direction for future research is to consider jump-diffusion processes on top of the Brownian motion to model the reserve level. Drawdown constraints, which allow the levels to decrease by a fixed proportion of the current level, could also be incorporated for both dividend and reinsurance levels.

Declaration of competing interest

The authors declare no competing interests.

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CRediT authorship contribution statement

Tim J. Boonen: Conceptualization, Formal analysis, Supervision, Writing - original draft, Writing - review and editing. **Engel John C. Dela Vega**: Conceptualization, Formal analysis, Software, Writing - original draft, Writing - review and editing

A Proofs of Section 2, and Intermediate Lemmas and Propositions

Proof of Proposition 2.1. From Remark 2.4, since V is bounded above by V_{NC} and that $\lim_{x\to\infty} V_{NC}(x) = \frac{\bar{c}}{q}$, it follows that $V(x,a,c)$ is bounded by $\frac{\bar{c}}{q}$.

The monotonicity of V with respect to a and c follows from Remark 2.2. Given $0 \le x_1 < x_2$, let $\pi_1 := (A^1, C^1) \in \Pi_{x_1,a,c}^{\mathscr{A},\mathscr{C}}$ be an admissible strategy for any $a \in \mathscr{A}$ and $c \in \mathscr{C}$. Moreover, define $\pi_2 := (A^2, C^2) \in \Pi_{x_2,a,c}^{\mathscr{A},\mathscr{C}}$ as

$$A^2(t) := \begin{cases} A^1(t), & \text{if } t < \tau_{\pi_1}, \\ \underline{a}, & \text{if } t \ge \tau_{\pi_1}, \end{cases} \quad \text{and} \quad C^2(t) := \begin{cases} C^1(t), & \text{if } t < \tau_{\pi_1}, \\ \bar{c}, & \text{if } t \ge \tau_{\pi_1}, \end{cases}$$

where τ_{π_1} is the corresponding ruin time of the controlled process $\{X_t^{\pi_1}\}_{t\geq 0}$ with $X_0^{\pi_1} = x_1$. Then,

$$\begin{aligned} J(x_1; A^1, C^1) &= \mathbb{E}\left[\int_0^{\tau_{\pi_1}} e^{-qs} C^1(s) ds\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau_{\pi_1}} e^{-qs} C^1(s) ds\right] + \mathbb{E}\left[\int_{\tau_{\pi_1}}^{\tau_{\pi_2}} e^{-qs} \overline{c}\, ds\right] \\ &= J(x_2; A^2, C^2), \end{aligned}$$

where τ_{π_2} is the corresponding ruin time of the controlled process $\{X_t^{\pi_2}\}_{t\geq 0}$ with $X_0^{\pi_2} = x_2$. It then implies that $V(x,a,c)$ is nondecreasing in x. □

Proof of Proposition 2.2. By Proposition 2.1, $V(x,a,c)$ is nondecreasing in x and a and nonincreasing in c. Hence, we obtain the first inequality $0 \leq V(x_2,a_1,c_1) - V(x_1,a_2,c_2)$ for all $0 \leq x_1 \leq x_2$, $a_1, a_2 \in \mathscr{A}$ with $a_2 \leq a_1$, and $c_1, c_2 \in \mathscr{C}$ with $c_1 \leq c_2$. For the second inequality, we divide the proof into three parts.

(Part I) We want to show that there exists $L_1 > 0$ such that

$$V(x_2,a,c) - V(x_1,a,c) \leq L_1\,(x_2 - x_1) \tag{11}$$

for all $0 \leq x_1 \leq x_2$. Let $\epsilon_1 > 0$ and $\pi^1 := (A,C) \in \Pi_{x_2,a,c}^{\mathscr{A},\mathscr{C}}$ such that

$$J(x_2; A, C) \geq V(x_2,a,c) - \epsilon_1. \tag{12}$$

Let τ_1 be the ruin time of the associated process $\{X_t^{\pi^1}\}_{t\geq 0}$. Define $\tilde{\pi} := (\tilde{A},\tilde{C}) \in \Pi_{x_1,a,c}^{\mathscr{A},\mathscr{C}}$ as $\tilde{\pi}_t = \pi_t^1$. Let $\tilde{\tau} \leq \tau_1$ be the ruin time of the associated process $\{X_t^{\tilde{\pi}}\}_{t\geq 0}$. It then holds that $X_t^{\pi^1} - X_t^{\tilde{\pi}} = x_2 - x_1$ for $t \leq \tilde{\tau}$. Using the properties of conditional expectations and a shift in stopping times, we obtain

$$\begin{aligned} J(x_2; A, C) - J(x_1; \tilde{A}, \tilde{C}) &= \mathbb{E}\left[e^{-q\tilde{\tau}}\mathbb{E}\left[\int_0^{\tau_1-\tilde{\tau}} e^{-qu} C(\tilde{\tau}+u) du \middle| \mathcal{F}_{\tilde{\tau}}\right]\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_1-\tilde{\tau}} e^{-qu} C(\tilde{\tau}+u) du \middle| \mathcal{F}_{\tilde{\tau}}\right]\right]. \end{aligned} \tag{13}$$

From Theorem 2 of Claisse et al. [2016], we have

$$\mathbb{E}\left[\int_0^{\tau_1-\tilde{\tau}} e^{-qu} C(\tilde{\tau}+u) du \middle| \mathcal{F}_{\tilde{\tau}}\right] = J\left(x_2 - x_1; \{A(t+\tilde{\tau})\}_{t\geq 0}, \{C(t+\tilde{\tau})\}_{t\geq 0}\right), \quad \mathbb{P}-a.s.$$

By the inclusion property discussed in Remark 2.2, we have $(\{A(t+\tilde{\tau})\}_{t\geq 0}, \{C(t+\tilde{\tau})\}_{t\geq 0}) \in \Pi_{x_2-x_1,a,c}^{\mathscr{A},\mathscr{C}}$. Write $\overline{a} := \arg\max \mathscr{A}$ and $\underline{c} := \arg\min \mathscr{C}$. Then,

$$\begin{aligned} J(x_2; A, C) - J(x_1; \tilde{A}, \tilde{C}) &\leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_1-\tilde{\tau}} e^{-qu} C(\tilde{\tau}+u) du \middle| \mathcal{F}_{\tilde{\tau}}\right]\right] \\ &= \mathbb{E}\left[J\left(x_2 - x_1; \{A(t+\tilde{\tau})\}_{t\geq 0}, \{C(t+\tilde{\tau})\}_{t\geq 0}\right)\right] \\ &\leq \mathbb{E}\left[V\left(x_2 - x_1; a, c\right)\right] \\ &\leq V(x_2 - x_1, \overline{a}, \underline{c}). \end{aligned} \tag{14}$$

By Remark 2.4, we then have

$$\begin{aligned} V(x_2,a,c) - V(x_1,a,c) &\overset{(12)}{\leq} J(x_2; A, C) - J(x_1; \tilde{A}, \tilde{C}) + \epsilon_1 \\ &\overset{(14)}{\leq} V(x_2 - x_1, \overline{a}, \underline{c}) + \epsilon_1 \\ &\leq V_{NC}(x_2 - x_1) + \epsilon_1 \\ &\overset{(3)}{\leq} L_0(x_2 - x_1) + \epsilon_1. \end{aligned}$$

Since ϵ_1 is arbitrary, (11) holds with $L_1 = L_0$.

(Part II) We now want to show that there exists $L_2 > 0$ such that

$$V(x,a,c_1) - V(x,a,c_2) \leq L_2\,(c_2 - c_1) \tag{15}$$

for all $c_1, c_2 \in \mathscr{C}$ with $c_1 \le c_2$. Take $\epsilon_2 > 0$ and $\pi^2 := (A, C) \in \Pi_{x,a,c_1}^{\mathscr{A},\mathscr{C}}$ such that

$$J(x; A, C) \ge V(x, a, c_1) - \epsilon_2. \tag{16}$$

Define the stopping time $\bar{T} = \inf\{t : C(t) \ge c_2\}$ and denote τ_2 the ruin time of the associated process $\{X_t^{\pi^2}\}_{t\ge0}$. Consider $\bar{\pi} := (A, \bar{C}) \in \Pi_{x,a,c_2}^{\mathscr{A},\mathscr{C}}$ such that $\bar{C}(t) = c_2 \mathbf{1}_{\{t<\bar{T}\}} + C(t)\mathbf{1}_{\{t\ge\bar{T}\}}$. Denote by $\{X_t^{\bar{\pi}}\}_{t\ge0}$ the associated reserve process and by $\bar{\tau} \le \tau_2$ the corresponding ruin time. Then,

$$\bar{C}(t) - C(t) = \begin{cases} c_2 - C(t), & \text{if } t < \bar{T}, \\ 0, & \text{if } t \ge \bar{T}. \end{cases}$$

It then follows that $\bar{C}(t) - C(t) \le c_2 - c_1$ for any $t \ge 0$. Hence,

$$X_{\bar{\tau}}^{\pi^2} = X_{\bar{\tau}}^{\pi^2} - X_{\bar{\tau}}^{\bar{\pi}} = \int_0^{\bar{\tau}} \left[\bar{C}(s) - C(s)\right] ds \le \int_0^{\bar{\tau}} [c_2 - c_1]\, ds = (c_2 - c_1)\bar{\tau}. \tag{17}$$

Using similar arguments made in (13) and (14), we then obtain

$$\mathbb{E}\left[\int_{\bar{\tau}}^{\tau_2} e^{-qs} C(s) ds\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-q\bar{\tau}} \int_0^{\tau_2-\bar{\tau}} e^{-qu} C(u+\bar{\tau}) du \middle| \mathcal{F}_{\bar{\tau}}\right]\right] \le \mathbb{E}\left[e^{-q\bar{\tau}} V(X_{\bar{\tau}}^{\pi^2}, \overline{a}, \underline{c})\right].$$

Together with Remark 2.4, we then have

$$\begin{aligned} V(x, a, c_1) - V(x, a, c_2) &\overset{(16)}{\le} J(x; A, C) - J(x; A, \bar{C}) + \epsilon_2 \\ &= \mathbb{E}\left[\int_0^{\bar{\tau}} e^{-qs}\left[C(s) - \bar{C}(s)\right] ds\right] + \mathbb{E}\left[\int_{\bar{\tau}}^{\tau_2} e^{-qs} C(s) ds\right] + \epsilon_2 \\ &\le 0 + \mathbb{E}\left[\int_{\bar{\tau}}^{\tau_2} e^{-qs} C(s) ds\right] + \epsilon_2 \\ &\le \mathbb{E}\left[e^{-q\bar{\tau}} V(X_{\bar{\tau}}^{\pi^2}, \overline{a}, \underline{c})\right] + \epsilon_2 \\ &\le \mathbb{E}\left[e^{-q\bar{\tau}} V_{NC}(X_{\bar{\tau}}^{\pi^2})\right] + \epsilon_2 \\ &\overset{(17)}{\le} \mathbb{E}\left[e^{-q\bar{\tau}} V_{NC}\left((c_2 - c_1)\bar{\tau}\right)\right] + \epsilon_2 \\ &\overset{(3)}{\le} L_0(c_2 - c_1)\mathbb{E}\left[e^{-q\bar{\tau}}\bar{\tau}\right] + \epsilon_2. \end{aligned}$$

Since ϵ_2 is arbitrary, choosing $L_2 := L_0 \max_{t\ge0}\{e^{-qt}t\} = \frac{L_0}{qe}$ yields (15).

(Part III) Lastly, we want to show that there exists $L_3 > 0$ such that

$$V(x, a_1, c) - V(x, a_2, c) \le L_3 (a_1 - a_2), \tag{18}$$

for all $a_1, a_2 \in \mathscr{A}$ with $a_2 \le a_1$. Take $\epsilon_3 > 0$ and $\pi^3 := (A, C) \in \Pi_{x,a_1,c}^{\mathscr{A}\mathscr{C}}$ such that

$$J(x; A, C) \ge V(x, a_1, c) - \epsilon. \tag{19}$$

Define the stopping time $\hat{T} := \inf\{t : A(t) \le a_2\}$ and denote τ_3 the ruin time of the associated process $\{X_t^{\pi^3}\}_{t\ge0}$. Consider $\hat{\pi} := (\hat{A}, C) \in \Pi_{x,a_2,c}^{\mathscr{A},\mathscr{C}}$ such that $\hat{A}(t) = a_2 \mathbf{1}_{\{t<\hat{T}\}} + A(t)\mathbf{1}_{\{t\ge\bar{T}\}}$. Denote by $\{X_t^{\hat{\pi}}\}_{t\ge0}$ the associated reserve process and by $\hat{\tau} \le \tau_3$ the corresponding ruin time. Then,

$$A(t) - \hat{A}(t) = \begin{cases} A(t) - a_2, & \text{if } t < \hat{T}, \\ 0, & \text{if } t \ge \hat{T}. \end{cases}$$

It then follows that $A(t) - \hat{A}(t) \le a_1 - a_2$ for any $t \ge 0$. Hence,

$$\begin{aligned} X_{\hat{\tau}}^{\pi^3} &= X_{\hat{\tau}}^{\pi^3} - X_{\hat{\tau}}^{\hat{\pi}} \\ &= \int_0^{\hat{\tau}} \mu\left[A(s) - \hat{A}(s)\right] ds + \int_0^{\hat{\tau}} \sigma\left[A(s) - \hat{A}(s)\right] dW_s \\ &\le \int_0^{\hat{\tau}} \mu\left[a_1 - a_2\right] ds + \int_0^{\hat{\tau}} \sigma\left[A(s) - \hat{A}(s)\right] dW_s \\ &= \mu(a_1 - a_2)\hat{\tau} + \int_0^{\hat{\tau}} \sigma\left[A(s) - \hat{A}(s)\right] dW_s. \end{aligned} \tag{20}$$

Using similar arguments made in (13) and (14), we then obtain

$$\mathbb{E}\left[\int_{\hat\tau}^{\tau_3} e^{-qs}C(s)ds\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-q\hat\tau}\int_0^{\tau_3-\hat\tau} e^{-qu}C(u+\hat\tau)du\middle|\mathcal{F}_{\hat\tau}\right]\right] \le \mathbb{E}\left[e^{-q\hat\tau}V(X_{\hat\tau}^{\pi^3},\overline{a},\underline{c})\right].$$

Together with Remark 2.4, we then have

$$\begin{aligned}
V(x,a_1,c)-V(x,a_2,c) &\overset{(19)}{\le} J(x;A,C)-J(x;\hat A,C)+\epsilon_3\\
&= \mathbb{E}\left[\int_{\hat\tau}^{\tau_3} e^{-qs}C(s)ds\right]+\epsilon_3\\
&\le \mathbb{E}\left[e^{-q\hat\tau}V(X_{\hat\tau}^{\pi^3},\overline{a},\underline{c})\right]+\epsilon_3\\
&\le \mathbb{E}\left[e^{-q\hat\tau}V_{NC}(X_{\hat\tau}^{\pi^3})\right]+\epsilon_3\\
&\overset{(20)}{\le} \mathbb{E}\left[e^{-q\hat\tau}V_{NC}\left(\mu(a_1-a_2)\hat\tau+\int_0^{\hat\tau}\sigma\left[A(s)-\hat A(s)\right]dW_s\right)\right]+\epsilon_3\\
&\overset{(3)}{\le} L_0(a_1-a_2)\mathbb{E}\left[e^{-q\hat\tau}\mu\hat\tau\right]+\sigma L_0\mathbb{E}\left[e^{-q\hat\tau}\int_0^{\hat\tau}\sigma\left[A(s)-\hat A(s)\right]dW_s\right]+\epsilon_3\\
&\le L_0(a_1-a_2)\mathbb{E}\left[e^{-q\hat\tau}\mu\hat\tau\right]+\sigma L_0\mathbb{E}\left[\int_0^{\hat\tau}\sigma\left[A(s)-\hat A(s)\right]dW_s\right]+\epsilon_3\\
&= L_0(a_1-a_2)\mathbb{E}\left[e^{-q\hat\tau}\mu\hat\tau\right]+\epsilon_3.
\end{aligned}$$

The term $\mathbb{E}\left[\int_0^{\hat\tau}\sigma\left[A(s)-\hat A(s)\right]dW_s\right]$ vanishes since the integrand is bounded above by $\sigma(a_1-a_2)$. Since ϵ_3 is arbitrary, choosing $L_3 := L_0\mu\max_{t\ge0}\{e^{-qt}t\} = \frac{L_0\mu}{qe}$ yields (18). Take $L=\max\{L_1,L_2,L_3\}$. The proof is complete. ☐

Proposition A.1. *The following properties hold:*

(i) $\frac{\partial}{\partial c}\theta_1(a,c), \frac{\partial}{\partial c}\theta_2(a,c) > 0$,

(ii) $\frac{\partial}{\partial a}\theta_1(a,c) < 0$,

(iii) $\frac{\partial}{\partial a}\theta_2(a,c) < 0$ *if* $\mu a-2(b+c)<0$ *and* $\sigma < \sqrt{\frac{-\mu(\mu a-2(b+c))}{2qa}}$,

(iv) $\frac{\partial}{\partial a}\theta_2(a,c) \ge 0$ *if* $\mu a-2(b+c)\ge 0$ *<u>or</u> if* $\mu a-2(b+c)<0$ *and* $\sigma \ge \sqrt{\frac{-\mu(\mu a-2(b+c))}{2qa}}$. *Equality holds if and only if* $\mu a-2(b+c)<0$ *and* $\sigma = \sqrt{\frac{-\mu(\mu a-2(b+c))}{2qa}}$.

Proof. Taking the partial derivative with respect to c, we have

$$\frac{\partial}{\partial c}\theta_1(a,c) = \frac{1}{\sigma^2a^2}\left[1+\frac{b+c-\mu a}{\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2}}\right]$$

and

$$\frac{\partial}{\partial c}\theta_2(a,c) = \frac{1}{\sigma^2a^2}\left[1-\frac{b+c-\mu a}{\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2}}\right].$$

Since $\frac{b+c-\mu a}{\sqrt{(b+c-\mu a)^2+2q\sigma^2a^2}} < 1$, the first result follows.

Write $\overline{b} := b+c$. Taking the partial derivative of $\theta_2(a,c)$ with respect to a, we have

$$\frac{\partial}{\partial a}\theta_2(a,c) = \frac{1}{\sigma^2a^3}\left[\mu a-2\overline{b}+\frac{(\mu a-\overline{b})(\mu a-2\overline{b})+2q\sigma^2a^2}{\sqrt{(\overline{b}-\mu a)^2+2q\sigma^2a^2}}\right].$$

If $\mu a-2\overline{b}\ge 0$, then $\mu a-\overline{b}>0$. Consequently, we have $\frac{\partial}{\partial a}\theta_2(a,c)>0$, which is the first "if" of the fourth result.

We now consider the case where $\mu a - 2\bar{b} < 0$ and $\sigma < \sqrt{\frac{-\mu(\mu a - 2\bar{b})}{2qa}}$. Then,

$$\begin{aligned} 0 &> 2q\sigma^2 a^2 + \mu a(\mu a - 2\bar{b}) \\ &= 4q^2\sigma^4 a^4 + 4q\sigma^2 a^2(\mu a - \bar{b})(\mu a - 2\bar{b}) + (\mu a - \bar{b})^2(\mu a - 2\bar{b})^2 - (\mu a - 2\bar{b})^2\left[(\mu a - \bar{b})^2 + 2q\sigma^2 a^2\right]. \end{aligned} \tag{21}$$

Hence,

$$\mu a - 2\bar{b} + \frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}} < 0,$$

which proves the third result.

Suppose $\mu a - 2\bar{b} < 0$ and $\sigma \geq \sqrt{\frac{-\mu(\mu a - 2\bar{b})}{2qa}}$. Similar to the previous case, we have

$$\begin{aligned} 0 &\leq 2q\sigma^2 a^2 + \mu a(\mu a - 2\bar{b}) \\ &= 4q^2\sigma^4 a^4 + 4q\sigma^2 a^2(\mu a - \bar{b})(\mu a - 2\bar{b}) + (\mu a - \bar{b})^2(\mu a - 2\bar{b})^2 - (\mu a - 2\bar{b})^2\left[(\mu a - \bar{b})^2 + 2q\sigma^2 a^2\right]. \end{aligned} \tag{22}$$

Hence,

$$|\mu a - 2\bar{b}| \leq \left|\frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}}\right|. \tag{23}$$

From the first line of (22), we have that $0 \leq 2q\sigma^2 a^2 + \mu a(\mu a - 2\bar{b})$. Then,

$$0 < 2q\sigma^2 a^2 + \mu a(\mu a - 2\bar{b}) - \bar{b}(\mu a - 2\bar{b}) = (\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2. \tag{24}$$

Combining (23) and (24) yields

$$\mu a - 2\bar{b} + \frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}} \geq 0,$$

which proves the second "if" of the fourth result.

Taking the partial derivative of $\theta_1(a, c)$ with respect to a, we have

$$\frac{\partial}{\partial a}\theta_1(a, c) = -\frac{1}{\sigma^2 a^3}\left[2\bar{b} - \mu a + \frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}}\right].$$

We want to prove that $\frac{\partial}{\partial a}\theta_1(a, c) < 0$. Suppose otherwise, that is,

$$2\bar{b} - \mu a + \frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}} \leq 0.$$

Write $d := \frac{(\mu a - \bar{b})(\mu a - 2\bar{b}) + 2q\sigma^2 a^2}{\sqrt{(\bar{b} - \mu a)^2 + 2q\sigma^2 a^2}}$. The inequality holds if one of the following cases is true: (1) $2\bar{b} - \mu a \leq 0$ and $d \geq 0$ with $(2\bar{b} - \mu a)^2 \geq d^2$, (2) $2\bar{b} - \mu a \geq 0$ and $d \leq 0$ with $(2\bar{b} - \mu a)^2 \leq d^2$, or (3) $2\bar{b} - \mu a \leq 0$ and $d \leq 0$. For case (1), we obtain in the reverse direction of (21) that $\sigma^2 < \frac{-\mu(\mu a - 2\bar{b})}{2qa} < 0$, which is a contradiction. For case (2), it must be the case that $\mu a - \bar{b} > 0$ since $\mu a - \bar{b} \leq 0$ will contradict the premise that $d \leq 0$. We obtain (22) in a reverse direction. From the first line of (22), we have $0 \leq 2q\sigma^2 a^2 + \mu a(\mu a - 2\bar{b}) < 2q\sigma^2 a^2 + (\mu a - \bar{b})(\mu a - 2\bar{b})$, which leads to a contradiction since it must be the case that $d \leq 0$. For case (3), since $\mu a - 2\bar{b} > 0$, we have $\mu a - \bar{b} > 0$. This is also a contradiction since it must be the case that $d \leq 0$. Therefore, we obtain the second result, which completes the proof. □

The next result is a comparison principle for the finite case and is used to prove the uniqueness of the viscosity solution to the HJB equation (7).

Proposition A.2. *Suppose the following conditions hold:*

(i) $\underline{u}$ *is a viscosity subsolution and* $\overline{u}$ *is a viscosity supersolution of the HJB equation* (7) *for all* $x > 0$,

(ii) $\underline{u}$ *and* $\overline{u}$ *are nondecreasing in the variable* x *and Lipschitz in* $[0, \infty)$, *and*

(iii) $\underline{u}(0) = \overline{u}(0)$ *and* $\lim_{x\to\infty} \underline{u}(x) \le \frac{c_n}{q} \le \lim_{x\to\infty} \overline{u}(x)$.

Then, $\underline{u} \le \overline{u}$ *in* $[0, \infty)$.

Proof of Proposition A.2. Suppose that there is a point $x_0 \in [0, \infty)$ such that

$$\underline{u}(x_0) - \overline{u}(x_0) > 0.$$

For every $\gamma > 1$, define $h_j := 1 + \eta e^{-\frac{c_j}{c_n}}$ and $\overline{u}^\gamma(x) = \gamma h_j \overline{u}(x)$, where

$$\eta = \frac{\underline{u}(x_0) - \overline{u}(x_0)}{2\overline{u}(x_0)} > 0.$$

Then, φ is a test function for supersolution of $\overline{u}$ at x if and only if $\varphi^\gamma := \gamma h_j \varphi$ is a test function for supersolution of $\overline{u}^\gamma$ at x. Using (8) and the fact that $1 - \gamma h_j < 1 - \gamma < 0$,

$$\begin{aligned}\mathcal{L}^{a_i,c_j}(\varphi^\gamma)(x) &= \frac{\sigma^2 a_i^2}{2}\gamma h_j \varphi_{xx}(x) + (\mu a_i - b - c_j)\gamma h_j \varphi_x(x) - q\gamma h_j \varphi(x) + c_j \\ &= \gamma h_j \mathcal{L}^{a_i,c_j}(\varphi)(x) + c(1 - \gamma h_j) < 0,\end{aligned}$$

and, provided $\varphi(x) > 0$,

$$\begin{aligned}V^{a_{i+1},c_j}(x) - \varphi^\gamma(x) &= V^{a_{i+1},c_j}(x) - \gamma h_j \varphi(x) < V^{a_{i+1},c_j}(x) - \varphi(x) \le 0, \\ V^{a_i,c_{j+1}}(x) - \varphi^\gamma(x) &= V^{a_i,c_{j+1}}(x) - \gamma h_j \varphi(x) < V^{a_i,c_{j+1}}(x) - \varphi(x) \le 0.\end{aligned}$$

Take $\gamma_0 > 1$ such that

$$\underline{u}(x_0) - \overline{u}^{\gamma_0}(x_0) > 0. \tag{25}$$

Define

$$M := \sup_{x \ge 0} \left[\underline{u}(x) - \overline{u}^{\gamma_0}(x)\right]. \tag{26}$$

Since $\lim_{x\to\infty} \underline{u}(x) \le \frac{c_n}{q} \le \lim_{x\to\infty} \overline{u}(x)$, there exists $\tilde{x} > x_0$ such that

$$\underline{u}(x) - \overline{u}^{\gamma_0}(x) \le 0 \quad \text{for } x \ge \tilde{x}. \tag{27}$$

We then have

$$0 \overset{(25)}{<} \underline{u}(x_0) - \overline{u}^{\gamma_0}(x_0) \overset{(26)}{\le} M = \max_{x \in [0,\tilde{x}]} \left[\underline{u}(x) - \overline{u}^{\gamma_0}(x)\right].$$

Define $x^* := \arg\max_{x\in[0,\tilde{x}]} \left[\underline{u}(x) - \overline{u}^{\gamma_0}(x)\right]$. Consider the set

$$\mathcal{S} := \{(x, y) : 0 \le x \le y \le \tilde{x}\}$$

and, for all $\xi > 0$, the functions

$$\begin{aligned}\Phi^\xi(x, y) &= \frac{\xi}{2}(x - y)^2 + \frac{2m}{\xi^2(y - x) + \xi}, \\ \Sigma^\xi(x, y) &= \underline{u}(x) - \overline{u}^{\gamma_0}(y) - \Phi^\xi(x, y),\end{aligned} \tag{28}$$

where $m > 0$ is a Lipschitz constant satisfying

$$\begin{aligned}|\underline{u}(x) - \underline{u}(y)| &\le m|x - y|, \\ |\overline{u}^{\gamma_0}(x) - \overline{u}^{\gamma_0}(y)| &\le m|x - y|.\end{aligned} \tag{29}$$

The partial derivatives of Φ^ξ satisfy

$$\Phi^\xi_x(x, y) = \xi(x - y) + \frac{2m}{(\xi(y - x) + 1)^2} = -\Phi^\xi_y(x, y). \tag{30}$$

Define $M^\xi = \max_{\mathcal{S}} \Sigma^\xi$ and $(x_\xi, y_\xi) = \arg\max_{\mathcal{S}} \Sigma^\xi$. We then obtain

$$M^\xi \geq \Sigma^\xi(x^*, x^*) = M - \Phi^\xi(x^*, x^*) = M - \frac{2m}{\xi}.$$

Hence,

$$\liminf_{\xi \to \infty} M^\xi \geq M. \tag{31}$$

It can be shown that the maximum is not achieved on the boundary $y = x$, and similar arguments apply to the boundaries $x = 0$ and $y = \tilde{x}$. Thus, there exists ξ_0 large enough such that if $\xi \geq \xi_0$, then $(x_\xi, y_\xi) \notin \partial\mathcal{S}$.

Using the inequality $\Sigma^\xi(x_\xi, x_\xi) + \Sigma^\xi(y_\xi, y_\xi) \leq 2\Sigma^\xi(x_\xi, y_\xi)$ and (29), we obtain

$$\xi |x_\xi - y_\xi|^2 \leq 6m|x_\xi - y_\xi|. \tag{32}$$

We can then find a sequence $\xi_n \to \infty$ such that $(x_{\xi_n}, y_{\xi_n}) \to (\hat{x}, \hat{y}) \in \mathcal{S}$. From (32), we get

$$|x_{\xi_n} - y_{\xi_n}| \leq \frac{6m}{\xi_n}, \tag{33}$$

which gives $\hat{x} = \hat{y}$ as $n \to \infty$.

Since Σ^ξ reaches the maximum in (x_ξ, y_ξ) in the interior of the set $\mathcal{S}$, we have

$$0 \leq \Sigma^\xi(x_\xi, y_\xi) - \Sigma^\xi(x, y_\xi) = \underline{u}(x_\xi) - \underline{u}(x) - \Phi^\xi(x_\xi, y_\xi) + \Phi^\xi(x, y_\xi).$$

Hence, the function $\psi(x) := \Phi^\xi(x, y_\xi) - \Phi^\xi(x_\xi, y_\xi) + \underline{u}(x_\xi)$ is a test for subsolution for $\underline{u}$ at x_ξ, and so,

$$\max\{\mathcal{L}^{a_i, c_j}(\psi)(x_\xi), V^{a_{i+1}, c_j}(x_\xi) - \psi(x_\xi), V^{a_i, c_{j+1}}(x_\xi) - \psi(x_\xi)\} \geq 0.$$

Similarly, the function $\varphi(x) := -\Phi^\xi(x_\xi, y) + \Phi^\xi(x_\xi, y_\xi) + \overline{u}^{\gamma_0}(y_\xi)$ is a test for supersolution for $\overline{u}^{\gamma_0}$ at y_ξ, and so,

$$\max\{\mathcal{L}^{a_i, c_j}(\varphi)(y_\xi), V^{a_{i+1}, c_j}(y_\xi) - \varphi(y_\xi), V^{a_i, c_{j+1}}(y_\xi) - \varphi(y_\xi)\} \leq 0.$$

Assume first that the functions $\underline{u}(x)$ and $\overline{u}^{\gamma_0}(y)$ are twice continuously differentiable at x_ξ and y_ξ, respectively. Since Σ^ξ defined in (28) reaches a local maximum at $(x_\xi, y_\xi) \notin \partial\mathcal{S}$, by the classical maximum principle, we have $\Sigma^\xi_x(x_\xi, y_\xi) = \Sigma^\xi_y(x_\xi, y_\xi) = 0$ and

$$H(\Sigma^\xi)(x_\xi, y_\xi) := \begin{bmatrix} A - \Phi^\xi_{xx}(x_\xi, y_\xi) & -\Phi^\xi_{xy}(x_\xi, y_\xi) \\ -\Phi^\xi_{xy}(x_\xi, y_\xi) & -B - \Phi^\xi_{yy}(x_\xi, y_\xi) \end{bmatrix} \preceq 0, \tag{34}$$

where $A = \underline{u}_{xx}(x_\xi)$, $B = \overline{u}^{\gamma_0}_{yy}(y_\xi)$, and $D \preceq 0$ means D is a negative semi-definite matrix. We also write $D_1 \preceq D_2$ to mean that $D_1 - D_2$ is a negative semi-definite matrix. Then, (34) can be rewritten as

$$\begin{aligned} \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} &\preceq H(\Phi^\xi)(x_\xi, y_\xi) \\ &:= \begin{bmatrix} \Phi^\xi_{xx}(x_\xi, y_\xi) & \Phi^\xi_{xy}(x_\xi, y_\xi) \\ \Phi^\xi_{xy}(x_\xi, y_\xi) & \Phi^\xi_{yy}(x_\xi, y_\xi) \end{bmatrix} \\ &\overset{(30)}{=} \left(\xi + \frac{4m\xi}{(\xi(y_\xi - x_\xi) + 1)^3}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Then,

$$\underline{u}_x(x_\xi) = \Phi^\xi_x(x_\xi, y_\xi) = \xi(x_\xi - y_\xi) + \frac{2m}{(\xi(y_\xi - x_\xi) + 1)^2} = -\Phi^\xi_y(x_\xi, y_\xi) = \overline{u}^{\gamma_0}_y(y_\xi).$$

Moreover, since $H(\Sigma^\xi)(x_\xi, y_\xi)$ is negative semi-definite,

$$0 \geq \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \left(\xi + \frac{4m\xi}{(\xi(y_\xi - x_\xi) + 1)^3}\right) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A - B.$$

In the case that $\underline{u}(x)$ and $\overline{u}^{\gamma_0}(y)$ are not twice continuously differentiable at x_ξ and y_ξ, respectively, we will use Theorem 3.2 of Crandall et al. [1992]. Since the assumptions of the theorem are satisfied, it holds that, for any $\delta > 0$, there exist real numbers A_δ and B_δ such that

$$\begin{bmatrix} A_\delta & 0 \\ 0 & -B_\delta \end{bmatrix} \preceq H(\Phi^\xi)(x_\xi, y_\xi) + \delta \left[H(\Phi^\xi)(x_\xi, y_\xi)\right]^2$$

and

$$\begin{aligned} \frac{\sigma^2 a_i^2}{2} A_\delta + (\mu a_i - b - c_j)\psi_x(x_\xi) - q\psi(x_\xi) + c_j \geq 0 \\ \frac{\sigma^2 a_i^2}{2} B_\delta + (\mu a_i - b - c_j)\varphi_y^{\gamma_0}(y_\xi) - q\varphi^{\gamma_0}(y_\xi) + c_j \leq 0. \end{aligned} \tag{35}$$

Then,

$$\begin{aligned} 0 &\geq \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} A_\delta & 0 \\ 0 & -B_\delta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix} \left[H(\Phi^\xi)(x_\xi, y_\xi) + \delta \left[H(\Phi^\xi)(x_\xi, y_\xi)\right]^2\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (A_\delta - B_\delta) \\ &\quad - \left(\xi + \frac{4m\xi}{(\xi(y_\xi - x_\xi) + 1)^3} + 2\delta\left(\xi + \frac{4m\xi}{(\xi(y_\xi - x_\xi) + 1)^3}\right)^2\right) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= A_\delta - B_\delta. \end{aligned} \tag{36}$$

Since $\varphi^{\gamma_0}(y_\xi) = \underline{u}^{\gamma_0}(y_\xi)$, $\psi(x_\xi) = \underline{u}(x_\xi)$, and

$$\varphi_y^{\gamma_0}(y_\xi) = -\Phi_y^\xi(x_\xi, y_\xi) = \Phi_x^\xi(x_\xi, y_\xi) = \psi_x(x_\xi),$$

we get

$$\underline{u}(x_\xi) - \overline{u}^{\gamma_0}(y_\xi) = \psi(x_\xi) - \varphi^{\gamma_0}(y_\xi) \overset{(35)}{\leq} \frac{\sigma^2 a_i^2}{2q}(A_\delta - B_\delta) \overset{(36)}{\leq} 0. \tag{37}$$

Hence,

$$\begin{aligned} 0 < M &\overset{(31)}{\leq} \liminf_{\xi \to \infty} M^\xi \leq \lim_{n \to \infty} M^{\xi_n} = \lim_{n \to \infty} \Sigma^{\xi_n}(x_{\xi_n}, y_{\xi_n}) \\ &= \lim_{n \to \infty} \left[\underline{u}(x_{\xi_n}) - \overline{u}^{\gamma_0}(y_{\xi_n}) - \frac{\xi_n}{2}(x_{\xi_n} - y_{\xi_n})^2 - \frac{2m}{\xi_n^2(y_{\xi_n} - x_{\xi_n}) + \xi_n}\right] \\ &\overset{(33)}{=} \underline{u}(\hat{x}) - \overline{u}^{\gamma_0}(\hat{x}) \overset{(37)}{\leq} 0, \end{aligned}$$

which is a contradiction. The proof is complete. □

Lemma A.3. *The functions $G_{i,j}^{\mathcal{A}}$, $G_{i,j}^{\mathcal{C}}$, and $G_{i,j}^{\mathcal{E}}$ attain their respective maxima in $[0, \infty)$. Moreover,* $\min\left[\arg\max_{x \in [0,\infty)} G_{i,j}^{\mathcal{A}}(x)\right]$, $\min\left[\arg\max_{x \in [0,\infty)} G_{i,j}^{\mathcal{C}}(x)\right]$, *and* $\min\left[\arg\max_{x \in [0,\infty)} G_{i,j}^{\mathcal{E}}(x)\right]$ *exist.*

Proof. Letting $x \to \infty$, we have the following inequality

$$\lim_{x \to \infty} W^{y^*, z^*}(x, a_i, c_j) = \frac{c_n}{q} > \frac{c_j}{q}.$$

Since $\theta_1(a_i, c_j) > 0 > \theta_2(a_i, c_j)$, it holds, for large enough x,

$$G_{i,j}^{\mathcal{A}}(x) = \frac{\frac{c_n}{q} - \frac{c_j}{q}\left(1 - e^{\theta_2(a_i,c_j)x}\right)}{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}} = \frac{\frac{c_n}{q} - \frac{c_j}{q} + \frac{c_j}{q} e^{\theta_2(a_i,c_j)x}}{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}} > 0. \tag{38}$$

Moreover,

$$\lim_{x \to \infty} G_{i,j}^{\mathcal{A}}(x) = \frac{\frac{c_n}{q}}{\lim_{x \to \infty}\left(e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}\right)} = 0. \tag{39}$$

Combining (38) and (39) with the definition of $G_{i,j}^{\mathcal{A}}(0)$ and the continuity of $G_{i,j}^{\mathcal{A}}$ on $[0, \infty)$ implies that $G_{i,j}^{\mathcal{A}}$ attains its maximum in $[0, \infty)$. Since $G_{i,j}^{\mathcal{A}}$ is continuous on $[0, \infty)$, the set $\arg\max_{\tilde{x} \in [0,\infty)} G_{i,j}^{\mathcal{A}}(\tilde{x})$ is closed. Hence, its minimum exists. The same analysis holds for $G_{i,j}^{\mathcal{C}}$ and $G_{i,j}^{\mathcal{E}}$. □

Lemma A.4. *$U_{\mathcal{A}}^*$, $U_{\mathcal{C}}^*$, and $U_{\mathcal{E}}^*$ exist.*

Proof. From (5), the solutions U of the equation $\mathcal{L}^{a_i,c_j}(U)=0$ in $[0,\infty)$ with boundary condition $U(0)=0$ are of the form

$$U_{k_{\mathcal{A}}^{i,j}}(x)=\frac{c_j}{q}\left[1-e^{\theta_2(a_i,c_j)x}\right]+k_{\mathcal{A}}^{i,j}\left[e^{\theta_1(a_i,c_j)x}-e^{\theta_2(a_i,c_j)x}\right].$$

Let $k_{\mathcal{A}}^{i,j}\geq 0$. From (10), $U_{k_{\mathcal{A}}^{i,j}}(x)\geq W^{y^*,z^*}(x,a_{i+1},c_j)$ for all $x\geq 0$ if and only if $k_{\mathcal{A}}^{i,j}\geq G_{i,j}^{\mathcal{A}}(x)$ for all $x\geq 0$. Using Lemma A.3, there exists $\hat{k}_{\mathcal{A}}^{i,j}=\max_{x\in[0,\infty)}G_{i,j}^{\mathcal{A}}(x)>0$. Hence, $U_{\mathcal{A}}^*=U_{\hat{k}_{\mathcal{A}}^{i,j}}$. The proof for $U_{\mathcal{C}}^*$, and $U_{\mathcal{E}}^*$ is similar. □

The following lemma establishes the differentiability and continuity of W^{y^*,z^*}.

Lemma A.5. *$W^{y^*,z^*}(x,a_i,c_j)$ is infinitely continuously differentiable at all $x\in[0,\infty)\backslash\{y_{k,j}^*:k=i,\ldots,m-1\}\cup\{z^*(c_{i,l}):l=j,\ldots,n-1\}$ and is continuously differentiable at the points $y_{k,j}^*$ and $z_{i,l}^*$ for $k=i,\ldots,m-1$ and $l=j,\ldots,n-1$.*

Proof. The result follows directly by construction and a recursive argument. □

The next lemma proves an inequality for $W_{xx}^{y^*,z^*}$ at a neighborhood of threshold values. The result can be used to prove that W^{y^*,z^*} is a viscosity solution since $W_{xx}^{y^*,z^*}$ may not exist as stated in Lemma A.5.

Lemma A.6. *For $1\leq i\leq m-1$ and $1\leq j\leq n-1$, we have $W_{xx}^{y^*,z^*}(y_{i,j}^{*-},a_i,c_j)-W_{xx}^{y^*,z^*}(y_{i,j}^{*+},a_i,c_j)\geq 0$ and $W_{xx}^{y^*,z^*}(z_{i,j}^{*-},a_i,c_j)-W_{xx}^{y^*,z^*}(z_{i,j}^{*+},a_i,c_j)\geq 0$.*

Proof. We prove the first inequality. By definition, it must hold that $U_{\mathcal{A}}^*(\cdot)-W^{y^*,z^*}(\cdot,a_{i+1},c_j)$ reaches the minimum at $y_{i,j}^*$. It then follows that $\frac{d}{dx}U_{\mathcal{A}}^*(y_{i,j}^*)-W_x^{y^*,z^*}(y_{i,j}^*,a_{i+1},c_j)=0$. Moreover, it holds that

$$W^{y^*,z^*}(x,a_i,c_j)=U_{\mathcal{A}}^*(x)\mathbf{1}_{\{x<y_{i,j}^*\}}+W^{y^*,z^*}(x,a_{i+1},c_j)\mathbf{1}_{\{x\geq y_{i,j}^*\}}.$$

Hence,

$$W_{xx}^{y^*,z^*}(y_{i,j}^*-,a_i,c_j)-W_{xx}^{y^*,z^*}(y_{i,j}^*+,a_i,c_j)=\frac{d^2}{dx^2}U_{\mathcal{A}}^*(y_{i,j}^*)-W_{xx}^{y^*,z^*}(y_{i,j}^*+,a_{i+1},c_j)\geq 0.$$

The proof for the second inequality is similar. □

B Proofs of the Results in Section 3

Proof of Theorem 3.2. (Part I) We first show that V^{a_i,c_j} is a viscosity supersolution. By Proposition 2.1, $V^{a_{i+1},c_j}(x)-V^{a_i,c_j}(x)\leq 0$ and $V^{a_i,c_{j+1}}(x)-V^{a_i,c_j}(x)\leq 0$ in $(0,\infty)$.

Consider an $x\in(0,\infty)$ and the admissible strategy $\pi:=(A,C)\in\Pi_{x,a_i,c_j}^{\mathscr{A},\mathscr{C}}$, which retains incoming claims at a constant rate a_i and pays dividends at a constant rate c_j up to the ruin time τ_π. Let $\{X_t^\pi\}_{t\geq 0}$ be the corresponding reserve process. Suppose there exists a test function φ for supersolution of (7) at x. Then, $\varphi\leq V^{a_i,c_j}$ and $\varphi(x)=V^{a_i,c_j}(x)$.

We want to prove that $\mathcal{L}^{a_i,c_j}(\varphi)(x)\leq 0$. Since $\mathcal{L}^{a_i,c_j}(\varphi)(\cdot)$ may be unbounded, we consider an auxiliary function $\tilde{\varphi}$ for the supersolution of (7) such that $\tilde{\varphi}\leq\varphi\leq V^{a_i,c_j}$ in $[0,\infty)$, $\tilde{\varphi}=\varphi$ in $[0,2x]$ and $\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(\cdot)$ is bounded in $[0,\infty)$. We construct $\tilde{\varphi}$ by considering a function $g:[0,\infty)\to[0,1]$ such that $g=0$ in $[2x+1,\infty)$ and $g=1$ in $[0,2x]$. Define $\tilde{\varphi}(w)=\varphi(w)g(w)$. Using Lemma 2.3, we obtain for $h>0$

$$\tilde{\varphi}(x)=V^{a_i,c_j}(x)\geq\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}c_j\,ds\right]+\mathbb{E}\left[e^{-q(\tau_\pi\wedge h)}\tilde{\varphi}\left(X_{\tau_\pi\wedge h}^\pi\right)\right].$$

Using Itô's formula and (4) yields

$$\begin{aligned}0&\geq\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}c_j\,ds\right]+\mathbb{E}\left[e^{-q(\tau_\pi\wedge h)}\tilde{\varphi}\left(X_{\tau_\pi\wedge h}^\pi\right)-\tilde{\varphi}(x)\right]\\&=\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}c_j\,ds\right]+\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}\tilde{\varphi}_x(X_s^\pi)\sigma dW_s\right]\\&\quad+\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}\left((\mu a_i-b-c_j)\tilde{\varphi}_x(X_s^\pi)+\frac{\sigma^2a_i^2}{2}\tilde{\varphi}_{xx}(X_s^\pi)-q\tilde{\varphi}(X_s^\pi)\right)ds\right]\\&=\mathbb{E}\left[\int_0^{\tau_\pi\wedge h}e^{-qs}\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(X_s^\pi)ds\right].\end{aligned}\tag{40}$$

The term $\mathbb{E}\left[\int_0^{\tau_\pi \wedge h} e^{-qs}\tilde{\varphi}_x(X_s^\pi)\sigma dW_s\right]$ vanishes to 0 because the integrand is bounded. Since φ is continuous and differentiable, it is locally bounded, which implies that it is locally Lipschitz. Consequently, $\varphi_x = \tilde{\varphi}_x$ is bounded. Moreover, the term $\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(x)$ is well-defined, even though g is not twice continuously differentiable everywhere, since $\tilde{\varphi}$ is twice continuously differentiable for any $w < 2x$.

Since $\tau_\pi > 0$ a.s. and the following results hold

$$\left|\frac{1}{h}\int_0^{\tau_\pi \wedge h} e^{-qs}\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(X_s^\pi)ds\right| \leq \sup_{w\in[0,\infty)} \left|\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(w)\right|,$$

and

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^{\tau_\pi \wedge h} e^{-qs}\mathcal{L}^{a_i,c_j}(\tilde{\varphi})(X_s^\pi)ds = \mathcal{L}^{a_i,c_j}(\tilde{\varphi})(x) \text{ a.s.},$$

then, via the bounded convergence theorem and inequality (40),

$$\mathcal{L}^{a_i,c_j}(\varphi)(x) = \mathcal{L}^{a_i,c_j}(\tilde{\varphi})(x) \leq 0.$$

Thus, V^{a_i,c_j} is a viscosity supersolution at x.

(Part II) We now show that V^{a_i,c_j} is a viscosity subsolution of (7) via contradiction. Suppose otherwise that V^{a_i,c_j} is not a viscosity subsolution. Then, there exist $\epsilon > 0$, $0 < h < \frac{x}{2}$ and a twice continuously differentiable function ψ with $\psi(x) = V^{a_i,c_j}(x)$ such that $\psi \geq V^{a_i,c_j}$,

$$\max\{\mathcal{L}^{a_i,c_j}(V^{a_i,c_j})(w), V^{a_{i+1},c_j}(w) - V^{a_i,c_j}(w), V^{a_i,c_{j+1}}(w) - V^{a_i,c_j}(w)\} \leq -q\epsilon < 0 \tag{41}$$

for $w \in [x-h, x+h]$, and

$$V^{a_i,c_j}(w) \leq \psi(w) - \epsilon \tag{42}$$

for $w \notin [x-h, x+h]$.

Consider the reserve process $\{X_t^\pi\}_{t\geq 0}$ corresponding to an admissible strategy $\pi \in \Pi_{x,a_i,c_j}^{\mathscr{A},\mathscr{C}}$. Define the stopping time τ^* given by

$$\tau^* := \inf\{t > 0 : X_t^\pi \notin [x-h, x+h]\}.$$

Taking expectations and using (41) yield

$$\begin{aligned}
\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)}\psi(X_{\tau^*}^\pi)\right] - \psi(x) &= \mathbb{E}\left[\int_0^{\tau_\pi\wedge\tau^*} e^{-qs}\mathcal{L}^{a_i,c_j}(\psi)(X_s^\pi)ds - \int_0^{\tau_\pi\wedge\tau^*} e^{-qs}c_j\, ds\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau_\pi\wedge\tau^*} e^{-qs}(-q\epsilon)ds + \frac{c_j}{q}\left(e^{-q(\tau_\pi\wedge\tau^*)} - 1\right)\right] \\
&= \left(\epsilon + \frac{c_j}{q}\right)\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)} - 1\right].
\end{aligned} \tag{43}$$

From (42) and (43),

$$\begin{aligned}
\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)}V^{a_i,c_j}(X_{\tau_\pi\wedge\tau^*}^\pi)\right] &\leq \mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)}\left(\psi(X_{\tau_\pi\wedge\tau^*}^\pi) - \epsilon\right)\right] \\
&\leq \psi(x) + \left(\epsilon + \frac{c_j}{q}\right)\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)} - 1\right] - \epsilon\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)}\right] \\
&= \psi(x) + \frac{c_j}{q}\mathbb{E}\left[e^{-q(\tau_\pi\wedge\tau^*)} - 1\right] - \epsilon.
\end{aligned}$$

Using Lemma 2.3, we have

$$V^{a_i,c_j}(x) = \sup_{\pi\in\Pi_{x,a_i,c_j}^{\mathscr{A},\mathscr{C}}} \mathbb{E}\left[-\frac{c_j}{q}\left(e^{-q(\tau_\pi\wedge\tau^*)} - 1\right) + e^{-q(\tau_\pi\wedge\tau^*)}V^{a_i,c_j}(X_{\tau_\pi\wedge\tau^*}^\pi)\right] \leq \psi(x) - \epsilon,$$

which contradicts the assumption that $V^{a_i,c_j}(x) = \psi(x)$. Therefore, V^{a_i,c_j} is a viscosity subsolution of (7).

The uniqueness result is a direct consequence of Propositions 2.2 and A.2. □

Proof of Theorem 3.3. By definition, $W^{y^*,z^*}(x, a_m, c_n) = V^{a_m,c_n}(x)$. Assume that $W^{y^*,z^*}(\cdot, a_k, c_l) = V^{a_k,c_l}$ for $k = i+1, \ldots, m$ and $l = j+1, \ldots, n$. Moreover, assume that $W^{y^*,z^*}(\cdot, a_{i+1}, c_j) = V^{a_{i+1},c_j}$ and $W^{y^*,z^*}(\cdot, a_i, c_{j+1}) = V^{a_i,c_{j+1}}$.

By construction, we know that

$$\begin{cases} V^{a_{i+1},c_j}(x) - W^{y^*,z^*}(x, a_i, c_j) = 0, & \text{if } x \geq y^*_{i,j}, \\ V^{a_{i+1},c_j}(x) - W^{y^*,z^*}(x, a_i, c_j) \leq 0, & \text{if } x < y^*_{i,j}, \\ V^{a_i,c_{j+1}}(x) - W^{y^*,z^*}(x, a_i, c_j) = 0, & \text{if } x \geq z^*_{i,j}, \\ V^{a_i,c_{j+1}}(x) - W^{y^*,z^*}(x, a_i, c_j) \leq 0, & \text{if } x < z^*_{i,j}, \\ \mathcal{L}^{a_i,c_j}\left(W^{y^*,z^*}\right)(x, a_i, c_j) = 0, & \text{if } x < y^*_{i,j} \wedge z^*_{i,j}. \end{cases}$$

Suppose Theorem 3.2 holds. It suffices to show that $W^{y^*,z^*}(\cdot, a_i, c_j)$ is a viscosity solution of (7). It remains to show the following:

$$\mathcal{L}^{a_i,c_j}\left(W^{y^*,z^*}\right)(x, a_i, c_j) \leq 0 \quad \text{for all } x \geq y^*_{i,j} \wedge z^*_{i,j}.$$

Fix j. Suppose $x \neq y^*_{k,j}$ for $k = i+1, \ldots, m-1$. Then, x belongs to one of the open intervals wherein $\mathcal{L}^{a_k,c_j}(W^{y^*,z^*})(x, a_i, c_j) = 0$. Suppose $\mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(x, a_i, c_j) > 0$. Then,

$$\begin{aligned} 0 &< \mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(x, a_i, c_j) - \mathcal{L}^{a_k,c_j}(W^{y^*,z^*})(x, a_i, c_j) \\ &= \frac{1}{2}\sigma^2(a_i^2 - a_k^2)W^{y^*,z^*}_{xx}(x, a_i, c_j) + \mu(a_i - a_k)W^{y^*,z^*}_x(x, a_i, c_j). \end{aligned} \tag{44}$$

There exist $\delta > 0$ and some $k > i$ such that $\mathcal{L}^{a_k,c_j}(W^{y^*,z^*})(x, a_i, c_j) = 0$ in $(y^*_{i,j}, y^*_{i,j} + \delta)$. Then,

$$\mathcal{L}^{a_k,c_j}(W^{y^*,z^*})(y^{*+}_{i,j}, a_i, c_j) = 0 \quad \text{and} \quad \mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(y^{*-}_{i,j}, a_i, c_j) = 0.$$

By Lemma A.5 and inequality (44),

$$\begin{aligned} 0 &= \mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(y^{*-}_{i,j}, a_i, c_j) - \mathcal{L}^{a_k,c_j}(W^{y^*,z^*})(y^{*+}_{i,j}, a_i, c_j) \\ &= \frac{1}{2}\sigma^2\left[a_i^2 W^{y^*,z^*}_{xx}(y^{*-}_{i,j}, a_i, c_j) - a_k^2 W^{y^*,z^*}_{xx}(y^{*+}_{i,j}, a_i, c_j)\right] + \mu(a_i - a_k)W^{y^*,z^*}_x(y^{*-}_{i,j}, a_i, c_j) \\ &> \frac{1}{2}\sigma^2\left[a_i^2 W^{y^*,z^*}_{xx}(y^{*-}_{i,j}, a_i, c_j) - a_k^2 W^{y^*,z^*}_{xx}(y^{*+}_{i,j}, a_i, c_j)\right] - \frac{1}{2}\sigma^2(a_i^2 - a_k^2)W^{y^*,z^*}_{xx}(y^{*-}_{i,j}, a_i, c_j) \\ &= \frac{1}{2}\sigma^2 a_k^2\left[W^{y^*,z^*}_{xx}(y^{*-}_{i,j}, a_i, c_j) - W^{y^*,z^*}_{xx}(y^{*+}_{i,j}, a_i, c_j)\right]. \end{aligned}$$

By Lemma A.6, $W^{y^*,z^*}_{xx}(y^{*-}_{i,j}, a_i, c_j) - W^{y^*,z^*}_{xx}(y^{*+}_{i,j}, a_i, c_j) \geq 0$, which is a contradiction to the inequality above. Thus, $\mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(x, a_i, c_j) \leq 0$ for $x \neq y^*_{k,j}$, where $k = i+1, \ldots, m-1$.

Consider now the case $x = y^*_{k,j}$ with $k = i+1, \ldots, m-1$ and $y^*_{k,j} \geq y^*_{i,j}$. Since $W^{y^*,z^*}(x, a_i, c_j)$ may not be twice differentiable at $x = y^*_{k,j}$, we prove that $\mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(x, a_i, c_j) \leq 0$ in the viscosity sense. Take a test function φ_1 as a supersolution at $y^*_{k,j}$. Since φ_1 is a supersolution, it holds that $W^{y^*,z^*}(\cdot, a_i, c_j) - \varphi_1(\cdot)$ achieves its minimum at $y^*_{i,j}$. Hence,

$$\varphi_1(y^*_{k,j}) = W^{y^*,z^*}(y^*_{k,j}, a_i, c_j) \quad \text{and} \quad \varphi_1'(y^*_{k,j}) = W^{y^*,z^*}_x(y^*_{k,j}, a_i, c_j). \tag{45}$$

Moreover,

$$W^{y^*,z^*}_{xx}(y^{*+}_{k,j}, a_i, c_j) - \varphi_1''(y^*_{k,j}) \geq 0 \quad \text{and} \quad W^{y^*,z^*}_{xx}(y^{*-}_{k,j}, a_i, c_j) - \varphi_1''(y^*_{k,j}) \geq 0,$$

or, equivalently,

$$\varphi_1''(y^*_{k,j}) \leq \min\left\{W^{y^*,z^*}_{xx}(y^{*-}_{k,j}, a_i, c_j), W^{y^*,z^*}_{xx}(y^{*+}_{k,j}, a_i, c_j)\right\}. \tag{46}$$

Using (45) and (46) yields

$$\begin{aligned} \mathcal{L}^{a_i,c_j}(\varphi_1)(y^*_{k,j}) &= \frac{1}{2}\sigma^2 a_i^2 \varphi_1''(y^*_{k,j}) + (\mu a_i - b - c_j)\varphi_1'(y^*_{k,j}) - q\varphi_1(y^*_{k,j}) + c_j \\ &= \frac{1}{2}\sigma^2 a_i^2 \varphi_1''(y^*_{k,j}) + (\mu a_i - b - c_j)W^{y^*,z^*}_x(y^*_{k,j}, a_i, c_j) \\ &\quad - qW^{y^*,z^*}(y^*_{k,j}, a_i, c_j) + c_j \\ &\leq \min\left\{\mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(y^{*-}_{k,j}, a_i, c_j), \mathcal{L}^{a_i,c_j}(W^{y^*,z^*})(y^{*+}_{k,j}, a_i, c_j)\right\} \\ &= 0. \end{aligned}$$

The proof for the dividend threshold levels is similar. □

C Value Function Derivation Using Scale Functions

We present an alternative derivation of the value function using scale functions. Write $X_t^{i,j} = x + (\mu a_i - b - c_j)t + \sigma a_i W_t$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For a fixed $\beta \geq 0$, we define the following passage times:

$$\tau_\beta^{i,j} := \inf\left\{t \geq 0 : X_t^{i,j} \geq \beta\right\}, \quad \tau_0^{i,j} := \inf\left\{t \geq 0 : X_t^{i,j} < 0\right\}.$$

We also define the functions $\mathbb{W}_{i,j}^q(x)$ and $\mathbb{Z}_{i,j}^q(x)$ of the process $X^{i,j} := \{X_t^{i,j}\}_{t\geq 0}$ as

$$\int_0^\infty e^{-ux}\, \mathbb{W}_{i,j}^{(q)}(x)\, dx = \frac{1}{\psi_{i,j}(u) - q}, \quad u > \theta_1(a_i, c_j),$$

$$\mathbb{Z}_{i,j}^{(q)}(x) = 1 + q\int_0^x \mathbb{W}_{i,j}^{(q)}(y)\, dy,$$

where $\psi_{i,j}(u) := \frac{1}{2}\sigma^2 a_i^2 u^2 + (\mu a_i - b - c_j)u$ is the Laplace exponent of $X^{i,j}$. The functions $\mathbb{W}_{i,j}^q$ and $\mathbb{Z}_{i,j}^q$ are referred to as the *q-scale functions* of $X^{i,j}$ in the literature of exit problems for spectrally negative Lévy processes.

Write $\kappa_{i,j} := \left((\mu a_i - b - c_j)^2 + 2q\sigma^2 a_i^2\right)^{-1/2}$. Using the method of partial fractions and the Laplace inverse transform of $(\psi_{i,j}(u) - q)^{-1}$, we obtain

$$\mathbb{W}_{i,j}^{(q)}(x) = \kappa_{i,j}\left(e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}\right).$$

Consequently, it can be shown that

$$\mathbb{Z}_{i,j}^{(q)}(x) = \frac{q}{\theta_1(a_i,c_j)}\mathbb{W}_{i,j}^{(q)}(x) + e^{\theta_2(a_i,c_j)x}.$$

Suppose $y_{i,j} < z_{i,j}$. For $x \in [0, y_{i,j})$, we have

$$\begin{aligned}
\upsilon^{i,j}(x) &= \mathbb{E}\left[\int_0^{\tau_{y_{i,j}}^{i,j}\wedge\tau_0^{i,j}} e^{-qt}\, c_j\, dt\right] + \mathbb{E}\left[e^{-q\tau_{y_{i,j}}^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} < \tau_0^{i,j}\}}\right]\upsilon^{i+1,j}(x) \\
&= \frac{c_j}{q}\left[1 - \mathbb{E}\left[e^{-q\left(\tau_{y_{i,j}}^{i,j}\wedge\tau_0^{i,j}\right)}\right]\right] + \mathbb{E}\left[e^{-q\tau_{y_{i,j}}^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} < \tau_0^{i,j}\}}\right]\upsilon^{i+1,j}(x) \\
&= \frac{c_j}{q}\left[1 - \mathbb{E}\left[e^{-q\tau_{y_{i,j}}^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} < \tau_0^{i,j}\}}\right] - \mathbb{E}\left[e^{-q\tau_0^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} > \tau_0^{i,j}\}}\right]\right] + \mathbb{E}\left[e^{-q\tau_{y_{i,j}}^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} < \tau_0^{i,j}\}}\right]\upsilon^{i+1,j}(x).
\end{aligned}$$

By Kyprianou [2014, Theorem 8.1(iii)], we have

$$\mathbb{E}\left[e^{-q\tau_{y_{i,j}}^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} < \tau_0^{i,j}\}}\right] = \frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y_{i,j})} = \frac{e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}}{e^{\theta_1(a_i,c_j)y_{i,j}} - e^{\theta_2(a_i,c_j)y_{i,j}}},$$

$$\mathbb{E}\left[e^{-q\tau_0^{i,j}} \cdot \mathbf{1}_{\{\tau_{y_{i,j}}^{i,j} > \tau_0^{i,j}\}}\right] = \mathbb{Z}_{i,j}^{(q)}(x) - \mathbb{Z}_{i,j}^{(q)}(y_{i,j})\frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y_{i,j})} = e^{\theta_2(a_i,c_j)x} + (1 - e^{\theta_2(a_i,c_j)y_{i,j}})\frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y_{i,j})}.$$

Then,

$$\begin{aligned}
\upsilon^{i,j}(x) &= \frac{c_j}{q}\left[1 - e^{\theta_2(a_i,c_j)x} - (1 - e^{\theta_2(a_i,c_j)y_{i,j}})\frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y_{i,j})}\right] + \frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y_{i,j})}\upsilon^{i+1,j}(x) \\
&= \frac{c_j}{q}\left(1 - e^{\theta_2(a_i,c_j)x}\right) + \left(e^{\theta_1(a_i,c_j)x} - e^{\theta_2(a_i,c_j)x}\right)\frac{\upsilon^{i+1,j}(x) - \frac{c_j}{q}(1 - e^{\theta_2(a_i,c_j)y_{i,j}})}{e^{\theta_1(a_i,c_j)y_{i,j}} - e^{\theta_2(a_i,c_j)y_{i,j}}},
\end{aligned}$$

which yields the form of the value function in Theorem 3.3 for $i < m$ or $j < n$. Moreover, by construction, if $x > y_{i,j}$, then $\upsilon^{i,j}(x) = \upsilon^{i+1,j}(x)$. Similar arguments apply for the cases $x < z_{i,j} < y_{i,j}$ and $x < y_{i,j} = z_{i,j}$.

For the case where $i = m$ and $j = n$, we have $y_{m,n} = z_{m,n} = \infty$. Moreover,

$$\lim_{y\to\infty}\frac{\mathbb{W}_{i,j}^q(x)}{\mathbb{W}_{i,j}^q(y)} = \frac{q}{\theta_1(a_m,c_n)}.$$

Hence, we obtain

$$\upsilon^{m,n}(x) = \mathbb{E}\left[\int_0^{\tau_0^{m,n}} e^{-qt}\, c_n\, dt\right] = \frac{c_n}{q}\left[1 - \mathbb{E}\left[e^{-q\tau_0^{m,n}}\right]\right] = \frac{c_n}{q}\left(1 - e^{\theta_2(a_m,c_n)x}\right),$$

which completes the form of the value function in Theorem 3.3.

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