Optimal reinsurance with multiple reinsurers: competitive pricing and coalition stability

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August 7, 2021

Abstract

This paper studies economic pricing of reinsurance contracts via competition of an insurer with multiple reinsurers. All firms are assumed to be endowed with distortion risk measures or expected exponential utilities. Reinsurance contracts are required to be Pareto optimal, individually rational, and satisfy a competition constraint that we call coalition stability. As shown in the literature, it holds that Pareto optimality is equivalent to a structure on the indemnities. This paper characterizes the corresponding premiums by a competition argument. The competition among reinsurers imposes constraints on the premiums that the reinsurers are able to charge and this may lead to a strictly positive profit for the insurer. When the firms use distortion risk measures, this constraint yields stability for subcoalitions, which is a condition akin to the core in cooperative game theory. The premiums and the profit of the insurer are derived in closed-form. This paper illustrates this premium function with the Mean Conditional Value-at-Risk and the GlueVaR. If the firms use expected exponential utilities, the premium is represented by an exponential premium.

Keywords: reinsurance, multiple reinsurers, competition, premiums, Mean Conditional Value-at-Risk, GlueVaR.

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1 Introduction

Optimal reinsurance contract design is a very popular subject since the seminal works of Borch (1960) and Arrow (1963). The problem is to determine optimal reinsurance contracts for a given risk of the insurer. Under the assumption that the reinsurance premium is calculated by the expected value principle, Borch (1960) and Arrow (1963) both show that a stop-loss contract is optimal even though the objective of the former paper is to minimize the variance of the retained loss of the insurer and the latter paper is to maximize the expected utility of the terminal wealth of a risk-averse insurer. Reinsurance contract design also gains popularity in actuarial science in a more recent past. See, for example, Denuit and Vermandele (1998), Young (1999), Gajek and Zagrodny (2000, 2004), Kaluszka (2001, 2005); Cai and Tan (2007), Balbás et al. (2009), Chi (2012), Cui et al. (2013), Bernard et al. (2015). These papers optimize a welfare criterion of the insurer who has the option to reinsure a part of its risk. However, these works are predominantly confined to study the optimal risk sharing between two parties, i.e., an insurer and a reinsurer.

A more realistic situation should involve multiple reinsurers available in the market. The insurer always could use more than one reinsurer to reinsure its risk in a well established reinsurance market. To the best of our knowledge, very few academic papers have been devoted to considering the optimal reinsurance problem in the presence of multiple reinsurers. These works include Asimit et al. (2013b), Chi and Meng (2014), Boonen et al. (2016b), and Cong and Tan (2016). However, all of these have in common that the premium principles for reinsurance are exogenously given. And they optimize a utility function of the insurer and characterize the optimal reinsurance indemnity functions. But, a very important aspect was ignored among these works. The multiple reinsurers in the real market could compete with each other and that insurer could exploit such competition. Therefore, an efficient reinsurance contract profile, and more importantly the welfare gains distribution among them are missing from the economic point of view. In this paper, we shed some light on this topic and introduce a novel economic approach to characterize an optimal reinsurance profile that takes into account competition among reinsurers. In particular, we study reinsurance profiles that satisfy Pareto optimality, individual rationality and a stability property.

First, we assume that all firms use dual utility as introduced by Yaari (1987). This is equivalent to firms minimizing distortion risk measures. Second, we study the case where all firms use expected exponential utilities. The Pareto optimal reinsurance contracts are given by a specific layering of the risk when the firms use distortion risk measures. Any layer is allocated to the one specific party for which the corresponding distortion function is minimal at a given quantile. When the firms use expected exponential utilities, the Pareto optimal reinsurance contracts are proportional to the risk. Hence, for both distortion risk measures and expected exponential utilities, Pareto optimality yields a structure on the indemnities, and the corresponding premiums can be chosen freely. In this paper, we focus on the corresponding premiums of these reinsurance indemnities by an economic stability criterion. Specifically speaking, the insurer has the option to select the reinsurers it wants to trade with. The insurer takes into account that there is an outside option to move to other reinsurers. The reinsurers maximize their profit. We show that this competition leads to a premium of the reinsurer. If the reinsurer charges a larger premium, it will be priced out of the reinsurance market by the others who behave rationally. For distortion risk measures, we derive this premium for each reinsurer in closed-form. Moreover, we demonstrate that such a reinsurance contract profile satisfies a core-type property (i.e. coalition stability), where the core is a well-known concept in cooperative game theory (Gillies, 1953; Scarf, 1967). In other words, no subgroup of reinsurers (*coalition*) and the insurer have a joint incentive to operate in the market without the other reinsurers.

While this paper is inspired by the work of Boonen et al. (2016a), it is important to point out the similarities and the differences between their work and the present paper. The key similarities are as follows. First, both papers study optimal risk sharing under the assumptions of Pareto optimality and individual rationality. Second, by using the indifference pricing arguments, lower and upper bounds of the Pareto optimal and individually rational contracts are similarly established. Third, both paper analyze welfare gain (i.e. hedged benefits) among the firms.

However the key differences (and hence highlighting the main contributions of the present paper) are as follows. First and foremost is the model specification. Boonen et al. (2016a) analyze optimal risk sharing between one insurer and one reinsurer; i.e. bilateral bargaining for reinsurance. The present paper extends Boonen et al. (2016a) by analyzing a more realistic setting with one insurer and multiple reinsurers. Second, the present paper studies the case where all firms use expected exponential utilities, in addition to distortion risk measures as in both papers. A third important distinction is that while Boonen et al. (2016a) models the behavior of the firms via bargaining, the present paper is based on competition. Unlike Boonen et al. (2016a)'s setup, the competition and the presence of multiple reinsurers imply that insurer in the present paper has the flexibility of trading with any reinsurer and with one or more reinsurers. As a result, the competition among reinsurers considerably complicates the pricing of reinsurance contracts. Because of competition it is necessary to impose the property of coalition stability to ensure the stability of market. Also, while bounds on the individual rational premiums of a specific Pareto optimal contract are derived under both models of Boonen et al. (2016a) and the present paper, the premium agreed upon by both insurer and reinsurer for the former model ultimately depends on the firms' relative bargaining power. We demonstrate in this paper that there is a vector of premiums that the firms will accept due to competition. Finally we show that if all reinsurers have the same preferences, a race to the bottom leads to reinsurers offering their indifference premiums (see, e.g., Bertrand, 1883). Therefore, all welfare gains in the market go to the insurer. As a result, competition may affect the premiums substantially. The current paper also provides a closed-form expression and the interpretation of the welfare gains for the insurer, each reinsurer and the aggregate reinsurers. The allocation of welfare gains among the insurer and multiple reinsurers is insightful for understanding the competition in the market.

The remaining paper is organized as follows. In Section 2, we state the model set-up. Section 3 shows the individual rational and Pareto optimal contracts. Sections 4 to 8 study distortion risk measures. Section 4 provides a characterization of the competitive premiums. Section 5 shows a characterization of stability, where stability is shown to be equivalent to coalition stability if we focus on Pareto optimal contracts. Section 6 characterizes the welfare gains. Sections 7 and 8 illustrate our premium function with the Mean Conditional Value-at-Risk and the GlueVaR, respectively. Section 9 shows the competitive premiums in case of expected exponential utilities and Section 10 concludes the paper.

2 Model Outline

Let $(\Omega, A, \mathbb{P})$ be a probability space. Moreover, we denote L^1 as the class of Lebesgue integrable random variables on $(\Omega, A, \mathbb{P})$. We assume that in this market there exists an insurer and n reinsurers. The insurer is indexed by I and the set of reinsurers is indexed by $N = \{1, \ldots, n\}$. The insurer is seeking an optimal reinsurance strategy to cede its risk to the n potential reinsurers. Any risk transfer decision between insurer and the reinsurers is based on a monetary utility function $\hat{V}_k$, which is defined as follows.

Definition 2.1 *A preference relation $\hat{V}_k$, $k = I, 1, 2, \ldots, n$, is monetary if it satisfies the following properties:*

- *Monotonicity: $\hat{V}_k(Y) \leq \hat{V}_k(Z)$ for all $Y, Z \in L^1$ such that $Y \leq Z$;*
- *Normalization: $\hat{V}_k(0) = 0$;*
- *Cash-invariance: $\hat{V}_k(Y + a) = \hat{V}_k(Y) + a$ for all $Y \in L^1$ and $a \in \mathbb{R}$.*

For the sake of presentation, we assume that all firms minimize

$$V_k(Y) := -\hat{V}_k(Y).$$

Due to the cash-invariance property, we obtain that the initial deterministic wealth of the insurer and reinsurers is irrelevant. Note that the value $V_k(Y)$ can be interpreted as a monetary cost or value due to cash-invariance and $V_k(0) = 0$. Another property of a preference relation V_k that plays an important role in this paper is comonotonic additivity, and is defined as follows:

- Comonotonic additivity: for all $X, Y \in L^1$ that are comonotonic, we have $V_k(X) + V_k(Y) = V_k(X + Y)$.

The insurer is endowed with a non-negative loss $X \in L^1$ that is such that $V_k(X) < \infty$ for all $k = I, 1, \ldots, n$. The insurer seeks to reinsure a part of this risk with some reinsurers. The insurer and the reinsurer i are bilaterally bargaining to agree on optimal reinsurance contracts $(f_i, \pi_i), i = 1, \ldots, n$, where $f_i(X)$ is the indemnity; i.e. the ceded loss function, and $\pi_i \in \mathbb{R}$ is the premium paid by the insurer to reinsurer i in exchange of the coverage $f_i(X)$. We assume that $f_i \in \mathcal{F}$, where

$$\mathcal{F} = \{ f : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, f(0) = 0, 0 \le f(x) - f(y) \le x - y, \forall x \ge y \ge 0 \} .$$

The aggregate indemnities $\sum_{i=1}^n f_i$ in $\mathcal{F}$ account for *ex post* moral hazard of the insurer (Huberman et al., 1983; Denuit and Vermandele, 1998; Young, 1999). For this reason, aggregate indemnities are often exogenously imposed to belong to the class $\mathcal{F}$ in the recent literature on reinsurance contract design (see, e.g., Asimit et al., 2013b; Chi and Meng, 2014; Assa, 2015; Xu et al., 2019). Therefore, we also assume $\sum_{i=1}^n f_i \in \mathcal{F}$ to eliminate double insurance.

3 Individual rationality and Pareto optimality

Given that we are concerned with an optimal risk transfer strategy between an insurer and N reinsurers, it is convenient to denote f as the tuple $(f_i)_{i=1}^n$ where $f_i \in \mathcal{F}$ for all i, and $\sum_{i=1}^n f_i \in \mathcal{F}$, and π as the tuple $(\pi_i)_{i=1}^n$ where π_i represents the premium charged by reinsurer i with corresponding indemnity contract f_i. It is also useful to use $\mathcal{F}_N$ to denote the collection of all such indemnity contracts. Before discussing the optimality of f_i and its premium π_i, the focus of this section is to describe two important properties for which a reinsurance contract profile $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ may satisfy. These two properties are known as individual rationality and Pareto optimality.

Individual rationality states that all firms are weakly better off from trading compared to the status quo. Hence, the property of individual rationality for the insurer I and all n reinsurers implies that the reinsurance contract profile $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ satisfies the following conditions:

$$V_I\left(X - \sum_{j=1}^{n}(f_j(X) - \pi_j)\right) \leq V_I\left(X - \sum_{j \neq i}(f_j(X) - \pi_j)\right), i = 1, \ldots, n, \quad (1)$$

$$V_i(f_i(X) - \pi_i) \leq 0, i = 1, \ldots, n. \quad (2)$$

Condition (1) ensures that the insurer's welfare is no worse off if it trades with all n reinsurers concurrently. Moreover, if V_I is comonotonic additive, then (1) implies $V_I(f_i(X) - \pi_i) \geq 0$ for all $i = 1, \ldots, n,$. This in turn leads to

$$V_I\left(X - \sum_{i=1}^{n}(f_i(X) - \pi_i)\right) \leq V_I(X).$$

Hence, not only there is an incentive for the insurer to trade with all n reinsurers, the above inequality further shows that the insurer's welfare of not ceding its risk cannot be better off.

Condition (2) focuses on the welfare of the reinsurers. Because of the normalization condition $V_i(0) = 0$, condition (2) asserts that the welfare of each reinsurer $i, i = 1, \ldots, n$, for accepting the ceded risk f_i is at least as great as not accepting it. Hence, there is an incentive for the reinsurers to trade with the insurer. The cash-invariance property also implies that $\pi_i \geq V_i(f_i(X))$.

We now discuss the Pareto optimality. A reinsurance contract profile $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is Pareto optimal if there does not exist another reinsurance contract profile $(\hat{f}, \hat{\pi}) \in \mathcal{F}_N \times \mathbb{R}^n$ such that

$$V_I\left(X - \sum_{j=1}^{n}(\hat{f}_j(X) - \hat{\pi}_j)\right) \leq V_I\left(X - \sum_{j=1}^{n}(f_j(X) - \pi_j)\right),$$

$$V_i(\hat{f}_i(X) - \hat{\pi}_i) \leq V_i(f_i(X) - \pi_i), i = 1, \ldots, n,$$

with at least one strict inequality. If a reinsurance contract profile is not Pareto optimal, then all firms will (weakly) benefit from selecting another reinsurance contract profile. Note that Pareto optimality does not imply individual rationality. For example, Pareto optimal reinsurance contracts may include contracts with some negative premiums, but it follows from (2) that such contracts are never individually rational.

The next proposition characterizes Pareto optimality for monetary preferences. It extends Proposition 2.2 of Boonen et al. (2016a) to the case with multiple reinsurers and to the case where the preferences do not need to be comonotonic additive. The proof is similar to that proposition, but for completeness we provide a self-contained proof in Appendix A.

Proposition 3.1 *Let $V_k, k \in \{I, 1, \ldots, n\}$ as in Definition 2.1. Reinsurance contract profile $(f^*, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is Pareto optimal if and only if f^* solves*

$$\min_{f \in \mathcal{F}_N} \left\{ V_I \left(X - \sum_{i=1}^{n} f_i(X) \right) + \sum_{i=1}^{n} V_i(f_i(X)) \right\}. \tag{3}$$

Note that $V_k(X) < \infty$ for all $k \in \{I, 1, \ldots, n\}$. This implies that solutions to (3) are finite.

4 Competitive pricing of reinsurance with distortion risk measures

First, we assume that all firms use specific monetary preferences given by dual utilities as introduced by Yaari (1987). In Section 9, we will extend our analysis to another class of monetary preferences known as the exponential utilities. Maximizing dual utility is equivalent to minimizing a distortion risk measure (Wang et al., 1997). Under the assumption that firms minimize distortion risk measures, we obtain the following definition.

Definition 4.1 *The preference relation V_k for firm $k \in \{I, 1, \ldots, n\}$ is a distortion risk measure when*

$$V_k(Y) = \mathbb{E}^{g_k}[Y] := \int_{-\infty}^{0} [1 - g_k(S_Y(z))]\, dz + \int_{0}^{\infty} g_k(S_Y(z))\, dz, \text{ for all } Y \in L^1, \tag{4}$$

where $S_Y(z) = 1 - F_Y(z)$ is the survival function of stochastic loss Y, and $g_k : [0,1] \to [0,1]$ is a non-decreasing, and left-continuous function such that $g_k(0) = 0$ and $g_k(1) = 1$.

A non-decreasing, and left-continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$ is called a distortion function. When $Y \geq 0$ *a.s.*, equation (4) can be written as

$$\mathbb{E}^{g_k}[Y] = \int_{0}^{\infty} g_k(S_Y(z))\, dz. \tag{5}$$

It is straightforward to see that $g(s) \geq \widetilde{g}(s)$ for all $s \in [0,1]$ implies $\mathbb{E}^{g}[Y] \geq \mathbb{E}^{\widetilde{g}}[Y]$ for all $Y \in L^1$. The Value-at-Risk (VaR) and all coherent risk measures satisfying law-invariance, comonotonic additivity, and a continuity-type property are distortion risk measures (Wang et al., 1997; Artzner et al., 1999). Wang et al. (1997) show that distortion risk measures $\mathbb{E}^g$ satisfy comonotonic additivity.

The optimal indemnity contracts that solve problem (3) with distortion risk measures and in the context of multiple reinsurers are given by the following proportion. Similar results have also been established by Cui et al. (2013) and Assa (2015) but for single reinsurer.[1] For completeness, we provide a self-contained proof in Appendix A.

Proposition 4.2 *Let $V_k, k \in \{I, 1, \dots, n\}$ be distortion risk measures, and define $A = \Big\{z \geq 0 : \min_{1\leq i\leq n}\big\{g_i(S_X(z)) - g_I(S_X(z))\big\} < 0\Big\}$ and $B = \Big\{z \geq 0 : \min_{1\leq i\leq n}\big\{g_i(S_X(z)) - g_I(S_X(z))\big\} = 0\Big\}$. Then, a profile $f^* \in \mathcal{F}_N$ is a solution to (3) if and only if it admits the following representation for $i = 1, \dots, n$:*

$$(f_i^*)'(z) = \begin{cases} \alpha_i(z) & \text{if } z \in A \text{ and } i \in \operatorname{argmin}_{1\leq j\leq n}\big\{g_j(S_X(z)) - g_I(S_X(z))\big\}, \\ \beta_i(z) & \text{if } z \in B \text{ and } i \in \operatorname{argmin}_{1\leq j\leq n}\big\{g_j(S_X(z)) - g_I(S_X(z))\big\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $z \geq 0$ a.s., $f_i^*(0) = 0$, $\alpha_i(z)$ and $\beta_i(z)$ are measurable and $[0,1]$-valued functions such that

$$\sum_{i=1}^n (f_i^*)'(z) = \begin{cases} 1 & \text{if } z \in A, \\ \phi(z) & \text{if } z \in B, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ is a measurable and $[0,1]$-valued function.

The above proposition implies that there exist Pareto optimal contracts, and it characterizes the corresponding indemnity contracts $f_i^*(X), i = 1, \dots, n$. The indemnities solving (3) are given by specific tranching of the insurer's risk, where every tranche is borne by the reinsurer that is endowed with the smallest distortion function on the corresponding quantiles. From (3) and Proposition 4.2, we obtain directly the reinsurance contract profile $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is Pareto optimal if and only if

$$\mathbb{E}^{g_I}\left[X - \sum_{i=1}^n f_i(X)\right] + \sum_{i=1}^n \mathbb{E}^{g_i}[f_i(X)] = \mathbb{E}^h[X], \tag{6}$$

where $h := \min\{g_I, g_1, \dots, g_n\}$.

[1] Cui et al. (2013) and Assa (2015) both study bilateral reinsurance contract design with a given distortion premium principle. Their objective function is however identical to (3), that characterizes Pareto optimality.

One of the objectives of this paper is to characterize the premiums corresponding to the indemnities satisfying (6). To do this, we first establish the bounds on the premiums that are acceptable to the reinsurer. From the bounds, we then argue a plausible premium that is acceptable to the firms in this market.

Suppose reinsurer i underwrites $f_i(X)$, where $f_i \in \mathcal{F}$. The minimum premium that is acceptable to the reinsurer is provided by the indifference premium; i.e. the critical premium such that it is indifferent for the reinsurer to underwrite or not underwrite the risk. By denoting $\underline{\pi}_i(f_i)$ as the minimum premium corresponding to a given indemnity contract $f_i(X)$, then the indifference premium is obtained by changing the inequality in (2) to equality so that

$$\underline{\pi}_i(f_i) := \mathbb{E}^{g_i}[f_i(X)].$$

The determination of the maximum reinsurance premium is considerably more involved. The premium charged by reinsurer i cannot be arbitrary large due to two reasons. First, it cannot exceed the maximum amount that the insurer is willing to pay. Second, there is competition; the presence of competition implies that the insurer can cede part or all of $f_i(X)$ to other reinsurers that offer better competitive pricing. Hence, the premium determined by a reinsurer i must be in such a way that prevents other reinsurers from rationally jointly offering a lower premium for the same risk.

To determine $\overline{\pi}_i(f_i)$, the maximum competitive premium that can be offered by reinsurer i for underwriting $f_i(X)$, let us consider the following feasible set of ceded loss function in the absence of reinsurer i

$$\mathcal{F}_{-i} := \left\{ (\widetilde{f}_j)_{j \neq i} : \widetilde{f}_j \in \mathcal{F}, j \neq i, \sum_{j \neq i} \widetilde{f}_j \in \mathcal{F} \right\},$$

and its Pareto optimality problem:

$$PO_1 := \min_{\widetilde{f} \in \mathcal{F}_{-i}} \left\{ V_I\Big(X - \sum_{j \neq i} \widetilde{f}_j(X)\Big) + \sum_{j \neq i} V_j\left(\widetilde{f}_j(X)\right) \right\}. \tag{7}$$

The above optimization problem is basically (3) except it seeks the Pareto optimal risk sharing among $n-1$ reinsurers by negating reinsurer i. Now let $X_{-i} = X - f_i(X)$ and consider the following optimization problem:

$$PO_2 := \min_{\widetilde{f} \in \mathcal{F}_{-i}} \left\{ V_I\Big(X_{-i} - \sum_{j \neq i} \widetilde{f}_j(X_{-i})\Big) + \sum_{j \neq i} V_j\left(\widetilde{f}_j(X_{-i})\right) \right\}. \tag{8}$$

The above formulation also seeks the Pareto optimal risk sharing among $n-1$ reinsurers without reinsurer i's participation. The key difference is that (7) is concerned with risk sharing X while (8) focuses on risk sharing $X - f_i(X)$; i.e. the residual risk assuming $f_i(X)$ is ceded to reinsurer i. As PO_1 and PO_2 correspond to the least cost of reinsuring the respective risk in the market, their difference represents the incremental cost for reinsuring $f_i(X)$. Hence, if reinsurer i were to be competitive in the market, its pricing on $f_i(X)$ cannot be more than $PO_1 - PO_2$; otherwise the risk $f_i(X)$ that is supposedly ceded to reinsurer i will be rationally jointly shared among the insurer and the remaining $n-1$ reinsurers. Consequently in a competitive market the above argument implies that the maximum competitive premium is provided by

$$\overline{\pi}_i(f_i) = PO_1 - PO_2. \tag{9}$$

The following proposition provides an additional characterization of $\overline{\pi}_i(f_i)$ under the additional assumption that the preference relations are governed by distortion risk measures.

Proposition 4.3 *Let $V_k, k \in \{I, 1, \ldots, n\}$ be distortion risk measures. The solution to* (9) *is given by*

$$\overline{\pi}_i(f_i) = \mathbb{E}^{h_i}[f_i(X)], \tag{10}$$

where $h_i := \min\{g_I, \min_{j\neq i} g_j\}$ and $f_i \in \mathcal{F}$.

Proof By fixing $f \in \mathcal{F}_N$, we have

$$\begin{aligned} PO_1 - PO_2 &= \min_{\widetilde{f}\in\mathcal{F}_{-i}} \left\{ \mathbb{E}^{g_I}\left[X - \sum_{j\neq i} \widetilde{f}_j(X)\right] + \sum_{j\neq i} \mathbb{E}^{g_j}\left[\widetilde{f}_j(X)\right] \right\} \\ &\quad - \min_{\widetilde{f}\in\mathcal{F}_{-i}} \left\{ \mathbb{E}^{g_I}\left[X_{-i} - \sum_{j\neq i} \widetilde{f}_j(X_{-i})\right] + \sum_{j\neq i} \mathbb{E}^{g_j}\left[\widetilde{f}_j(X_{-i})\right] \right\} \\ &= \mathbb{E}^{h_i}[X] - \mathbb{E}^{h_i}[X - f_i(X)] \qquad (11) \\ &= \mathbb{E}^{h_i}[f_i(X)]. \end{aligned}$$

Here, the second equality follows from (6) and the last equality follows from comonotonic additivity of the distortion risk measure $\mathbb{E}^{h_i}$. This concludes the proof. □

Remark 1. Consider that the reinsurer i is offering $f_i(X)$, where $f_i \in \mathcal{F}$, to the insurer. The insurer with distortion risk measure $\mathbb{E}^{g_I}$ values this risk as

$$\mathbb{E}^{g_I}\left[X - \sum_{j\neq i} f_j(X)\right] - \mathbb{E}^{g_I}\left[X - \sum_{j=1}^{n} f_j(X)\right] = \mathbb{E}^{g_I}[f_i(X)]. \tag{12}$$

This follows immediately from the comonotonic additivity of the distortion risk measure, $\sum_{j=1}^n f_j \in \mathcal{F}$ and $f_i \in \mathcal{F}$ for all $i = 1, \dots, n$, and that $X - \sum_{j=1}^n f_j(X)$ and $X - \sum_{j\neq i} f_j(X)$ are comonotonic for any i. Therefore, the functional form of the preferences of the insurer is similar to the ones of the reinsurers.

We now provide an alternate approach of justifying the maximum premium $\overline{\pi}_i(f_i)$. Consider an initial risk allocation $f_j(X)$ to reinsurer j, for $j = 1, \dots, n$. Now fixed i and suppose that $f_i(X)$ is to be re-distributed among the insurer I and the remaining $n-1$ reinsurers. Then, the maximum premium that reinsurer i determines is such that all other reinsurers have no incentive to underwrite a part of the risk $f_i(X)$, nor the insurer wants to keep part of it. This leads to

$$\begin{aligned}
&\min_{\widetilde{f}\in\mathcal{F}_{-i}} \Bigg\{ \left[\mathbb{E}^{g_I}\Big(X - \sum_{j\neq i}[f_j(X) + \widetilde{f}_j(f_i(X))]\Big) - \mathbb{E}^{g_I}\Big(X - \sum_{j=1}^n f_j(X)\Big) \right] \\
&\qquad + \left[\sum_{j\neq i} \mathbb{E}^{g_j}[f_j(X) + \widetilde{f}_j(f_i(X))] - \sum_{j\neq i} \mathbb{E}^{g_j}[f_j(X)] \right] \Bigg\} \\
&= \min_{\widetilde{f}\in\mathcal{F}_{-i}} \left\{ \mathbb{E}^{g_I}\Big(f_i(X) - \sum_{j\neq i} \widetilde{f}_j(f_i(X))\Big) + \sum_{j\neq i} \mathbb{E}^{g_j}[\widetilde{f}_j(f_i(X))] \right\} \\
&= \mathbb{E}^{h_i}[f_i(X)] = \overline{\pi}_i(f_i),
\end{aligned}$$

where the first equality follows from comonotonic additivity of the preferences, and the second equality follows from (6). The difference in the first and second bracket represent the incremental benefit to the insurer and to the remaining $n-1$ reinsurers, respectively. Therefore the sum of these differences captures the incremental benefits to the market from absorbing $f_i(X)$ that is otherwise ceded to reinsurer i. Hence, if reinsurer i were to capture the market share of underwriting $f_i(X)$, its pricing cannot exceed the above maximum incremental benefit (i.e. without reinsurer i's participation). Thus we recover (10) in Proposition 4.3. Note that the above premium upper bound depends on $f \in \mathcal{F}_N$ via f_i. In other words, while we fix f_j for $j \neq i$, the premium bound $\overline{\pi}_i$ does not depend on it.

If reinsurer i sets a premium that is higher than $\overline{\pi}_i(f_i)$ for underwriting $f_i(X)$, then this reinsurer will be phased out by the other reinsurers and insurer that jointly behave rationally. This follows from the above competitive pricing argument. This is also the reason why $\overline{\pi}_i(f_i)$ has been denoted as the maximum premium that can be charged by reinsurer i while still ensuring a "stable" market among the insurer and the n reinsurers. This is a useful characteristic of a reinsurance market and we highlight its importance by formally introducing the following definition of *stability*.

Definition 4.4 *A reinsurance contract profile* $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ *is called stable when* $\pi_i \leq \overline{\pi}_i(f_i)$ *for all* $i = 1, \ldots, n$.

We emphasize that the notion of stability property is attributed to the competition among the multiple reinsurers. The condition $\pi_i \leq \overline{\pi}_i(f_i)$ for all i implies that the pricing offered by all reinsurers are competitive. In the special case with only one reinsurer, the stability property leads to individual rationality.

Due to the fact that $g_I(s) \geq h_i(s)$ for every $s \in [0, 1]$, we have

$$\overline{\pi}_i(f_i) \leq \mathbb{E}^{g_I}[f_i(X)], \tag{13}$$

for every $f_i \in \mathcal{F}$. As $\mathbb{E}^{g_I}[f_i(X)]$ is the insurer's indifference premium, this implies that whenever a reinsurer i charges a weakly smaller premium than $\overline{\pi}_i(f_i)$ for covering $f_i(X)$, there is a welfare gain to the insurer. We will relegate the discussion of the welfare gains among the insurer and the reinsurers to Section 6.

For a given $f_i \in \mathcal{F}$, we get in case of individual rationality and stability that the premium of $f_i(X)$ is in the interval $\left[\mathbb{E}^{g_i}[f_i(X)], \mathbb{E}^{h_i}[f_i(X)]\right]$ whenever $\mathbb{E}^{g_i}[f_i(X)] \leq \mathbb{E}^{h_i}[f_i(X)]$. Zhuang et al. (2016, Lemma 2.1 therein) show that

$$\mathbb{E}^{g}[f_i(X)] = \int_0^\infty g(S_X(z)) f_i'(z)\, dz, \tag{14}$$

for any distortion function g and $f_i \in \mathcal{F}$. Let f^* solve (3), i.e., the reinsurance contract profile is Pareto optimal. It follows from Proposition 4.2 that

$$h_i(S_X(z)) \geq h(S_X(z)) = g_i(S_X(z)),$$

for all $z \geq 0$ such that $(f_i^*)'(z) > 0$. Then, from this and (14), we get

$$\mathbb{E}^{h_i}[f_i^*(X)] \geq \mathbb{E}^{g_i}[f_i^*(X)], \tag{15}$$

i.e., $\overline{\pi}_i(f_i^*) \geq \underline{\pi}_i(f_i^*)$. From (2), (3), Definition 4.4, (13) and (15), we immediately obtain the following result.

Theorem 4.5 *Let* $V_k, k \in \{I, 1, \ldots, n\}$ *be distortion risk measures. A reinsurance contract profile* $(f^*, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ *is Pareto optimal, individually rational and stable if and only if* f^* *solves* (3) *and*

$$\pi_i \in [\underline{\pi}_i(f_i^*), \overline{\pi}_i(f_i^*)], \text{ for all } i = 1, \ldots, n. \tag{16}$$

This theorem characterizes a subset of reinsurance contract profiles that satisfy three properties of a reinsurance market: Pareto optimal, individually rational and stable. The following corollary follows directly from Propositions 4.2 and 4.3.

Corollary 4.6 *Let $V_k, k \in \{I, 1, \ldots, n\}$ be distortion risk measures. If there exists two reinsurers $i \neq j$ with $g_i(S_X(z)) = g_j(S_X(z))$ for all z, then $\underline{\pi}_i(f_i^*) = \overline{\pi}_i(f_i^*)$ and $\underline{\pi}_j(f_j^*) = \overline{\pi}_j(f_j^*)$ where $f^* \in \mathcal{F}_N$ solves* (3).

The above corollary states that if there are two reinsurers with the same preferences, then both reinsurers will make no welfare gain when the reinsurance contracts are Pareto optimal, individual rational and stable. If they provide some reinsurance coverage, they will charge their indifference premiums. However the premiums, in general, are non-uniquely determined by Pareto optimality, individual rationality and stability.

To conclude this section, we introduce another important difference between Boonen et al. (2016a) and the present paper. In the context of an insurer and one reinsurer, Proportion 2.4 of Boonen et al. (2016a) establishes that there a range of premium that leads to Pareto optimal and individually rational. Likewise, in the context of one insurer and n reinsurers, the relation (16) in Theorem 4.5 affirms that there is a range of premium for which a reinsurance contract profile $(f^*, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ can be Pareto optimal, individually rational and stable. However, as opposed to Boonen et al. (2016a) which asserts that the final premium agreed upon by both insurer and reinsurer ultimately depends on their relative bargaining power, in the present paper we argue that there is a vector of premiums for which all firms will agree upon and this does not depend on the relative bargaining powers of the firms. More specifically, under the additional assumption that the reinsurers jointly and rationally maximize their welfare while still mindful of the competition, then each reinsurer i will seek an optimal reinsurance contract profile $(f^*, \pi^*) \in \mathcal{F}_N \times \mathbb{R}^n$ in such a way that f^* solves (3), with the premium π^* attaining its premium upper bound. In other words,

$$\pi_i^* = \overline{\pi}_i(f_i^*) = \mathbb{E}^{h_i}[f_i^*(X)] = \mathbb{E}^{\hat{g}}[f_i^*(X)] \tag{17}$$

for any $i = 1, \ldots, n$. Here, $\hat{g}$ is the second-lowest distortion function of the set of functions $\{g_I, g_1, \ldots, g_n\}$.[2] The last equality follows from (14) and the fact that

$$h_i(S_X(z)) \geq g_i(S_X(z)),$$

for all i, z such that $(f_i^*)'(z) > 0$ (cf. Proposition 4.2). So, reinsurer i prices risk via the second-lowest distortion function, i.e., the premium reinsurer i charges is the maximum premium the insurer would be willing to pay or the minimum premium the insurer would receive from a reinsurer when reinsurer i would not participate.

[2] The second-lowest function $\hat{g}$ of the set of functions $\{g_I, g_1, \ldots, g_n\}$ is defined as follows. For all $s \in [0, 1]$, there exist $i, j \in \{I, 1, \ldots, n\}$ such that $i \neq j$, $\hat{g}(s) = g_j(s)$, $g_i(s) \leq \hat{g}(s)$ and $g_k(s) \geq \hat{g}(s)$ for all $k \neq i, j$.

Hence, the reinsurers price their reinsurance contracts competitively, and the premium function mimics the premium principle of a hypothetical second-best agent. This second-best agent is hypothetical because its distortion premium principle uses the second-lowest distortion function in the market. This premium corresponds with the Nash equilibrium in a Bertrand competition (Bertrand, 1883), and is popular as a method to derive prices in welfare economics.

5 Coalition stability with distortion risk measures

In the preceding section, we define stability as a market phenomenon for which an individual reinsurer will not be phased out immediately by competition. To attain stability, each reinsurer's indemnity contract needs to be priced competitively. This translates into the condition that $\pi_i \leq \overline{\pi}_i(f_i)$ for every reinsurer i; i.e. imposing a competition constraint on the premium of an individual indemnity $f_i \in \mathcal{F}$. While this condition ensures the stability of each reinsurer, it says nothing about the possibility of establishing partnership (or coalition) among reinsurers in such a way that dominates the reinsurance market. By dominating we refer to the situation for which the coalition can offer a joint reinsurance contract that makes all its members and the insurer better off; thus phasing out other reinsurers who are not part of the coalition. If it is not possible to find such a coalition, then the market is said to attain coalition stability; i.e. the market is resilient to a coalition effect. The formal definition of coalition stability is provided below. Note that this property is related to the core in cooperative game theory (see, e.g., Gillies, 1953; Scarf, 1967).

Definition 5.1 *Let $V_k, k \in \{I, 1, \ldots, n\}$ be distortion risk measures. A reinsurance contract profile $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is called coalitional stable if for any subset $S \subseteq \{1, \ldots, n\}$ there does not exist $(\hat{f}, \hat{\pi}) \in \mathcal{F}_S \times \mathbb{R}^s$ such that*

$$\mathbb{E}^{g_I}\Big[X - \sum_{i \in S}(\hat{f}_i(X) - \hat{\pi}_i)\Big] \leq \mathbb{E}^{g_I}\Big[X - \sum_{i=1}^{n}(f_i(X) - \pi_i)\Big], \tag{18}$$

$$\mathbb{E}^{g_i}[\hat{f}_i(X) - \hat{\pi}_i] \leq \mathbb{E}^{g_i}[f_i(X) - \pi_i], i \in S, \tag{19}$$

with at least one strict inequality, where $\mathcal{F}_S = \{(\widetilde{f}_i)_{i \in S} : \widetilde{f}_i \in \mathcal{F}, i \in S, \sum_{i \in S} \widetilde{f}_i \in \mathcal{F}\}$ and $|S| = s$.

Inequality (18) stipulates that the insurer cannot be better off from the reinsurance contract profile provided by any coalition. Similarly, inequality (19) ensures that the welfare of any member of the coalition cannot be better off by trading the coalition's reinsurance contract profile.

Armed with the definition of coalition stability, it is therefore of interest to provide a further analysis on the viability of this property in a given reinsurance market. It turns out that the conditions of Pareto optimality and stability are sufficient to establish coalition stability, as shown in the following proposition.

Proposition 5.2 *Let $V_k, k \in \{I, 1, \dots, n\}$ be distortion risk measures. Then, $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is Pareto optimal and stable if and only if (f, π) is coalitional stable.*

Proof We start with the "only if" part. Let $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ be Pareto optimal and stable. We need to show that there does not exist a $(\hat{f}, \hat{\pi}) \in \mathcal{F}_S \times \mathbb{R}^s$ such that

$$\mathbb{E}^{g_I}\Big[X - \sum_{i \in S}(\hat{f}_i(X) - \hat{\pi}_i)\Big] \leq \mathbb{E}^{g_I}\Big[X - \sum_{i=1}^{n}(f_i(X) - \pi_i)\Big],$$
$$\mathbb{E}^{g_i}[\hat{f}_i(X) - \hat{\pi}_i] \leq \mathbb{E}^{g_i}[f_i(X) - \pi_i], i \in S,$$

for any coalition of reinsurers $S \subseteq \{1, \dots, n\}$, with at least one inequality strict. By the cash-invariance property of $\mathbb{E}^g$, coalition stability holds when the following the inequality holds

$$\mathbb{E}^{g_I}\Big[X - \sum_{i \in S}\hat{f}_i(X)\Big] + \sum_{i \in S}\mathbb{E}^{g_i}[\hat{f}_i(X)] \geq \mathbb{E}^{g_I}\Big[X - \sum_{i=1}^{n}(f_i(X))\Big] + \sum_{i \in S}\mathbb{E}^{g_i}[f_i(X)] + \sum_{i \in N \setminus S}\pi_i, \tag{20}$$

for all $S \subseteq \{1, \dots, n\}$.

Recall that $N = \{1, \dots, n\}$. This implies that (6) can equivalently be expressed as

$$\mathbb{E}^{g_I}\Big[X - \sum_{i \in N} f_i(X)\Big] + \sum_{i \in N}\mathbb{E}^{g_i}[f_i(X)] = \mathbb{E}^h[X].$$

For any $S \subseteq \{1, \dots, n\}$ and let $h_{N \setminus S} := \min\{g_I, \min_{j \notin S} g_j\}$, then we have

$$\mathbb{E}^{g_I}\Big[X - \sum_{i \in S}\hat{f}_i(X)\Big] + \sum_{i \in S}\mathbb{E}^{g_i}[\hat{f}_i(X)] \geq \mathbb{E}^{h_{N \setminus S}}[X].$$

Furthermore, by adding and subtracting $\sum_{i \in N \setminus S}\mathbb{E}^{g_i}[f_i(X)]$ from the right hand side of (20), it is sufficient to show that

$$\mathbb{E}^{h_{N \setminus S}}[X] \geq \mathbb{E}^h[X] - \sum_{i \in N \setminus S}\mathbb{E}^{g_i}[f_i(X)] + \sum_{i \in N \setminus S}\pi_i. \tag{21}$$

Since $h_{N\setminus S}(s) \geq h(s)$ for all $s \in [0,1]$, we can define the measurable and non-negative function

$$\Delta h(s) := h_{N\setminus S}(s) - h(s), s \in [0,1].$$

If $\Delta h(S_X(z)) > 0$, then the maximum of the functions $g_I(S_X(z)), g_1(S_X(z)), \dots, g_n(S_X(z))$ is not obtained by any firm in S nor I. By Proposition 4.2, this implies that $\sum_{i\in N\setminus S} f_i'(z) = 1$ so that $\Delta h(S_X(z)) = \Delta h(S_X(z)) \sum_{i\in N\setminus S} f_i'(z)$ and that

$$\begin{aligned}
\mathbb{E}^{h_{N\setminus S}}[X] - \mathbb{E}^h[X] &= \int_0^\infty h_{N\setminus S}(S_X(z))dz - \int_0^\infty h(S_X(z))dz \\
&= \int_0^\infty \Delta h(S_X(z))dz \\
&= \int_0^\infty \Delta h(S_X(z)) \sum_{i\in N\setminus S} f_i'(z)dz \\
&= \int_0^\infty h_{N\setminus S}(S_X(z)) \sum_{i\in N\setminus S} f_i'(z)dz - \int_0^\infty h(S_X(z)) \sum_{i\in N\setminus S} f_i'(z)dz \\
&= \mathbb{E}^{h_{N\setminus S}}\Big[\sum_{i\in N\setminus S} f_i(X)\Big] - \mathbb{E}^h\Big[\sum_{i\in N\setminus S} f_i(X)\Big] \\
&= \mathbb{E}^{h_{N\setminus S}}\Big[\sum_{i\in N\setminus S} f_i(X)\Big] - \sum_{i\in N\setminus S} \mathbb{E}^{g_i}[f_i(X)],
\end{aligned}$$

where the fifth equality is due to (14). Hence, (21) is equivalent to

$$\sum_{i\in N\setminus S} \pi_i \leq \mathbb{E}^{h_{N\setminus S}}\Big[\sum_{i\in N\setminus S} f_i(X)\Big].$$

It is easy to verify that the function $h_{N\setminus S}$ is a distortion function and hence $\mathbb{E}^{h_{N\setminus S}}$ is comonotonic additive. Then, if $(f,\pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is stable, we have

$$\sum_{i\in N\setminus S} \pi_i \leq \sum_{i\in N\setminus S} \mathbb{E}^{h_i}[f_i(X)] \leq \sum_{i\in N\setminus S} \mathbb{E}^{h_{N\setminus S}}[f_i(X)] = \mathbb{E}^{h_{N\setminus S}}[\sum_{i\in N\setminus S} f_i(X)].$$

where the first inequality follows from Proposition 4.3 and Definition 4.4, the second inequality is due to $h_i(s) \leq h_{N\setminus S}(s)$ for all $s \in [0,1]$ whenever $i \in S$. This concludes the proof of the "only if" part.

We now focus on the "if" part of the proof. Let $(f,\pi) \in \mathcal{F}_N \times \mathbb{R}^n$ be coalitional stable. This immediately implies that it is Pareto optimal (take subcoalition $S = N$ in Definition 5.1). Then,

$$\mathbb{E}^{g_I}\Big[X - \sum_{i\in S} \hat{f}_i(X)\Big] + \sum_{i\in S} \mathbb{E}^{g_i}[\hat{f}_i(X)] \geq \mathbb{E}^{g_I}\Big[X - \sum_{i=1}^{n}(f_i(X))\Big] + \sum_{i\in S} \mathbb{E}^{g_i}[f_i(X)] + \sum_{i\in N\setminus S} \pi_i, \tag{22}$$

for all $S \subseteq \{1, \ldots, n\}$, and all $(\hat{f}, \hat{\pi}) \in \mathcal{F}_S \times \mathbb{R}^s$.

Take $S = N \backslash \{j\}$. Then, from (22), we get

$$\begin{aligned}
\pi_j &\leq \min_{\hat{f} \in \mathcal{F}_S} \Big\{ \mathbb{E}^{g_I}\big[X - \sum_{i \in S} \hat{f}_i(X)\big] + \sum_{i \in S} \mathbb{E}^{g_i}[\hat{f}_i(X)] \Big\} \\
&\quad - \mathbb{E}^{g_I}\Big[X - \sum_{i=1}^{n} (f_i(X))\Big] - \sum_{i \in N \backslash \{j\}} \mathbb{E}^{g_i}[f_i(X)] \\
&= \mathbb{E}^{h_j}[X] - \mathbb{E}^{h}[X] + \mathbb{E}^{g_j}[f_j(X)] \\
&= \mathbb{E}^{h_j}[f_j(X)],
\end{aligned}$$

where the first equality follows from (6). So, (f, π) is stable, and this completes the proof. □

6 Welfare gains in closed form with distortion risk measures

Whenever there is a risk transfer from an insurer to a reinsurer, it is reasonable to assume that at least one party will benefit from the trade. As a result of the trades and the competition among the multiple reinsurers, it is therefore of interest to provide an in-depth understanding on the welfare gain, if any, to the insurer, the reinsurers, as well as the market as a whole. To do this, we first assume that the insurer I is trading with n reinsurers with respective indemnity contracts $f_i \in \mathcal{F}, i = 1, \ldots, n$, and that the insurer's and reinsurers' monetary preference relations are given by the distortion risk measures. With this setup, we then study the hedged benefits, the profit to the insurer, and the profits to the reinsurers. We now explain these concepts in turn.

Recall that if the insurer were to hedge its risk via indemnity contracts $f_i, i = 1, \ldots, n$, then the welfare of the market from the trading is given by

$$\mathbb{E}^{g_I}\left[X - \sum_{j=1}^{n} f_j(X)\right] + \sum_{j=1}^{n} \mathbb{E}^{g_j}[f_j(X)]. \tag{23}$$

What if the insurer decides not to trade with reinsurer i but still trade with the remaining $n-1$ reinsurers using the same indemnity contracts as before? In this case, the welfare of the market becomes

$$\mathbb{E}^{g_I}\left[X - \sum_{j \neq i} f_j(X)\right] + \sum_{j \neq i} \mathbb{E}^{g_j}[f_j(X)]. \tag{24}$$

The difference between these two welfare must be the hedged benefit attributed to the indemnity contract f_i. By using $HB_i(f_i)$ to represent the resulting hedged benefit, we obtain

$$\begin{aligned} HB_i(f_i) &= \mathbb{E}^{g_I}\left[X - \sum_{j\neq i} f_j(X)\right] - \mathbb{E}^{g_I}\left[X - \sum_{j=1}^{n} f_j(X)\right] - \mathbb{E}^{g_i}[f_i(X)] \\ &= \mathbb{E}^{g_I}[f_i(X)] - \mathbb{E}^{g_i}[f_i(X)], \end{aligned} \tag{25}$$

for any f_j, $j \neq i$ such that $f \in \mathcal{F}_N$. Here, the second equality follows from comonotonic additivity of the distortion risk measure $\mathbb{E}^{g_I}$. We emphasize that the monetary amount $HB_i(f_i)$ is the welfare gain (i.e. hedged benefit) that is obtained when the insurer transfers the risk $f_i(X)$ to reinsurer i. Note that HB_i only depends on $f \in \mathcal{F}_N$ via f_i. Furthermore, $HB_i(f_i)$ can be positive, zero, or negative for arbitrary f_i.

By summing up all the hedged benefits among all n reinsurers, we derive the total hedged benefit of the market:

$$\begin{aligned} HB(f) &:= \sum_{i=1}^{n} HB_i(f_i) \\ &= \sum_{i=1}^{n} \left[\mathbb{E}^{g_I}[f_i(X)] - \mathbb{E}^{g_i}[f_i(X)]\right] \\ &= \mathbb{E}^{g_I}\left[\sum_{i=1}^{n} f_i(X)\right] - \sum_{i=1}^{n} \mathbb{E}^{g_i}[f_i(X)], \end{aligned} \tag{26}$$

where $f \in \mathcal{F}_N$. Here, (26) follows from the comonotonic additivity of $\mathbb{E}^{g_I}$. It should be pointed out that (26) can alternatively be derived by subtracting $\mathbb{E}^{g_I}[X]$ from (23). This is not surprising since $\mathbb{E}^{g_I}[X]$ corresponds to the welfare of market when there is no hedging.

Recall that Proposition 3.1 characterizes Pareto optimality as the optimal contracts $f_i^*, i = 1, \ldots, n$ that are the solutions to the optimization problem (3). The characterization of the total hedged benefit (26) provides another (equivalent) formulation of Pareto optimality. More specifically, the Pareto optimal contracts can equivalently be defined as the contracts that solve

$$\max_{f \in \mathcal{F}_N} HB(f). \tag{27}$$

We remark that the optimal contracts $f_i^*, i = 1, \ldots, n$ are similarly given in Proposition 4.2. Furthermore, the optimization problem (27) relies on the comonotonic additivity property of the distortion risk measure $\mathbb{E}^{g_I}$.

Substituting the optimal contracts f^* from Proposition 4.2 into (26) yields

$$\begin{aligned} HB^* := HB(f^*) &= \mathbb{E}^{g_I}[X] - \sum_{i=1}^{n} \mathbb{E}^{g_i}[f_i^*(X)] - \mathbb{E}^{g_I}\left[X - \sum_{i=1}^{n} f_i^*(X)\right] \\ &= \mathbb{E}^{g_I}[X] - \mathbb{E}^{h}[X], \end{aligned} \tag{28}$$

where HB^* corresponds to the maximum in (27). It is easy to see that HB^* can equivalently be represented as $HB^* = \int_0^\infty \Delta g(S_X(z))\,dz$, where $\Delta g(s) := g_I(s) - h(s)$ for all $s \in [0,1]$. Note that $\Delta g(s) \geq 0$, and hence $HB^* \geq 0$.

Next, we address the profit of the reinsurers. It is natural to define the reinsurer's profit corresponding to reinsurance indemnity $f_i \in \mathcal{F}$ as follows:

$$RP_i(f_i) := \overline{\pi}_i(f_i) - \underline{\pi}_i(f_i) = \mathbb{E}^{h_i}[f_i(X)] - \mathbb{E}^{g_i}[f_i(X)], \tag{29}$$

where $i = 1, \dots, n$. The profit of the reinsurer is the difference of the premium charged and the indifference premium. From (14), we obtain

$$RP_i(f_i) = \int_0^\infty \Big[h_i(S_X(z)) - g_i(S_X(z))\Big] f_i'(z)\,dz.$$

From this result it is not difficult to see that

$$\max_{f_i \in \mathcal{F}} RP_i(f_i) = RP_i(f_i^*),$$

for any f^* solving (3). This follows from the fact that the function $h_i(S_X(z))$ is the second-lowest value of the set $\{g_I(S_X(z)), g_1(S_X(z)), \dots, g_n(S_X(z))\}$ for all z such that $(f_i^*)'(z) > 0$ (see Proposition 4.2).[3] We define $RP_i^* = RP_i(f_i^*)$.

Let us define $\tilde{f}_{-i}^*$ as the indemnities for Pareto optimal contracts when only the reinsurers in $N \backslash \{i\}$ are in the market, and

$$HB(\tilde{f}_{-i}^*) := \sum_{j \neq i} HB_j(\tilde{f}_j^*)$$

as the aggregate hedge benefit in the market when the indemnities are given by $\tilde{f}_{-i}^*$. The following proposition demonstrates that RP_i^* has a specific interpretation.

Proposition 6.1 *Let $V_k, k \in \{I, 1, \dots, n\}$ be distortion risk measures. For all $i = 1, \dots, n$, it holds that*

$$RP_i^* = HB^* - HB(\tilde{f}_{-i}^*), \tag{30}$$

where $\tilde{f}_{-i}^* \in \mathcal{F}_{-i}$ solves (3) for the set of reinsurers given by $N \backslash \{i\}$.

[3] Note that if the minimum of $\{g_I(s), g_1(s), \dots, g_n(s)\}$ is attained by two functions, then the second-lowest and minimum function coincide at s.

Proof Let $f^* \in \mathcal{F}_N$ solve (3). From Proposition 4.2, we get that $(f_i^*)'(z) > 0$ implies

$$g_i(S_X(z)) = h(S_X(z)) = h_j(S_X(z))$$

for all $j \neq i$. From this, we get

$$\begin{aligned}
RP_i^* &= \mathbb{E}^{h_i}[f_i^*(X)] - \mathbb{E}^{g_i}[f_i^*(X)] \\
&= \mathbb{E}^{h_i}[f_i^*(X)] + \sum_{j\neq i} \mathbb{E}^{g_j}[f_j^*(X)] - \mathbb{E}^{g_i}[f_i^*(X)] - \sum_{j\neq i} \mathbb{E}^{g_j}[f_j^*(X)] \\
&= \mathbb{E}^{h_i}[f_i^*(X)] + \sum_{j\neq i} \mathbb{E}^{h_i}[f_j^*(X)] - \mathbb{E}^{h}[f_i^*(X)] - \sum_{j\neq i} \mathbb{E}^{h}[f_j^*(X)] \\
&= \mathbb{E}^{h_i}\left[\sum_{i=1}^n f_i^*(X)\right] - \mathbb{E}^{h}\left[\sum_{i=1}^n f_i^*(X)\right] \\
&= \mathbb{E}^{g_I}[X] - \mathbb{E}^{h}[X] - \Big(\mathbb{E}^{g_I}[X] - \mathbb{E}^{h_i}[X]\Big) \\
&= HB^* - HB(\tilde{f}_{-i}^*),
\end{aligned}$$

where the fourth equality follows from comonotonic additivity of $\mathbb{E}^g$ for any given distortion function g, and the fifth equality follows from (14) and

$$h(S_X(z)) = h_i(S_X(z)) = g_I(S_X(z)),$$

for all $z \geq 0$ such that $\sum_{i=1}^n (f_i^*)'(z) < 1$ (see Proposition 4.2), and the last equation follows from (28). This concludes the proof. □

The expression of RP_i^* in Proposition 6.1 provides us with an economic interpretation. It is the aggregate hedge benefit that disappears if reinsurer i decides to leave the market. This is the maximum welfare gain that reinsurer i can claim. We next provide a closed-form expression of the maximum aggregate reinsurer's profit, which is defined by

$$RP^* := \sum_{i=1}^n RP_i^*.$$

Proposition 6.2 *Let $V_k, k \in \{I, 1, \ldots, n\}$ be distortion risk measures. The maximum aggregate reinsurer's profit is given by*

$$RP^* = \mathbb{E}^{\hat{g}}\left[\sum_{i=1}^n f_i^*(X)\right] - \sum_{i=1}^n \mathbb{E}^{g_i}[f_i^*(X)]. \tag{31}$$

Proof The result follows directly from

$$RP^* = \sum_{i=1}^n \mathbb{E}^{h_i}[f_i^*(X)] - \sum_{i=1}^n \mathbb{E}^{g_i}[f_i^*(X)] \tag{32}$$

$$= \sum_{i=1}^n \mathbb{E}^{\hat{g}}[f_i^*(X)] - \sum_{i=1}^n \mathbb{E}^{g_i}[f_i^*(X)] \tag{33}$$

$$= \mathbb{E}^{\hat{g}}\left[\sum_{i=1}^n f_i^*(X)\right] - \sum_{i=1}^n \mathbb{E}^{g_i}[f_i^*(X)], \tag{34}$$

where (33) follows from (17), and (34) follows from comonotonic additivity of the distortion risk measure $\mathbb{E}^{\hat{g}}$. This concludes the proof. □

We now address the profit of the insurer. The insurer's profit can be defined as the difference between the insurer's indifference premium and the premium charged by the reinsurer. Let $IP_i(f_i)$ be the insurer's profit for ceding $f_i(X)$ to reinsurer i. Assuming reinsurer i charges the maximal premium that yields stability and individual rationality, we have

$$IP_i(f_i) = \mathbb{E}^{g_I}[f_i(X)] - \mathbb{E}^{h_i}[f_i(X)].$$

Under the additional assumption that f_i^* solves (3), the above equation becomes

$$IP_i^* := IP_i(f_i^*) = \mathbb{E}^{g_I}[f_i^*(X)] - \mathbb{E}^{\hat{g}}[f_i^*(X)] \tag{35}$$

$$= \int_0^\infty \Big[g_I(S_X(z)) - \hat{g}(S_X(z))\Big](f_i^*)'(z)\,dz \tag{36}$$

$$= HB_i(f_i^*) - RP_i(f_i^*). \tag{37}$$

Here, (36) follows from (14), and (37) follows trivially from the definitions of $HB_i(f_i^*)$ and $RP_i(f_i^*)$. Analogously, the aggregate insurer's profit by trading with all n reinsurers, denoted by IP^*, is easily shown to be

$$IP^* := HB^* - RP^* = \mathbb{E}^{g_I}\left[\sum_{i=1}^n f_i^*(X)\right] - \mathbb{E}^{\hat{g}}\left[\sum_{i=1}^n f_i^*(X)\right]. \tag{38}$$

The following proposition asserts that $IP_i^* \geq 0$, and hence $IP^* \geq 0$.

Proposition 6.3 *Let $V_k, k \in \{I, 1, \ldots, n\}$ be distortion risk measures. We have $IP_i^* \geq 0$ for all $i = 1, \ldots, n$, and hence $IP^* \geq 0$, where $f^* \in \mathcal{F}_N$ solves (3). Moreover, we have $IP^* = 0$ if and only if $\hat{g}(S_X(z)) \geq g_I(S_X(z))$ for all $z \geq 0$ almost everywhere.*

Proof We start by proving the first result. Let $i \in \{1, \dots, n\}$ and $z \geq 0$. From Proposition 4.2, we get that $(f_i^*)'(z) > 0$ implies $g_i(S_X(z)) \leq g_I(S_X(z))$, which in turn leads to

$$\hat{g}(S_X(z)) \leq g_I(S_X(z)).$$

Hence, $IP_i^* \geq 0$, which is an immediate consequence of this and (36).

We continue with proving the second result. If $\hat{g}(S_X(z)) \geq g_I(S_X(z))$ for all $z \geq 0$ almost everywhere, then we get $\mathbb{E}^{\hat{g}}[f(X)] \geq \mathbb{E}^{g_I}[f(X)]$ for any $f \in \mathcal{F}$, and, so,

$$\mathbb{E}^{\hat{g}}\left[\sum_{i=1}^{n} f_i^*(X)\right] \geq \mathbb{E}^{g_I}\left[\sum_{i=1}^{n} f_i^*(X)\right].$$

Hence, $IP^* \leq 0$. Combining this with the first result yields $IP^* = 0$. If $IP^* = 0$, we obtain

$$\mathbb{E}^{\hat{g}}\left[\sum_{i=1}^{n} f_i^*(X)\right] = \mathbb{E}^{g_I}\left[\sum_{i=1}^{n} f_i^*(X)\right].$$

Then, we get $\hat{g}(S_X(z)) = g_I(S_X(z))$ for all $z \geq 0$ such that $\sum_{i=1}^{n}(f_i^*)'(z) > 0$. If $\sum_{i=1}^{n}(f_i^*)'(z) = 0$, then we get from Proposition 4.2 that

$$\hat{g}(S_X(z)) \geq g_I(S_X(z)).$$

Hence, we get $\hat{g}(S_X(z)) \geq g_I(S_X(z))$ for all $z \geq 0$ almost everywhere. This concludes the result. □

We conclude this section by drawing the following two observations:

- The decomposition (37) provides a useful insight to the allocation of hedged benefit. Recall that $HB_i(f_i^*)$ denotes the welfare gain from optimally trading f_i^* between the insurer and reinsurer i. This also represents the maximum hedged benefit can be jointly claimed by both insurer and reinsurer i. Consequently $IP_i(f_i^*)$ and $RP_i(f_i^*)$ capture the welfare gain that is allocated to the insurer and reinsurer i, respectively. Similar interpretation can be applied to the decomposition (38) except at the aggregate level. In aggregate, reinsurer i is allocated a welfare gain of $RP_i(f_i^*), i = 1, \dots, n$ while the insurer is allocated IP^*.

- Since $IP^* \geq 0$, this implies that as the market adds more reinsurers, the welfare gain to the insurer increases. This is to be expected due to the increased competition aggravated by the additional reinsurers.

7 Illustration with the Mean Conditional Value-at-Risk

In this section, we provide some numerical results where we assume that there is one insurer I, and reinsurers 1 and 2. All three firms are risk-neutral, but face costs of holding capital given by $CoC_i(\rho_i(Y) - \mathbb{E}[Y])$ for holding risk $Y \in L^1$, where $i = I, 1, 2$, $CoC_i \in [0,1]$ and $\rho_i := CVaR_{\beta_i}$. Here, $CoC_i \in [0,1]$ represents the relative cost of capital for holding a buffer, and $CVaR_{\beta_i}$ with $\beta_i \in (0,1)$ is the Conditional Value-at-Risk which is the distortion risk measure with distortion function $\breve{g}_i(s) = \min\{\frac{s}{1-\beta_i}, 1\}$ (see Dhaene et al., 2006). The Conditional Value-at-Risk, also called the Expected Shortfall, has received considerable attention after the introduction of the Basel III regulations and the Swiss Solvency Test (see, e.g., Eling et al., 2008; Basel Committee on Banking Supervision, 2012). It is also often used in the literature on optimal reinsurance contract design (see, e.g., Chi and Tan, 2011; Chi, 2012; Asimit et al., 2013a; Cheung and Lo, 2017).

Define $\gamma_i = 1 - CoC_i$. Let the preferences of reinsurer $i \in \{1, 2\}$ be given by a distortion risk measure *Mean Conditional Value-at-Risk* with the following representation:

$$MCVaR_{\beta_i,\gamma_i}(Y) := \gamma_i \mathbb{E}[Y] + (1-\gamma_i) CVaR_{\beta_i}(Y), \text{ for all } Y \in L^1. \tag{39}$$

These preferences are generated by the distortion function

$$g_i(s) = \gamma_i s + (1-\gamma_i)\min\left\{\frac{s}{1-\beta_i}, 1\right\}, \text{ for all } s \in [0,1].$$

We study the distortion function $\min\{g_I, g_1, g_2\}$ that is used in Proposition 4.2. This is a piecewise linear function, where minimum is attained for small s by a firm $i^* \in \text{argmin}\{\frac{\beta_i\gamma_i}{1-\beta_i} : i = I, 1, 2\}$, and the minimum is attained for large s by a firm $j^* \in \text{argmin}\{\gamma_i : i = I, 1, 2\}$, where it is possible that $i^* = j^*$. The only kink of $\min\{g_I, g_1, g_2\}$ is located at

$$s^* = \frac{\gamma_{j^*}}{\gamma_{j^*} + \beta_{i^*}\gamma_{i^*}/(1-\beta_{i^*})}. \tag{40}$$

Consequently $g_{i^*}(s) = \min\{g_I(s), g_1(s), g_2(s)\}$ for all $0 \le s \le s^*$, and $g_{j^*}(s) = \min\{g_I(s), g_1(s), g_2(s)\}$ for all $s^* \le s \le 1$.

Let $X \in L^1$ be a continuous random variable with cumulative distribution function $F_X(\cdot)$. From Proposition 4.2, we derive that $(f^*, \pi) \in \mathcal{F}_{\{1,2\}} \times \mathbb{R}^2$ is Pareto optimal when $f^* \in \mathcal{F}_2$ is given by

$$f_1^*(X) = \begin{cases} (X - F_X^{-1}(s^*))^+ & \text{if } i^* = 1, j^* \neq 1, \\ \min\{X, F_X^{-1}(s^*)\} & \text{if } i^* \neq 1, j^* = 1, \\ X & \text{if } i^* = 1, j^* = 1, \\ 0 & \text{if } i^* \neq 1, j^* \neq 1, \end{cases} \tag{41}$$

$$f_2^*(X) = \begin{cases} (X - F_X^{-1}(s^*))^+ & \text{if } i^* = 2, j^* \neq 2, \\ \min\{X, F_X^{-1}(s^*)\} & \text{if } i^* \neq 2, j^* = 2, \\ X & \text{if } i^* = 2, j^* = 2, \\ 0 & \text{if } i^* \neq 2, j^* \neq 2, \end{cases} \tag{42}$$

where $x^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$, i.e., firm i^* bears the risk $(X - F_X^{-1}(s^*))^+$ and firm j^* bears the risk $\min\{X, F_X^{-1}(s^*)\}$, where firm i^* or j^* may be the insurer that retains the risk.

We conclude this section with the following example.

Example 7.1 Let $X \sim Exp(\mu)$, $\mathbb{E}^{g_I}[Y] = MCVaR_{0.8,0.8}(Y)$, $\mathbb{E}^{g_1}[Y] = MCVaR_{0.5,0.5}(Y)$, and $\mathbb{E}^{g_2}[Y] = MCVaR_{0.2,0.2}(Y)$ for $Y \in L^1$. Since the risk X is exponentially distributed with parameter $\mu > 0$, we have

$$\begin{aligned} & CVaR_{\beta_i}(X) \\ & = \int_0^{F_X^{-1}(\beta_i)} 1 dz + \int_{F_X^{-1}(\beta_i)}^{\infty} \frac{\exp(-\mu z)}{1 - \beta_i} dz = F_X^{-1}(\beta_i) + \frac{\exp(-\mu F_X^{-1}(\beta_i))}{\mu(1 - \beta_i)} && (43) \\ & = F_X^{-1}(\beta_i) + \mathbb{E}[X], && (44) \end{aligned}$$

where (43) follows from (5), and (44) follows from the fact that $F_X^{-1}(\beta_i) = -\ln(1-\beta_i)/\mu$. Note that the quantile $F_X^{-1}(\beta_i)$ is also known as the Value-at-Risk.

The distortion functions g_I, g_1, g_2, and $\hat{g}$ are displayed in Figure 1. From (40), we get $s^* = 0.5$. In this example, the firms i^* and j^* are uniquely determined, and given by $i^* = 2$ and $j^* = I$. Here, $i^* = 2$ because $g_2(s) \leq \min\{g_I(s), g_1(s)\}$ for all $0 \leq s \leq s^*$, and $j^* = I$ because $g_I(s) \leq \min\{g_1(s), g_2(s)\}$ for all $s^* \leq s \leq 1$. The second-lowest function is given by

$$\hat{g}(s) = \begin{cases} 1.5s & \text{if } 0 \leq s \leq 2/7, \\ 0.2 + 0.8s & \text{if } 2/7 < s \leq 0.5, \\ 1.2s & \text{if } 0.5 < s \leq 5/7, \\ 0.5s + 0.5 & \text{if } 5/7 < s \leq 1, \end{cases}$$

which is not concave. We get from (41)-(42) that $(f^*, \pi) \in \mathcal{F}_2 \times \mathbb{R}^2$ is Pareto optimal when $f_1^*(X) = 0$ and $f_2^*(X) = (X - \frac{1}{\mu}\ln(2))^+$. We readily derive

$$\overline{\pi}_2(f_2^*) = \mathbb{E}^{\hat{g}}[f_2^*(X)] = MCVaR_{5/7,0.2}[f_2^*(X)] = (0.6 + 0.2\ln(1.75))/\mu.$$

By individual rationality of the insurer, we get that $\overline{\pi}_1(f_1^*) = 0$. Moreover, we obtain $\underline{\pi}_2(f_2^*) = \mathbb{E}^{g_2}[f_2^*(X)] = MCVaR_{0.2,0.8}[f_2^*(X)] = 0.6/\mu$. So, we readily verify that $\underline{\pi}_2(f_2^*) \leq \overline{\pi}_2(f_2^*)$ and, also, $\overline{\pi}_2(f_2^*) \leq \mathbb{E}^{g_I}[f_2^*(X)] = (0.6 + 0.2\ln(2.5))/\mu$.

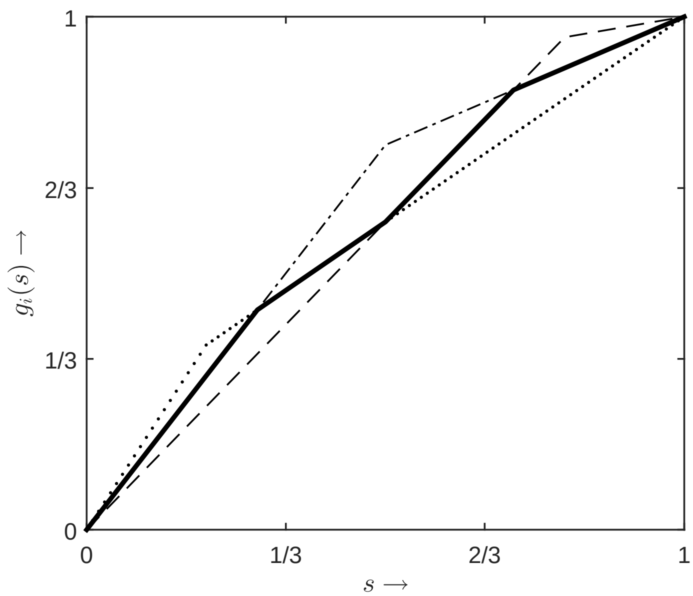

Figure 1: Construction of the function $\hat{g}$ via the distortion functions g_I, g_1 and g_2 corresponding to Example 7.1. The function g_I is the dotted line, g_1 is the dashed-dotted line, g_2 is the dashed line, and $\hat{g}$ is the solid line.

Hence, $RP_1^* = 0$ and $RP^* = RP_2^* = 0.2\ln(1.75)/\mu$ and the welfare gain for the insurer is given by $IP^* = HB^* - RP^* = 0.2(\ln(2.5) - \ln(1.75))/\mu > 0$.

Let $\mu = 1$. Then, we find that $\underline{\pi}_2(f_2^*) \approx 0.6, \overline{\pi}_2(f_2^*) \approx 0.71$ and $\mathbb{E}^{g_I}[f_2^*(X)] \approx 0.78$, so that $IP^* \approx 0.07$ and $RP^* \approx 0.11$. Note that in contrast to the case with $n = 1$, this distortion function does not need to be concave whenever all distortion functions $g_I, g_1, \ldots, g_n$ are concave. So, the premium does not need to increase when the indemnity is increased by mean-preserving spreads. The insurer makes a welfare gain of approximately 0.13, which is significantly larger than the welfare gain of Reinsurer 2, which is approximately 0.06. This difference follows from the competition between the reinsurers, that leads to the insurer's profit $IP^* \approx 0.07$. Hence, even when Reinsurer 1 is not reinsuring any risk, the presence of Reinsurer 1 in the market leads to a significant reduction in the welfare gain of Reinsurer 2.

8 Illustration with the GlueVaR risk measure

In this section, we study a particular choice of a non-coherent distortion risk measure known as the GlueVaR (Belles-Sampera et al., 2014a). Formally GlueVaR is defined by

$$GlueVaR^{h_1,h_2}_{\beta,\alpha}(Y) := \omega_1 CVaR_\beta(Y) + \omega_2 CVaR_\alpha(Y) + \omega_3 F_Y^{-1}(\alpha), \text{ for all } Y \in L^1,$$

where $0 \leq \alpha \leq \beta \leq 1$, $0 \leq h_1 \leq h_2 \leq 1$, $\omega_1 := h_1 - (h_2 - h_1)(1-\beta)/(\beta-\alpha)$, $\omega_2 := (h_2 - h_1)(1-\alpha)/(\beta-\alpha)$, and $\omega_3 := 1 - \omega_1 - \omega_2 = 1 - h_2$. Recall that $F_Y^{-1}(\alpha)$ is also called the Value-at-Risk. The GlueVaR is a distortion risk measure (Belles-Sampera et al., 2014a), with distortion function:

$$g(s) = \begin{cases} \frac{h_1}{1-\beta}s & \text{if } 0 \leq s < 1-\beta, \\ h_1 + \frac{h_2-h_1}{\beta-\alpha}(s-(1-\beta)) & \text{if } 1-\beta \leq s < 1-\alpha, \\ 1 & \text{if } 1-\alpha \leq s \leq 1. \end{cases}$$

The GlueVaR is not coherent as defined in Artzner et al. (1999), since it is not necessarily sub-additive.

As in the previous section, we assume there are two reinsurers: Reinsurers 1 and 2. Moreover, let $\mathbb{E}^{g_I}[Y] = GlueVaR^{h_{1,I},h_{2,I}}_{\beta,\alpha}(Y)$, $\mathbb{E}^{g_1}[Y] = GlueVaR^{h_{1,1},h_{2,1}}_{\beta,\alpha}(Y)$, and $\mathbb{E}^{g_2}[Y] = GlueVaR^{h_{1,2},h_{2,2}}_{\beta,\alpha}(Y)$ for $Y \in L^1$ and a fixed $0 < \alpha < \beta < 1$. A representation of the corresponding distortion functions is displayed in Figure 2.

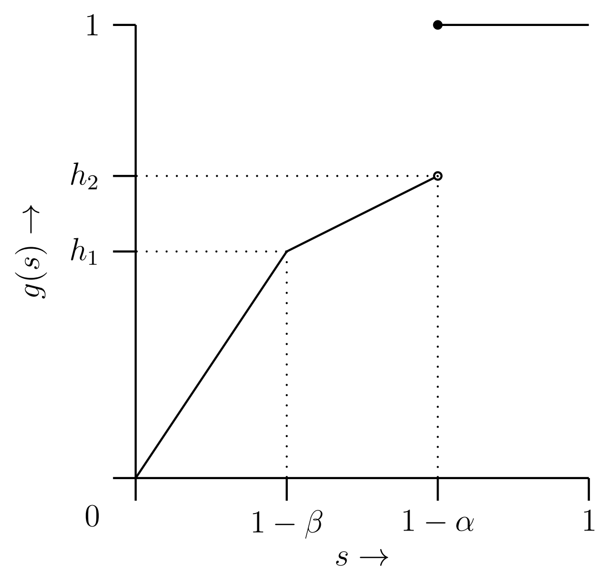

Figure 2: Graphical illustration of the distortion function $g(s)$ of $GlueVaR^{h_1,h_2}_{\beta,\alpha}$.

Note that over the interval $[1-\alpha, 1]$ all distortions are identical. Let $X \in L^1$ again be a continuous random variable with cumulative distribution function $F_X(\cdot)$.

It follows from Proposition 4.2 that the risk $\min\{X, F_X^{-1}(\alpha)\}$ is shared among the three firms in any comonotonic way. Also, $\hat{g}(s) = 1$ for all $s \in [1-\alpha, 1]$. Moreover, for $i^* \in \operatorname{argmin}\{h_{1,i} : i = I, 1, 2\}$, we obtain $g_{i^*}(s) = \min\{g_I(s), g_1(s), g_2(s)\}$ for all $0 \le s \le 1-\beta$. This implies that the risk $(X - F_X^{-1}(\beta))^+$ is allocated to firm i^*. Moreover, $\hat{g}(s) = g_j(s)$ for all $s \in [0, 1-\beta]$, where $j \in \{I, 1, 2\}$ is the firm that yields the second-lowest value of $\{h_{1,i} : i = I, 1, 2\}$. For the risk-layer $(\min\{X, F_X^{-1}(\beta)\} - F_X^{-1}(\alpha))^+$, different indemnity patterns can be optimal.

To provide a more explicit numerical illustration, we now consider the following example.

Example 8.1 In this example, the parameters $h_{1,i}, h_{2,i}$ of the agents are chosen to coincide with the parameter choices in Belles-Sampera et al. (2014b), and are given by $h_{1,I} = 11/30, h_{2,I} = 2/3, h_{1,1} = 0, h_{2,1} = 1, h_{1,2} = 1/20$, and $h_{2,2} = 1/4$. Moreover, we select the parameters $\beta = 2/3$ and $\alpha = 1/3$. The distortion functions g_I, g_1, g_2, and $\hat{g}$ are displayed in Figure 3.

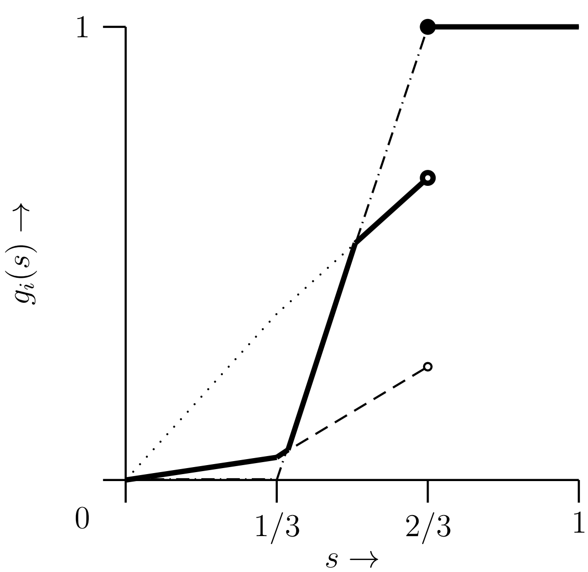

Figure 3: Construction of the function $\hat{g}$ via the distortion functions g_I, g_1 and g_2 corresponding to Example 8.1. The function g_I is the dotted line, g_1 is the dashed-dotted line, g_2 is the dashed line, and $\hat{g}$ is the solid line.

Let $X \sim Exp(1)$. Then, the indemnities $f_1(X) = (X - F_X^{-1}(31/48))^+ = (X - \ln(48/17))^+ \approx (X - 1.04)^+$ and $f_2(X) = (\min\{X, F_X^{-1}(31/48)\} - F_X^{-1}(1/3))^+ = (\min\{X, \ln(48/17)\} - \ln(3/2))^+ \approx (\min\{X, 1.04\} - 0.41)^+$ solve (3), and we assign the risk $\min\{X, \ln(3/2)\} \approx \min\{X, 0.41\}$ to the insurer. We obtain $\underline{\pi}_1(f_1^*) =$

$\mathbb{E}^{g_1}[f_1^*(X)] = GlueVaR^{0,1}_{2/3,1/3}(f_1^*(X)) \approx 0.002$ and $\overline{\pi}_1(f_1^*) = \mathbb{E}^{\hat{g}}[f_1^*(X)]) \approx 0.053$. Moreover, $\underline{\pi}_2(f_2^*) = \mathbb{E}^{g_2}[f_2^*(X)] = GlueVaR^{1/20,1/4}_{2/3,1/3}(f_2^*(X)) \approx 0.093$ and $\overline{\pi}_2(f_2^*) = \mathbb{E}^{\hat{g}}[f_2^*(X)]) \approx 0.262$. So, we readily verify that $\underline{\pi}_i(f_i^*) \leq \overline{\pi}_i(f_i^*)$. We derive $RP_1^* \approx 0.052$, $RP_2^* \approx 0.169$, $RP^* = RP_1^* + RP_2^* = 0.221$. For the insurer, we obtain from (38) that $IP^* = \mathbb{E}^{g_I}[f_1^*(X) + f_2^*(X)] - \mathbb{E}^{\hat{g}}[f_1^*(X) + f_2^*(X)] \approx \mathbb{E}^{g_I}[(X - 0.41)^+] - \mathbb{E}^{\hat{g}}[(X - 0.41)^+] \approx 0.713 - (0.053 + 0.262) = 0.398$.

9 Exponential Utility Framework

In this section, we discuss another class of monetary preferences that is based on the exponential utility. We derive the competitive premium $\bar{\pi}_i(f_i)$ as defined in (9). Interestingly, we find that the premium $\bar{\pi}_i(f_i)$ does not only depend on f_i but also depend on X.

Formally the exponential utility is defined as $U_i(z) = -\gamma_i \exp(-\frac{1}{\gamma_i} z)$, where γ_i captures the risk tolerance of firm i. Then, we have

$$U_i^{-1}(\mathbb{E}[U_i(Y + c)]) = U_i^{-1}(\mathbb{E}[U_i(Y)]) + c,$$

for any constant $c \in \mathbb{R}$. Accordingly, we assume that the preference for firm $k, k \in \{I, 1, 2, \ldots, n\}$, is given by

$$V_k(Y) = U_i^{-1}(\mathbb{E}[U_i(Y)]) = \gamma_k \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma_k} Y\right)\right], \tag{45}$$

which, as defined in Definition 2.1, is a monetary utility function. The risk measure (45) is also known as the entropic risk measure (see Barrieu and El Karoui, 2005). One of the key properties is that the cash-invariance property still holds, i.e. $V_k(Y + c) = V_k(Y) + c$. Note that this preference does not satisfy the comonotonic additivity, but it is additive for *independent* risks, and super-additive for comonotonic risks (Wang and Dhaene, 1998). The comonotonic super-additivity property is defined as follows:

- comonotonic super-additivity: for all $X, Y \in L^1$ that are comonotonic, we have $V_k(X) + V_k(Y) \leq V_k(X + Y)$.

By applying Theorem 3.9 of Barrieu and El Karoui (2005), it can be shown that a solution to (3) is given by

$$f_i^*(X) = \frac{\gamma_i}{\gamma_I + \sum_{j=1}^n \gamma_j} X, \text{ for all } i = 1, 2, \ldots, n.$$

Hence, the Pareto optimal reinsurance contracts are proportional.

Similarly, the reinsurer i will charge a premium that avoids other reinsurers jointly to rationally provide a lower premium for the same risk. Similar to (9), we determine competitive premium of $f_i(X)$ as the minimum value that the insurer and other reinsurers assign jointly to this risk. Then, we get

$$\begin{aligned}
\overline{\pi}_i(f_i) :=& \min_{\widetilde{f}\in\mathcal{F}_{-i}} \left\{ V_I\Big(X - \sum_{j\neq i} \widetilde{f}_j(X)\Big) + \sum_{j\neq i} V_i\left(\widetilde{f}_j(X)\right) \right\} \\
&- \min_{\widetilde{f}\in\mathcal{F}_{-i}} \left\{ V_I\Big(X_{-i} - \sum_{j\neq i} \widetilde{f}_j(X_{-i})\Big) + \sum_{j\neq i} V_i\left(\widetilde{f}_j(X_{-i})\right) \right\} \\
=& V_I\Big(\frac{\gamma_I}{\gamma_I + \sum_{j\neq i}\gamma_j} X\Big) + \sum_{j\neq i} V_i\left(\frac{\gamma_j}{\gamma_I + \sum_{j\neq i}\gamma_j} X\right) \\
&- V_I\Big(\frac{\gamma_I}{\gamma_I + \sum_{j\neq i}\gamma_j}(X - f_i(X))\Big) - \sum_{j\neq i} V_i\left(\frac{\gamma_j}{\gamma_I + \sum_{j\neq i}\gamma_j}(X - f_i(X))\right) \\
=& (\gamma_I + \sum_{j\neq i}\gamma_j)\ln\mathbb{E}\Big[\exp(\frac{1}{\gamma_I + \sum_{j\neq i}\gamma_j} X)\Big] \\
&- (\gamma_I + \sum_{j\neq i}\gamma_j)\ln\mathbb{E}\Big[\exp\big(\frac{1}{\gamma_I + \sum_{j\neq i}\gamma_j}(X - f_i(X))\big)\Big] \\
=& (\gamma_I + \sum_{j\neq i}\gamma_j)\ln\frac{\mathbb{E}\Big[\exp\big(\frac{1}{\gamma_I + \sum_{j\neq i}\gamma_j} X\big)\Big]}{\mathbb{E}\Big[\exp(\frac{1}{\gamma_I + \sum_{j\neq i}\gamma_j}(X - f_i(X)))\Big]}.
\end{aligned}$$

The above result can be simplified as

$$\overline{\pi}_i(f_i) = H_{\alpha_i}(X) - H_{\alpha_i}(X - f_i(X)), \tag{46}$$

where $H_\alpha(X) = \alpha\ln\mathbb{E}[\exp(\frac{1}{\alpha}X)]$ is the exponential premium principle and $\alpha_i := \gamma_I + \sum_{j\neq i}\gamma_j$.

Observe that (46) under the exponential utility framework is isomorphic to (11) under the distortion risk measure framework. As the preference relation in (45) does not satisfy comonotonic additivity, the competitive premium depends not only on $f_i(X)$, but also on X. In (11), the competitive premium in the distortion risk measure framework depends only on $f_i(X)$ by virtue of comonotonic additivity. But due to the comonotonic super-additive property, it follows that $\overline{\pi}_i(f_i) \geq H_{\alpha_i}(f_i(X))$.

10 Conclusion

In practice, insurers can typically reinsure their risk with more than one reinsurer. Optimal indemnities for reinsurance with fixed premium functions have been studied in the literature (Asimit et al., 2013b; Chi and Meng, 2014; Boonen et al., 2016b; Cong and Tan, 2016). This paper studies the case where premiums are not pre-determined via a premium principle, but instead determined via modeling the competition. We assume that all firms minimize distortion risk measures, or maximize exponential utilities. Pareto optimality for insurance contracts leads to a specific structure of the indemnities. When reinsurers are individually rational and there is competitive pricing, we characterize the premiums by taking into account potential competition among the reinsurers. This yields welfare gains for the insurer and the reinsurers. In case of distortion risk measures, we show this welfare gain in closed form. If all reinsurers have similar preferences, competition leads to a large welfare gain for the insurer. This welfare gain is generated by paying relatively low premiums.

As a suggestion for further research, we propose to study an appropriate definition of competition when there are multiple insurers, and their risks have a known multivariate distribution. This problem is mathematically challenging, and akin to the case of background risk for the reinsurers. Background risk is studied by Dana and Scarsini (2007) in case the firms are endowed with expected utilities, but it is to the best of our knowledge not studied for distortion risk measures.

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A Appendix A

In this appendix, we provide the detailed proofs for Proposition 3.1 and Proposition 4.2.

Proof of Proposition 3.1:

First, we prove for "only if" part. We suppose that $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is Pareto optimal, but f is not an element of the set (3). Then, there exists an $\hat{f} \in \mathcal{F}_N$ such that

$$V_I\left(X - \sum_{i=1}^{n} f_i(X)\right) + \sum_{i=1}^{n} V_i(f_i(X)) > V_I\left(X - \sum_{i=1}^{n} \hat{f}_i(X)\right) + \sum_{i=1}^{n} V_i(\hat{f}_i(X)).$$

Define $\hat{\pi}_i := V_i(\hat{f}_i(X)) - V_i(f_i(X) - \pi_i)$ for $i = 1, 2, \cdots, n$. The cash-invariance property of V_i implies that

$$V_i(\hat{f}_i(X) - \hat{\pi}_i) = V_i(f_i(X) - \pi_i).$$

As π and $\hat{\pi}$ will cancel out due to cash-invariance of V_I and V_i, it follows that

$$V_I\left(X - \sum_{i=1}^{n} f_i(X) + \sum_{i=1}^{n} \pi_i\right) > V_I\left(X - \sum_{i=1}^{n} \hat{f}_i(X) + \sum_{i=1}^{n} \hat{\pi}_i\right),$$

since $V_i(\hat{f}_i(X) - \hat{\pi}_i) = V_i(f_i(X) - \pi_i)$ for all $i = 1, \ldots, n$. This is a contradiction with $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ being Pareto optimal and thus f must be an element of the set (3).

Second, we show the "if" part. We suppose that f is an element of the set (3), but $(f, \pi) \in \mathcal{F}_N \times \mathbb{R}^n$ is not Pareto optimal. Then, there exists another reinsurance contract profile $(\hat{f}, \hat{\pi}) \in \mathcal{F}_N \times \mathbb{R}^n$ such that

$$V_I\left(X - \sum_{j=1}^{n}(\hat{f}_j(X) - \hat{\pi}_j)\right) \leq V_I\left(X - \sum_{j=1}^{n}(f_j(X) - \pi_j)\right),$$
$$V_i(\hat{f}_i(X) - \hat{\pi}_i) \leq V_i(f_i(X) - \pi_i), i = 1, \ldots, n,$$

with at least one strict inequality. Thus, we obtain from cash-invariance of V_I and V_i that

$$V_I\left(X - \sum_{i=1}^{n}\hat{f}_i(X)\right) + \sum_{i=1}^{n} V_i(\hat{f}_i(X)) < V_I\left(X - \sum_{i=1}^{n} f_i(X)\right) + \sum_{i=1}^{n} V_i(f_i(X)),$$

which leads to a contradiction with f being an element of the set (3). The proposition is thus proved. □

Proof of Proposition 4.2:

For any $f \in \mathcal{F}_N$, we have

$$\begin{aligned}
&V_I\left(X - \sum_{i=1}^{n} f_i(X)\right) + \sum_{i=1}^{n} V_i(f_i(X)) \\
&= V_I(X) + \sum_{i=1}^{n}\Big(V_i(f_i(X)) - V_I(f_i(X))\Big) \\
&= V_I(X) + \sum_{i=1}^{n}\Big(\mathbb{E}^{g_i}[f_i(X)] - \mathbb{E}^{g_I}[f_i(X)]\Big) \\
&= V_I(X) + \sum_{i=1}^{n}\int_0^{\infty}\Big(g_i(S_X(z)) - g_I(S_X(z)\Big)df_i(z),
\end{aligned}$$

where the last equality is due to the fact that $\mathbb{E}^{g_k}[f_i(X)] = \int_0^{\infty} g_k(S_X(z))df_i(z)$ (see Lemma 2.1 in Zhuang et al. (2016)).

We denote h_i^* as the density of f_i^* for $i = 1, 2, \cdots, n$, satisfying $f_i^*(z) = \int_0^z h_i^*(x)dx$ for all $z \geq 0$. Because $f_i^* \in \mathcal{F}$ for all i and $\sum_{i=1}^n f_i^* \in \mathcal{F}$, we must have $h_i^* \in \mathcal{H}$ for all i and $\sum_{i=1}^n h_i^* \in \mathcal{H}$, where $\mathcal{H} := \big\{h : [0, \infty) \to [0, 1] \mid 0 \leq h(z) \leq 1, a.s.\big\}$. Then, a profile $f^* \in \mathcal{F}_N$ is a solution to (3) if f^* (or equivalently h^*) solves the following optimization problem

$$\begin{aligned}
\min \quad & \int_0^{\infty}\sum_{i=1}^n \Big(g_i(S_X(z)) - g_I(S_X(z)\Big)h_i^*(z)dz \\
\text{s.t.} \quad & h_i^* \in \mathcal{H}, \forall i = 1, \ldots, n, \ \sum_{i=1}^n h_i^* \in \mathcal{H}.
\end{aligned}$$

Notice that, if $z \in A$, we should set $\sum_{i=1}^n h_i^*(z)$ to be 1 and $h_i^*(z) = 0$ for $i \notin \operatorname{argmin}_{1\le j\le n}\left\{g_j(S_X(z)) - g_I(S_X(z))\right\}$. And, if $z \in B$, we can set $\sum_{i=1}^n h_i^*(z)$ to be any value in $[0,1]$, but $h_i^*(z) = 0$ for $i \notin \operatorname{argmin}_{1\le j\le n}\left\{g_j(S_X(z)) - g_I(S_X(z))\right\}$. Finally, if $z \notin A \cup B$, we should set $h_i^*(z)$ to be 0 for all i. Therefore, h_i^* should satisfy

$$h_i^*(z) = \begin{cases} \alpha_i(z) & \text{if } z \in A \textit{ and } i \in \operatorname{argmin}_{1\le j\le n}\left\{g_j(S_X(z)) - g_I(S_X(z))\right\}, \\ \beta_i(z) & \text{if } z \in B \textit{ and } i \in \operatorname{argmin}_{1\le j\le n}\left\{g_j(S_X(z)) - g_I(S_X(z))\right\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $z \ge 0$, where $\alpha_i(z)$ and $\beta_i(z)$ are measurable and $[0,1]$-valued functions such that

$$\sum_{i=1}^n h_i^*(z) = \begin{cases} 1 & \text{if } z \in A, \\ \phi(z) & \text{if } z \in B, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ is a measurable and $[0,1]$-valued function. The proof is thus complete.□