Competitive insurance pricing in a duopoly

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Abstract

A single-period stochastic insurance duopoly is formulated to examine the pre-assignment of roles to the insurance game's players. This paper considers two information structures. In the first structure, one insurer assumes the role of the Stackelberg leader by setting the premium first, while the competitor, acting as the Stackelberg follower, responds after observing the leader's premium. In the second structure, both insurers act as Nash players, setting premiums simultaneously without considering the competitor's premium. This paper shows the existence of Stackelberg and Nash equilibria in these settings and identifies which information structure leads to superior utility when the decision to disclose the premium to the competitor is endogenous. A decision game is developed to determine the conditions under which both insurers prefer sequential over simultaneous premium setting in terms of utility.

Keywords: Risk Management; Game theory; Stackelberg equilibrium; Nash equilibrium; Insurance duopoly.

JEL classification: C72; G22.

1 Introduction

1.1 Game-theoretic models for competitive insurance pricing

Calculating competitive prices is a crucial aspect of the insurance underwriting process. The price includes the actuarial gross premium, which accounts for potential losses, expenses, competitor pricing, and customer behavior. Insurers aim to gain a competitive advantage while simultaneously increasing their revenue and profitability.

This paper proposes a game-theoretic approach to determine competitive premiums, with a particular focus on comparing sequential- and simultaneous-move games among competing insurers in the market. These games are motivated by the observation that insurance prices are not always set simultaneously. Instead, they may be announced sequentially, allowing later premiums to be influenced by earlier premium announcements made by competitors.

A practical example of this dynamic can be observed in the premium announcements for healthcare insurance in the Netherlands. All citizens have the option to switch their healthcare insurer before

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January 1st of the new year, and premiums are typically announced between September and November of the previous year. However, insurers do not announce their premiums on the same day. Some insurers choose to announce early, while others delay their announcements, enabling them to observe competitors' pricing strategies before finalizing their own premiums.

The insurance literature has extensively examined game-theoretic approaches to pricing optimization. Within these contexts, a game consists of a group of insurers and their premium-setting strategies, which are influenced by their information structure. The way insurers compute premiums, whether simultaneously or sequentially, is associated with a game's information structure. The optimal premiums are determined by appropriate equilibrium concepts that differ depending on the insurance game in question. Emms (2012) derives the Nash equilibrium premiums in an n-player, non-cooperative, differential game. Dutang et al. (2013) apply two distinct information structures to a static, non-cooperative game and, accordingly, prove the existence of Nash and Stackelberg equilibrium premiums of the game. Wu and Pantelous (2017) derive the Nash equilibrium premiums in an n-player potential game, while Boonen et al. (2018) determine the open-loop Nash equilibrium premium strategies in an n-player differential game. Asmussen et al. (2019) consider the customer's problem with market frictions and develop a stochastic differential game between two insurance companies of varying sizes. Mourdoukoutas et al. (2021) demonstrate the Nash equilibrium premium profile in a single-period, stochastic game for two distinct exponential demand functions. A stochastic insurance market with multiple insurers competing in various lines of business over a multi-period time horizon is studied by Mourdoukoutas et al. (2025). They characterize open- and closed-loop equilibrium premiums, in which premiums for distinct lines of business under the same policyholder are bundled together and offered at a discount. Mourdoukoutas et al. (2024) introduce uncertainty on the beliefs about the risk aversion of competitors. Specifically, they examine a stochastic, single-period insurance market with incomplete information. The level of insurers' risk aversion is assumed to be private information, which results in premium strategies being defined as mappings between risk-aversion types and premium rates. Optimal premium strategies are defined by the pure-strategy Bayesian Nash equilibrium.

The Stackelberg equilibrium between a central monopolistic insurer and multiple policyholders is studied by Ghossoub and Zhu (2024). Stackelberg games have been utilized in reinsurance contracting problems as well. Cao et al. (2023) study a market of a single insurer and two reinsurers, and resolve the reinsurance contracting problem by applying Stackelberg differential games. Considering two insurers and investment components, Deng et al. (2018) derive the Nash equilibrium reinsurance-investment policies. Bai et al. (2022) study a market with one reinsurer and two insurers, and a financial market with a risk-free and a risky asset. Using a Stackelberg differential subgame and a non-zero-sum stochastic differential subgame, they derive the optimal reinsurance-investment strategies. For modeling the premium competition between insurers, the Stackelberg equilibrium is yet unexplored. This paper aims to fill that gap in the literature.

Most importantly, in all of the existing game-theoretic insurance literature, any information structure is defined *a priori* and the equilibrium results are considered optimal. However, in the present study, this assumption is challenged and tested endogenously. For the first time in the existing literature, we explore two distinct information structures, each corresponding to Stackelberg or Nash equilibria, and develop a decision game to determine when both insurers prefer one information structure over the other in terms of utility.

1.2 Nash versus Stackelberg equilibrium

The Nash and Stackelberg equilibria have been well-studied in industrial organization as concepts to understand the pricing strategies of firms in general production economies. A Nash game is a game where players operate simultaneously without observing their competitors' actions in advance. Conversely, the name Stackelberg refers to a game where players act sequentially. Specifically, certain players disclose their actions to their competitors. The former are referred to as leaders, while the latter are the followers in the context of the Stackelberg game. Furthermore, a Cournot solution is a Nash equilibrium in which the firms compete on quantity, rather than on price as in a so-called Bertrand competition.

Dowrick (1986) constructs a game in which two firms decide their role of leader or follower in a Stackelberg duopoly model. Their decision is solely determined by the slope of the firms' best response function in either the price or quantity space. He finds that both firms prefer the Stackelberg solutions (Stackelberg equilibria) than the Cournot solution (Nash equilibrium), however, there is a conflict over the choice of roles. Albaek (1990) assumes cost uncertainty in a Stackelberg duopoly model where information sharing is prohibited and investigates whether firms agree on a distribution of Stackelberg roles. He defines the *Natural Stackelberg Situation* (NSS) in which one firm prefers, in terms of expected profits, being the leader to being the follower and playing (Bayesian) Nash, while the competitor prefers being the follower to being the leader and playing (Bayesian) Nash. When firms are quantity setters, he establishes the existence of an NSS, in which the firm with the greater cost variance has the role of leader. However, an NSS never exists when prices are the strategic variables. Amir and Grilo (1999) study a Cournot duopoly (i.e., both players are quantity setters). They argue that the players' decision on moving simultaneously or sequentially, as well as the assignment of roles to the players in the latter case, should be endogenous. They consider an extended game of two layers. In the first-layer game, the players decide whether they move early or late. If both players move at the same time (whether early or late), the second-layer game, the basic game, is a simultaneous play, otherwise the second-layer game is a sequential play under perfect information (with leader being the player who decides to move earlier). Under different sets of general conditions on the demand and cost function, they derive all possible timing and equilibrium outcomes of the extended game. Colombo and Labrecciosa (2019) study an infinite-horizon differential oligopoly game by considering multiple Stackelberg leaders and followers at each point in time. They show that the traditional Stackelberg-Cournot welfare ranking (i.e., Stackelberg is more efficient than Cournot) does not necessarily hold in a Markovian environment.

1.3 Our contribution

This paper studies a single-period stochastic insurance duopoly. Insurers are presumed to have a risk-averse attitude and aim to optimize exponential utility functions. The number of policyholders follows a Poisson distribution, with market competition affecting the intensity. We investigate games under two different information structures: the simultaneous-move structure and the sequential-move structure. In the first scenario, both insurers assume the role of a Nash player. In the second scenario, the insurer that establishes the premium initially takes the role of the Stackelberg leader, while the competitor acts as the Stackelberg follower, observing and promptly reacting to the leader's premium. We prove and characterize the Nash and Stackelberg equilibria. Additionally, we prove the existence of the former under the less strict condition of quasi-convexity for the insurers' payoff function. The main feature of our research is considering the allocation of roles to insurers as endogenous. In order to achieve this objective, we provide a more specific definition of a decision game where insurers have two options: either disclosing or not disclosing the premium to their competitor. Any pair of decisions leads to either a Nash or Stackelberg game. Our aim is to provide conditions leading to equilibria in a decision game, defined as the information structure that both insurers prefer in terms of utility.

The insurance duopoly model we use is particularly comparable to the ones described by Dutang et al. (2013) and Mourdoukoutas et al. (2021). Similarly, the collective risk model defines insurers' terminal surplus, which is determined by market premiums. Mourdoukoutas et al. (2021) define insurers' risk aversion using the exponential utility function. The number of policies follows a Poisson and negative binomial distributions, with pricing competition determining the intensity. We differ from both in the following ways: we focus on two insurers in the insurance sector, and the two insurers face an expense rate on incoming revenue. Moreover, we show the existence of Nash equilibrium under more general strategy spaces, which is shown via the quasi-convexity of the log-negative objective functions. Furthermore, we provide an accurate description of the Stackelberg equilibrium and perform a mathematical analysis of the insurers' strategic motivations in the sequential-move game. Dutang et al. (2013) also demonstrate the existence of a Stackelberg equilibrium; however, no strategic interaction between insurers is provided. We consider an insurer's decision to disclose or not disclose premiums to a competitor as endogenous. Insurers' optimal actions are determined by the equilibrium of a decision game.

This paper is organized as follows. Section 2 defines the insurance duopoly and the objective of the insurers, who are considered exponential utility maximizers. Section 3 characterizes the Stackelberg and Nash equilibria, which are used in the decision game in Section 4 to determine the optimal information structure in terms of utility. Section 5 illustrates our results with examples, and Section 6 concludes. All proofs are delegated to Appendix A.

2 Insurance model

Let $\mathcal{I} = \{1, 2\}$ denote the set of two insurers in the market. We focus on insurers' competition for a line of business over a single period of time. Selecting only two insurers may seem restrictive and unrealistic. However, there is evidence from insurance markets, where two insurance companies capture a significant share of the market in particular lines of business. For example, the report conducted by Bhat (2023) includes an overview of motor vehicle and home and contents insurances in Australia. The findings of the report are presented in Table 1 and illustrate the market share in these two lines of business by insurance provider. Thus, it is reasonable for these insurers to optimally respond to their major competitor's underwriting strategy, without being concerned about the smaller market competitors. Moreover, the single-period approach is particularly applicable to general (non-life) insurance, whose products are characterized by short lifespans.

The current number of policyholders $n_{i,0}$ and surplus value $w_{i,0}$ are the result of each Insurer i setting a premium per policy at the beginning of the previous period. When determining premium rates for the next period, all insurers possess this information as common knowledge.

When insurers determine premium rates in the next period, they have two potential information structures to consider. In the first setting, both insurers establish a premium at the beginning of the period, and neither insurer has the ability to anticipate the set premium of their competitor. Thus, we designate that as simultaneous move. An alternative approach is for an insurance company, known as the leader, to disclose the premium rate directly at the start of each period. Having received this value, the other insurer, known as the follower, promptly reacts by determining its premium rate. This phenomenon is commonly known as sequential-move information structure.

Let p_i denote the premium per policy, on the line of business under question, set by Insurer i at the beginning of the period. The insurer's number of policies N_i and generated surplus W_i, both realized at the end of the period, are stochastic and depend on the premium profile $p = (p_i)_{i \in \mathcal{I}}$. Details on how the number of policies underwritten is driven by the pricing competition among the insurers are delegated to Section 2.2. Figure 1 illustrates the market dynamics in our model formulation.

In the following sections, we define the surplus realized by an insurer at the end of the period and explain how the premium strategies determine the number of policies underwritten in the same period. Finally, we assume that the insurers' premium decisions are made based on exponential utility.

2.1 Surplus for insurers

This paper focuses only on the underwriting performance of insurers. Thus, we investigate changes in insurers' surplus attributed to premium income, underwriting expenses and policy claims.

To emphasize Insurer i, we decompose the premium vector p as $p = (p_i, p_{-i})$, where $p_{-i} = p_j$ is the

Australian general insurance market			
Motor vehicle insurance		Home and contents insurance	
Provider	Market share	Provider	Market share
Suncorp Group	18%	Insurance Australia Group Ltd	28.3%
Insurance Australia Group Ltd	16%	Suncorp Group	23.1%
QBE Insurance Group Ltd	7%	QBE Insurance Group Ltd	22.9%
Allianz Australia Ltd	6%	Allianz Australia Ltd	10.3%
Youi Holdings Ltd	4%	Other	15.4%
Open Insurance Ltd	1%		
Other	48%		

Table 1: Market share by company in Motor vehicle and Home and contents insurances in the Australian general insurance market.

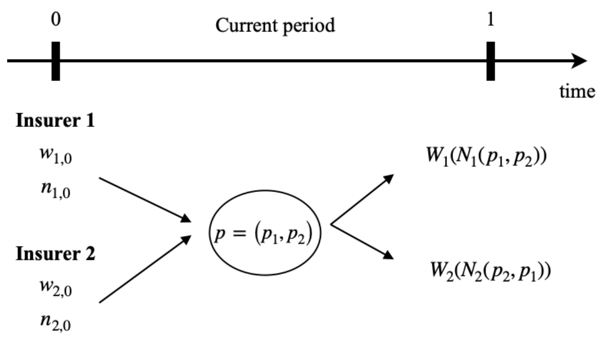


Figure 1: Market dynamics in our model. Information of Insurer $i \in \mathcal{I}$ from the previous period ($w_{i,0}$ and $n_{i,0}$), is known to both insurers. Insurer i sets the premium p_i at the beginning of the period, and the distribution of N_i is affected by the profile $p = (p_i)_{i \in \mathcal{I}}$. The end-of-period surplus W_i depends on N_i.

premium rate of Insurer i's competitor j, i.e., $j \in \mathcal{I}$ and $j \neq i$, and thus $j := 3 - i$. Hereafter, the notations p_{-i} and p_j will be used interchangeably, with the former emphasizing on the fact that the premium refers to Insurer i's competitor. The surplus realized by Insurer i at the end of the period is defined as

$$W_i \left(N_i(p_i, p_{-i})\right) = w_{i,0} + (1 - e_i) \left(p_i N_i(p_i, p_{-i}) - \sum_{k=1}^{N_i(p_i, p_{-i})} X_{i,k} \right), \tag{1}$$

where the parameter $e_i \in [0, 1)$ represents per unit expense of incoming revenue. Moreover, the random variable $X_{i,k}$ denotes the total claims arising from policy k of Insurer i in the current period, and N_i is the random variable that denotes the total number of policies realized by Insurer i at the end of the period. We impose the following two assumptions on these random variables:

(A1) The total claim amounts that arise from distinct policyholders are independent and identically distributed. That is, $(X_{i,k})_k$ are i.i.d. and follow the same distribution as the generic random variable X_i, whose expectation is finite, i.e., $\mathbb{E}[X_i] < +\infty$.

(A2) The total number of policies in the current period and the individual policy claims are independent. That is, $(X_{i,k})_k$ and N_i are independent random variables.

In the realized surplus in Eq. (1), we stress the dependence of N_i on the premium profile p. In the section that follows, we characterize how the differences in insurers' premium rates shape the demand for policies. We consider only a price-driven competition, whereas other factors that might affect insurers' ability to attract or hold clients are disregarded from our analysis.

2.2 Market dynamics

Another assumption stated in the paper is that the insurers are competing for a line of business that displays positive price elasticity of demand. That is, when an insurer charges a higher premium rate than its competitor over a period of time, we expect the insurer to experience a decrease in the number of policies it offers. The extent of the decrease depends not only on the premium differences but also upon the insurer's market power.

Let the number of policies underwritten by Insurer i in the current period follow a Poisson distribution whose intensity depends on the percentage difference in insurers' premium rates and is proportional to the current number of policies of Insurer i. Specifically, we formulate it as

$$N_i(p_i, p_{-i}) \sim \text{Poisson}\left(f_i(p_i, p_{-i}) n_{i,0}\right), \tag{2}$$

where $f_i(p_i, p_{-i})$ is the relative change in Insurer i's expected number of policies in the current period. We define f_i as in Taylor (1986, 1987), namely,

$$f_i(p_i, p_{-i}) = \exp\left(-a_i \frac{p_i - p_{-i}}{p_{-i}}\right). \tag{3}$$

The parameter a_i represents the price sensitivity for Insurer i in the current period and is linked with the point price *elasticity of demand* (EoD) as follows:

$$\text{EoD}(p_i) = -\frac{\partial \ln \mathbb{E}\left[N_i(p_i, p_{-i})\right]}{\partial p_i} p_i = \frac{a_i}{p_{-i}} p_i. \tag{4}$$

Considering positive price elasticity of demand, Eq. (4) implies that $a_i > 0$ for all $i \in \mathcal{I}$. The market power of Insurer i is reflected in a_i. If Insurer i charges r_i% above the competitor's premium, the proportion of policies expected to be gained in the current period is given by

$$\mathbb{E}\left[\frac{N_i(p_i, p_{-i}) - n_{i,0}}{n_{i,0}}\right] = \exp\left(-r_i a_i\right) - 1 < 0. \tag{5}$$

The last equality results from the relations in (2) and (3) by setting $p_i = (1 + r_i)p_{-i}$. For a fixed value of r_i, higher values of the price sensitivity result in maintaining less proportion of the current number of policies. It is reasonable to deduce that an insurer with greater market power than the competitor will be able to hold a higher proportion of the current policyholders for fixed r. Thus, a lower value is attached to the insurer's price sensitivity parameter compared to the competitor's.

From (3), we have that the relative change in Insurer i's expected number of policies in the current period is infinitely differentiable with respect to the insurer's premium. Consistently with the assumption of positive price elasticity of demand, the first partial derivative of f_i with respect to p_i is negative and given by

$$\frac{\partial f_i(p_i, p_{-i})}{\partial p_i} = -\frac{a_i}{p_{-i}} f_i(p_i, p_{-i}). \tag{6}$$

Having constructed a probabilistic model to characterize how the price competition affects the number of policies gained or lost, we can talk about the profit an insurer expects to realize at the end of the period. Particularly, given the distributional form of N_i in (2), the value of surplus that Insurer i expects at the end of the period is equal to

$$\begin{aligned} \mathbb{E}\left[W_i\left(N_i(p_i, p_{-i})\right)\right] &= w_{i,0} + (1 - e_i)\mathbb{E}\left[N_i(p_i, p_{-i})\right](p_i - \mu_{X_i}) \\ &= w_{i,0} + (1 - e_i)\exp\left(-a_i \frac{p_i - p_{-i}}{p_{-i}}\right) n_{i,0}\left(p_i - \mu_{X_i}\right), \end{aligned} \tag{7}$$

where μ_{X_i} is the expectation of the generic random variable X_i that denotes an individual claim amount for Insurer i in the current period.

2.3 Utility for risk-averse insurers

In a risk-neutral framework, an insurer's optimal premium decision would be based on maximising the expected end-of-period surplus given in (7). However, we adopt the perception that insurers display risk aversion towards undertaking the risks associated with policies. Hence, insurers decide premium rates on the basis of a utility function that captures their risk-averse behavior.

Similar to Emms (2012) and Mourdoukoutas et al. (2021), Insurer i's risk aversion is characterized by an exponential utility defined as $U_i(x) = -\exp(-\lambda_i x)$, for $\lambda_i > 0$ and $i \in \mathcal{I}$. Firstly, the choice of exponential utility assists in mathematical tractability. Secondly, it can be argued that the exponential utility possesses the property of *constant absolute risk aversion* (CARA). That is, an insurer's level of risk aversion remains constant regardless of the insurer's surplus level.

In our risk-based approach, insurers are treated as expected utility maximizers. The following proposition describes insurers' objective function when individual claim amounts follow non-heavy-tailed distributions.

Proposition 1 *Assume that the moment generating function (MGF) of X_i evaluated at $\lambda_i(1-e_i)$ exists, namely,*

$$M_{X_i}(t_0) < +\infty, \text{ for } t_0 = \lambda_i(1-e_i). \tag{8}$$

For premium rates that satisfy

$$p_j \geq p_j^L := \frac{1}{\lambda_j(1-e_j)} \ln M_{X_j}\left(\lambda_j(1-e_j)\right), \quad j \in \mathcal{I}, \tag{9}$$

Insurer i's objective function is defined as the expected utility of the end-of-period surplus and given by

$$o_i(p_i, p_{-i}) = -\exp\Big(-\lambda_i w_{i,0} + f_i(p_i, p_{-i}) n_{i,0}\Big(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\Big)\Big). \tag{10}$$

Proof. See Appendix A.1. ■

In the proof of Proposition 1, the lower premium bound p_i^L may look only like a necessary mathematical condition for the existence of the *probability generating function* (PGF) of N_i. In terms of utility, however, it serves as the minimum premium rate required by Insurer i in order to sell insurance. To see that, we notice the inequality in (9) arises from the rational participation argument that $\mathbb{E}\left[U_i\left(w_{i,0} + (1-e_i)(p_i - X_i)\right)\right] \geq \mathbb{E}\left[U_i\left(w_{i,0}\right)\right]$. The lower premium bound p_i^L is known as the indifference premium for Insurer i.

Instead of maximizing $o_i(p_i, p_{-i})$, Insurer i can minimize the logarithm of the (additive) inverse of the objective function. Given the conditions (8) and (9) in Proposition 1, we define the log-negative objective function of Insurer i as follows:

$$c_i(p_i, p_{-i}) = \ln\left(-o_i(p_i, p_{-i})\right) = -\lambda_i w_{i,0} + f_i(p_i, p_{-i}) n_{i,0}\Big(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\Big). \tag{11}$$

Next, we assume that the number of policies is Poisson distributed, as indicated by (2), and that the Assumptions (A1)-(A2) and (8) are always satisfied. The online supplementary appendix provides a unified approach for the policyholders' number. We assume that the number of policyholders belongs to the Panjer $(a, b, 0)$ class, derive the associated log-negative objective function, and connect it to its counterpart in Eq. (11). The relationship between the two log-negative objective functions demonstrates that our results for the Poisson case are also applicable to the Panjer class.

In the following sections, we use game-theoretic arguments to investigate the strategic interaction in a sequential-move and simultaneous-move insurance duopoly, as well as the utility-maximizing information structure preferred by insurers. In that analysis, we use features related to the curvature of an insurer's log-negative objective function. The following two lemmata describe the desired characteristics.

Lemma 1 *Given Insurer j's premium p_j, Insurer i's log-negative objective function is quasi-convex in p_i.*

Lemma 2 *Given Insurer j's premium p_j, Insurer i's log-negative objective function attains a unique global minimum greater than the indifference premium p_i^L. The optimal premium is equal to*

$$p_i^*(p_j) = -\frac{1}{\lambda_i(1-e_i)}\left[\ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right)\right]. \tag{12}$$

Proof. The proof of Lemmata 1 and 2 is provided in Appendix A.2. ■

Given the competitor j's premium, Lemma 2 provides Insurer i's premium rate that minimizes the log-negative objective function. The optimal premium clearly depends on the competitor's premium, and the result of this lemma is used to characterize the follower's best response in Stackelberg and Nash equilibria. Lemma 1 shows the weak condition of quasi-convexity for the log-negative objective function, which is required for the existence of Nash equilibrium.

3 Insurance games with distinct information structures

Initially, we investigate the optimal premium strategies for insurers under two distinct information structures: the sequential-move structure and the simultaneous-move structure. The equilibrium outputs of this section are invoked by Section 4 to support a decision game that investigates in which information structure insurers perform better in terms of utility.

Market regulators impose minimum requirements on insurers' capital to guarantee their solvent operation. At the same time, shareholders demand the profitable management of insurers' portfolios. On top of these restrictions, insurers should consider the competition pressures, i.e., an unreasonable overpricing will result in underperformance in terms of utility. Therefore, the insurers' feasible premium region is reasonably assumed to be a compact set denoted by $\mathcal{P}_i = \left[p_i^L, p_i^U\right]$, for $i \in \mathcal{I}$, where p_i^L is the indifference premium defined in Eq. (9) and p_i^U represents an upper premium bound, which is typically considered to be arbitrarily large.

The behavior of the log-negative objective function can mathematically assuage any doubts about the compactness of the feasible premium region, particularly the upper bound. When the premium constraint in (9) is binding, the term in parenthesis in Eq. (11) becomes zero, and hence, we have $c_i(p_i^L, p_j) = -\lambda_i w_{i,0}$ for all p_j. On the other hand, given the competitor's premium p_j, when Insurer i's premium tends to infinity, the log-negative objective function satisfies $\lim_{p_i \to +\infty} c_i(p_i, p_j) = -\lambda_i w_{i,0}$. To see that, notice $\lim_{p_i \to +\infty} f_i(p_i, p_j) = \lim_{p_i \to +\infty} \exp(-a_i \frac{p_i - p_j}{p_j}) = 0$, and the second exponential in Eq. (11) also tends to zero, that is, $\lim_{p_i \to +\infty} \exp(-\lambda_i(1-e_i)p_i) = 0$. At the same time, any $p_i > p_i^L$ satisfies $c_i(p_i, p_j) < -\lambda_i w_{i,0}$.

In summary, the log-negative objective function of Insurer i attains, or approaches, the negative value of $-\lambda_i w_{i,0}$ when the insurer's premium is equal to p_i^L or close to infinity, whereas it is strictly less than $-\lambda_i w_{i,0}$ at any other intermediate point. Moreover, Lemma 2 demonstrates the uniqueness of a minimum point $p_i^*(p_j)$ greater than p_i^L, given the competitor's premium p_j. The proof of Lemma 1 verifies that $p_i \mapsto c_i(p_i, p_j)$ is decreasing on $[p_i^L, p_i^*(p_j)]$ and increasing for $p_i > p_i^*(p_j)$. In conjunction with the smoothness of $p_i \to c_i(p_i, p_j)$, we anticipate the existence of an upper bound p_i^U sufficiently large such that the turning point $p_i^*(p_j)$ is within the interior of the feasible premium region.

3.1 Sequential-move insurance game: Stackelberg equilibrium

Let $i, j \in \mathcal{I}$ and $i \neq j$. Throughout this section, we assume, without loss of generality, that Insurer i sets the premium rate first, and the competitor Insurer j, having observed Insurer i's action, decides its premium rate subsequently.

In terms of strategies, Insurer i selects a premium rate p_i within $\mathcal{P}_i$. Insurer j's premium strategy is a mapping from $\mathcal{P}_i$ to $\mathcal{P}_j$, denoted by s_j, i.e., $s_j : \mathcal{P}_i \mapsto \mathcal{P}_j$. Let $\mathcal{S}_j = \{s_j | s_j(p_i) \in \mathcal{P}_j \text{ for } p_i \in \mathcal{P}_i\}$ be the strategy space of Insurer j. The definition of Insurer j's strategy is a consequence of the sequential-move information structure, meaning Insurer i's premium choice is observable to Insurer j.

The rationality of both insurers indicates that the follower, Insurer j, will optimally respond to any premium selection by Insurer i. In turn, the leader, Insurer i, chooses the optimal premium anticipating the follower's best response. The next definition presents the equilibrium concept based on these conjectures (we refer interested readers to Fudenberg and Tirole, 1991).

Definition 1 *Let $\mathcal{G}^S = \left\langle \mathcal{I}, \left(\mathcal{P}_i, \mathcal{S}_j\right), (c_k)_{k\in\mathcal{I}} \right\rangle$ denote the insurance game with sequential-move information structure, in which Insurer i is the leader and the competitor Insurer j is the follower. Here, $\mathcal{P}_i$ and $\mathcal{S}_j$ are the strategy spaces of the two insurers, and $(c_k)_{k\in\mathcal{I}}$ are the insurers' log-negative objective functions as defined in Eq. (11). The premium profile $p^S = \left(p_i^S, BR_j\left(p_i^S\right)\right)$ is a Stackelberg equilibrium (SE) if, for all $p_i \in \mathcal{P}_i$, the best response of Insurer j satisfies*

$$BR_j\left(p_i\right) \in \underset{p_j \in \mathcal{P}_j}{\arg\min}\, c_j\left(p_j, p_i\right), \tag{13}$$

and Insurer i's optimal premium rate is given by

$$p_i^S \in \underset{p_i \in \mathcal{P}_i}{\arg\min}\, c_i\left(p_i, BR_j\left(p_i\right)\right). \tag{14}$$

The conditions in (13) and (14) are the two steps in a backward induction argument. At the second step, the follower Insurer j selects a premium that minimizes its log-negative objective function for a given premium selection by the leader. Then, the leader Insurer i minimizes its log-negative objective function, knowing that Insurer j will best respond to any premium p_i. The following proposition provides the characterization of the SE.

Theorem 1 *Let $\mathcal{G}^S = \left\langle \mathcal{I}, \left(\mathcal{P}_i, \mathcal{S}_j\right), (c_k)_{k\in\mathcal{I}} \right\rangle$ be the insurance game with sequential-move information structure as defined in Definition 1. A SE premium profile $p^S = \left(p_i^S, BR_j\left(p_i^S\right)\right)$ exists. Moreover, a SE premium profile $p^S = \left(p_i^S, BR_j\left(p_i^S\right)\right)$ is characterized by*

$$BR_j(p_i) = -\frac{1}{\lambda_j(1-e_j)}\left[\ln\left(\frac{a_j}{a_j + \lambda_j(1-e_j)p_i}\right) - \ln M_{X_j}\left(\lambda_j(1-e_j)\right)\right], \tag{15}$$

and $p_i = p_i^S$ satisfies the first-order condition (FOC)

$$\begin{aligned} a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)} &\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right) \\ &+ \lambda_i(1-e_i)e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) = 0, \end{aligned} \tag{16}$$

if the following condition is satisfied

$$\left.\frac{d^2 f_i(p_i, BR_j(p_i))}{dp_i^2} + 2\frac{d}{dp_i}\left(a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)}\right) f_i(p_i, BR_j(p_i))\right|_{p_i = p_i^S} \geq 0. \tag{17}$$

Here, we use the notation $BR_j'(p_i)$ for $\frac{dBR_j(p_i)}{dp_i}$.

Proof. See Appendix A.3. ■

Theorem 1 identifies the SE premium profile under a *second-order condition* (SOC) in (17). Provided the leader's premium, Lemma 2 assures that the best response of the follower satisfies the SOC of the minimization problem in (13). Regarding the leader, we cannot refer to Lemma 2 for the SOC since the competitor's premium is not fixed but it depends on the leader's premium. Instead, we need the additional condition in (17) which regards the curvature and slope of the relative change in the leader's number of policyholders. Specifically, the second term in condition (17) may be linked with the price elasticity of the leader's demand. Recalling the definition of EoD in (4) we derive

$$\mathrm{EoD}_i(p_i) = -\frac{d \ln f_i(p_i, BR_j(p_i))}{dp_i} p_i = a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)} p_i.$$

Hence, (17) is related to the relative change in the leader's expected number of policies.

In games where a player discloses its action to the competitor, it is important to investigate the player's strategic incentives. In our case, we focus on how deviations in the leader's premium affect the follower's best response and preferences. The following proposition demonstrates the strategic interaction between the leader and the follower insurers.

Proposition 2 *Consider the insurance game* $\mathcal{G}^S = \left\langle \mathcal{I}, (\mathcal{P}_i, \mathcal{S}_j), (c_k)_{k\in\mathcal{I}} \right\rangle$ *and its SE premium profile* $p^S = \left(p_i^S, BR_j\left(p_i^S\right)\right)$ *as characterized in Theorem 1. Then, the follower's best response is increasing with respect to the leader's premium, whereas the follower's log-negative objective function is decreasing in the leader's premium.*

Proof. See Appendix A.4. ■

The result of Proposition 2 is in accordance with our intuition. That is, an aggressive premium strategy from the leader's side (i.e., implementation of low premium rates) yields higher values for the log-negative objective function of the follower. In terms of utility, the follower's performance deteriorates (recall that both insurers minimize the log-negative objective function). At the same time, as long as the leader is decreasing the premium rate, the follower's optimal premium response to the leader is decreasing as well. Interestingly, the follower cannot gain competitive advantage and improve the log-negative objective function despite the premium reduction.

In Eq. (5) we saw that larger values of the price-sensitivity parameter penalize insurers by losing a higher proportion of their current number of policies. In order to offset this loss, they are expected to decrease their premium. Our intuition is justified by the following proposition, which investigates how the market power of insurers, reflected in the price-sensitivity parameter, affects the SE premium profile.

Proposition 3 *Consider the insurance game* $\mathcal{G}^S = \left\langle \mathcal{I}, (\mathcal{P}_i, \mathcal{S}_j), (c_k)_{k\in\mathcal{I}} \right\rangle$ *and its SE premium profile* $p^S = \left(p_i^S, BR_j\left(p_i^S\right)\right)$ *as characterized in Theorem 1. Then, the leader's and follower's price-sensitivity parameter affects* p^S *as follows:*

1. p_i^S *and* $BR_j(p_i^S)$ *are decreasing in the leader's price sensitivity* a_i*;*
2. p_i^S *and* $BR_j(p_i^S)$ *are decreasing in the follower's price sensitivity* a_j *when the price sensitivity of the follower satisfies*

$$a_j > \frac{p_i^S}{BR_j(p_i^S)}. \tag{18}$$

Proof. See Appendix A.5. ■

3.2 Simultaneous-move insurance game: Nash equilibrium

In this section, we assume that both insurers set premium rates simultaneously at the start of the period, with neither insurer able to observe the competitor's choice in advance.

In the simultaneous-move information structure, an insurer cannot condition on the competitor's premium decision since it is not observable. Therefore, the premium strategy of Insurer i, for $i \in \mathcal{I}$, is any premium rate p_i within $\mathcal{P}_i$.

In this framework, an insurer anticipates the competitor to optimally respond to any premium choice, and the competitor also expects an optimal premium response from the insurer. The insurers' conjectures form the so called intersection of best responses which is the equilibrium concept defined next (we refer interested readers to Fudenberg and Tirole, 1991).

Definition 2 *Let $\mathcal{G}^N = \left\langle \mathcal{I}, (\mathcal{P}_i)_{i\in\mathcal{I}}, (c_i)_{i\in\mathcal{I}} \right\rangle$ denote the insurance game with simultaneous-move information structure. Here, $(\mathcal{P}_i)_{i\in\mathcal{I}}$ are the strategy spaces of the two insurers, and $(c_i)_{i\in\mathcal{I}}$ are the insurers' log-negative objective functions as defined in Eq. (11). The premium profile $p^N = \left(p_1^N, p_2^N\right)$ is a (pure-strategy) Nash equilibrium (NE) if, for all $i \in \mathcal{I}$, it holds*

$$p_i^N \in \underset{p_i \in \mathcal{P}_i}{\arg\min}\; c_i\left(p_i, p_{-i}^N\right). \tag{19}$$

Provided that the competitor chooses the NE premium, an insurer cannot deviate from its NE premium and be better off. Our focus is on pure strategies, as random (also called mixed) premium strategies are unrealistic in insurance markets. The following proposition provides the characterization of the NE premium profile in our insurance duopoly.

Theorem 2 *Let $\mathcal{G}^N = \left\langle \mathcal{I}, (\mathcal{P}_i)_{i\in\mathcal{I}}, (c_i)_{i\in\mathcal{I}} \right\rangle$ be the insurance game with simultaneous-move information structure as defined in Definition 2. A NE premium profile $p^N = \left(p_1^N, p_2^N\right)$ exists and, for all $i \in \mathcal{I}$, satisfies the first-order conditions*

$$p_i = -\frac{1}{\lambda_i(1-e_i)}\left[\ln\left(\frac{a_i}{a_i + \lambda_i(1-e_i)p_{-i}^N}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right)\right], \tag{20}$$

where $p_{-i}^N = p_j^N$ for $j \in \mathcal{I}$ and $j \neq i$.

Proof. See Appendix A.6.■

The arguments in this proof can be applied to extend Proposition 22 in Mourdoukoutas et al. (2021) to allow for a large compact strategy space, and show the existence of a NE in an n-insurer game.

Comparing Eqs. (15) and (20), we can see that the NE premium profile is the intersection of insurers' best responses. Considering the simultaneous-move information structure and the single-period horizon, the NE is a reasonable and credible outcome of the insurance game $\mathcal{G}^N$. In the following proposition, we demonstrate our expectation that higher values of the price sensitivity cause insurers to lower premium rates in order to mitigate the penalties of maintaining a smaller number of policyholders.

Proposition 4 *Consider the insurance game $\mathcal{G}^N = \left\langle \mathcal{I}, (\mathcal{P}_i)_{i\in\mathcal{I}}, (c_i)_{i\in\mathcal{I}} \right\rangle$ and its NE premium profile $p^N = \left(p_1^N, p_2^N\right)$ as characterized in Theorem 2. Then, the NE premium of both insurers is decreasing with respect to an insurer's price-sensitivity parameter.*

Proof. See Appendix A.7. ■

4 Decision game for assessing information structures

This section focuses on the most fundamental aspect of a game: the information structure. As we saw in the previous section, insurers' knowledge is predetermined, which determines their premium strategies and, as a result, the game's equilibrium outcome. We do not intend to compare the SE and NE outcomes separately. This comparison is conceptually inappropriate because SE and NE are reasonable outcomes within their own particular information structures. Instead, we view the insurers' decision to reveal or not reveal the premium rate to the competitor as endogenous. Thus, we construct a decision game to investigate the conditions in our insurance duopoly that make one information structure preferable over another. As with any game, we must determine the actions and payoffs for the two insurers.

In this section, we construct a decision game where Insurer i can undertake one of the following actions:

D_i	Insurer i *discloses* the premium rate to the competitor without knowing the competitor's premium,
ND_i	Insurer i does *not disclose* the premium rate and observe instead the competitor's premium.

The two-agent, two-action framework allows us to analyze this decision problem as a matrix-form game. An action profile in the matrix-form game is denoted by a two-element vector whose first component is Insurer i's decision and the second is the decision of Insurer j. The action profiles of the two insurers determine the information structure of the insurance game. Particularly, we have the following four outcomes:

(D_i, D_j)	a simultaneous-move game,
(D_i, ND_j)	a sequential-move game in which Insurer i is the leader and Insurer j the follower,
(ND_i, D_j)	a sequential-move game in which Insurer i is the follower and Insurer j the leader,
(ND_i, ND_j)	there exists no insurance game; no insurer sets a premium while waiting for the competitor's premium.

The payoff function in the current decision game is inherited from the induced insurance game, $\mathcal{G}^S$ or $\mathcal{G}^N$, depending on the resulting information structure. Let $A_i = \{D_i, ND_i\}$ denote the action set of Insurer i. The payoff function of Insurer i is defined on the set of action profiles, $A = A_i \times A_j$, and denoted by π_i. The action profile (D_i, D_j) indicates that both insurers decide the premium rate simultaneously and, hence, the payoff of Insurer i is equal to the NE outcome provided by Theorem 2. We write $\pi_i(D_i, D_j) = c_i(p_i^N, p_j^N)$ and $\pi_j(D_j, D_i) = c_j(p_j^N, p_i^N)$. The decision profile (D_i, ND_j) implies that Insurer i reveals the premium rate to the competitor, while the competitor decides the premium rate after having observed Insurer i's premium. The resulting information structure is a sequential-move game in which Insurer i is the leader and Insurer j is the follower. Therefore, the payoffs of the two insurers are equal to the SE outcome provided by Theorem 1, i.e., $\pi_i(D_i, ND_j) = c_i(p_i^S, BR_j(p_i^S))$ and $\pi_j(ND_j, D_i) = c_j(BR_j(p_i^S), p_i^S)$. Similarly, the profile (ND_i, D_j) yields a sequential-move game in which Insurer i is the follower and Insurer j the leader, and the payoffs are equal to $\pi_i(ND_i, D_j) = c_i(BR_i(p_j^S), p_j^S)$ and $\pi_j(D_j, ND_i) = c_j(p_j^S, BR_i(p_j^S))$.

Lastly, when the outcome is (ND_i, ND_j), neither insurer derives any benefit from this decision. In this situation, no strategic interaction takes place between the insurers, as both refrain from actively setting premium rates within the proposed game-theoretic framework. This does not imply that premiums are not set at all; rather, they are determined without any strategic (i.e., game-theoretic) justification. We assume that such non game-theoretically determined premiums are not preferred by the insurers in terms of their utility. Since the payoff function reflects insurers' preferences, a sufficiently large value of π can be assigned.[1]

The equilibrium depends on which outcome is better, not the payoff levels. We therefore normalize as follows:

$$\pi_i(ND_i, ND_j) = \pi_j(ND_j, ND_i) = 0,$$

meaning that neither firm gains anything from the game if both withhold. Note that the log-negative function ensures the payoff is always negative. The actions and corresponding payoffs of the decision game are summarized in Table 2, where the first entry in each payoff vector corresponds to the "row

[1]Since the payoff is tied to the log-negative objective function, insurers aim to minimize the payoff.

player" i, and the second to the "column player" j.

	D_j	ND_j
D_i	$[c_i(p_i^N, p_j^N), c_j(p_j^N, p_i^N)]$	$[c_i(p_i^S, BR_j(p_i^S)), c_j(BR_j(p_i^S), p_i^S)]$
ND_i	$[c_i(BR_i(p_j^S), p_j^S), c_j(p_j^S, BR_i(p_j^S))]$	$[0, 0]$

Table 2: Decision game in matrix form for the insurance duopoly. Insurer i is the "row player" and Insurer j is the "column player". The actions mean the insurer discloses (D) or not discloses (ND) the premium. The function c is defined in Eq. (11), whereas p^S and p^N represent the SE and NE premium profiles, respectively.

Whether an insurer is willing to disclose the premium is private information to the insurer; the competitor does not know the insurer's intentions in advance. We aim to find the decision profiles from which no insurer has an incentive to deviate. This context leads to the following theorem.

Theorem 3 *Let $\mathcal{G} = \left\langle \mathcal{I}, (A_k)_{k\in\mathcal{I}}, (\pi_k)_{k\in\mathcal{I}} \right\rangle$ denote the decision game in which $\mathcal{I}$ is the set of the two insurers, $(A_k)_{k\in\mathcal{I}}$ are the action sets, and $(\pi_k)_{k\in\mathcal{I}}$ are the payoff functions. Let $(p_k^S, BR_{-k}(p_k^S))$ denote the SE premium profile of a sequential-move game $\mathcal{G}^S$, in which Insurer $k \in \mathcal{I}$ is the leader and its competitor $-k$ the follower, whereas (p_k^N, p_{-k}^N) denotes the NE premium profile of a simultaneous-move game $\mathcal{G}^N$. If, for all $k \in \mathcal{I}$, the following two conditions:*

$$p_k^N \leq p_k^S, \tag{21}$$

$$\frac{BR_k(p_{-k}^S)}{p_{-k}^S} < \frac{p_k^N}{p_{-k}^N}, \tag{22}$$

are satisfied, then the two NE profiles of $\mathcal{G}$ correspond to the sequential-move games, i.e., (D_i, ND_j) and (ND_i, D_j), for $i, j \in \mathcal{I}$.

Proof. See Appendix A.8. ■

Theorem 3 provides conditions in the insurance duopoly under which both insurers would prefer to get involved in a sequential-move game rather than in a simultaneous-move game. However, the theorem is not conclusive regarding which player prefers being the leader and which the follower. In other words, a natural Stackelberg situation, as defined by Albaek (1990), is not guaranteed in our two-insurer market.[2]

The ranking of the insurers' decisions is based on the two conditions of Theorem 3. The implication of the inequality in (22) is that Insurer k, in the follower's role, gains a greater competition advantage compared to participating in a Nash game. Now, the inequality in (21) is a reasonable consequence considering the information structure of Stackelberg and Nash games. That is, the lack of information in the simultaneous-move game leads an insurer to adopt a conservative premium strategy and set a relatively lower premium compared to the premium set when the insurer possesses the leader's role in a Stackelberg game.

Our paper avoids a direct comparison of the two distinct information structures, unlike Albaek (1990), who ranks Stackelberg and Nash equilibrium outcomes by comparing the size of the associated expected profits. What we rank is the insurers' decision to move first by disclosing the premium to the competitor or waiting for the competitor's premium choice. In Amir and Grilo (1999), players choose between an early or late response to their competitor. In our case, the insurers' decision is whether to disclose or not disclose the premium to the competitor, with the time between the insurers' actions being negligible. Figure 2 illustrates the possible insurance games that happen at time zero. The treatment of response time as endogenous should recognize the fact that the early-move insurer is gaining policyholders during the intermediate time.

[2]In a natural Stackelberg situation, a player prefers being the leader over being the follower and to playing Nash, and the competitor prefers being the follower over being the leader and to playing Nash.

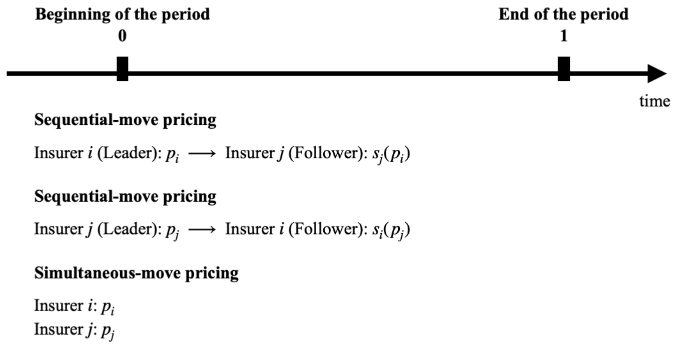


Figure 2: The insurers' decision to disclose or not disclose the premium to the competitor happens at the start of the period. Depending on the pair of decisions, we obtain three potential insurance games.

5 Examples

In this section, we apply our theoretical results to illustrative examples. We study both a symmetric market, where the same set of parameter values characterizes both insurers, and a non-symmetric market, where differences in insurers arise according to their market power.

The individual claim amounts are derived from an exponential distribution with a mean of $\mu_{X_i} = 100$. Regarding the condition in (8), the MGF of X_i exists for all values of λ_i that are less than $\frac{1}{(1-e_i)\mu_{X_i}} = 0.0108$. In the symmetric and non-symmetric markets, we assume that insurers are exposed to the same underlying risk X, face the same rate of underwriting expenses e and display the same risk-aversion level λ. The pricing analysis is presented separately for the symmetric and non-symmetric markets.

5.1 Symmetric insurance duopoly

When insurers charge $r = 20\%$ above the competitor's premium, they are expected to maintain only $R = 70\%$ of their current policyholders. Thus, the insurers' price-sensitivity parameter is derived from Eq. (5) by solving the equation $\exp(-0.2a_i) - 1 = -0.3$, i.e., $\exp(-0.2a_i) = 0.7$. All the model parameters of the symmetric, two-insurer market are summarized in Table 3.

Parameter	Notation	Insurer 1 Insurer 2
Current surplus	$w_{i,0}$	2,000,000
Current number of policies	$n_{i,0}$	2,500
Expense rate	e_i	0.07
Expected individual claim size	μ_{X_i}	100
Risk-aversion parameter	λ_i	0.004
Holdings (%) of previous policies	R_i	70
Price-sensitivity parameter	a_i	1.7834

Table 3: Basic model parameters in the symmetric, two-insurer market.

The symmetry between the two insurers results in identical NE outcomes for both insurers. Likewise, the leader and follower of the Stackelberg games share the same SE outcomes as their counterparts. The equilibrium premiums in the simultaneous-move and sequential-move information structures are characterized by Theorems 2 and 1, respectively. Our numerical results associated with the equilibrium premiums are provided in Table 4. Regarding the characterization of the leader's SE premium, Figure 3 verifies that the SOC, as well as the more restrictive sufficient condition in (17), are satisfied. In particular, the latter is satisfied in the leader's feasible premium region.

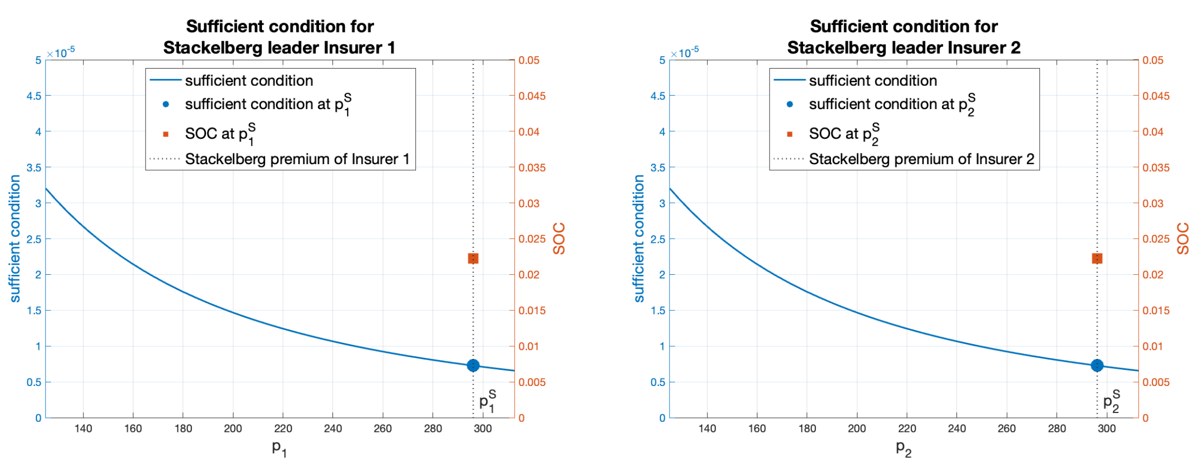
In the Nash game, both insurers expect no changes in the number of policies at the end of the period since the equilibrium premiums are of the same size. However, they generate a considerable profit margin, i.e., the NE premium is well above the indifference premium. Consequently, they expect to realize a 15.19% increase in their surplus.

In the Stackelberg games, the follower sets a premium lower than the leader's, but still greater than the indifference premium. Figure 4 captures this result by illustrating both the best response and the constant premium strategy of the follower across the leader's feasible premium region. Therefore, the follower expects an increase of 28.59% in the policy numbers at the end of the period, whereas the leader is losing 25.38% of policies on average. However, both insurers expect an increase in their surplus, with the greater increase in the follower's surplus.

The insurers' SE premiums are higher compared to the Nash game. Interestingly, both insurers' surplus growth is larger in a sequential-move information structure than in a simultaneous-move information structure.

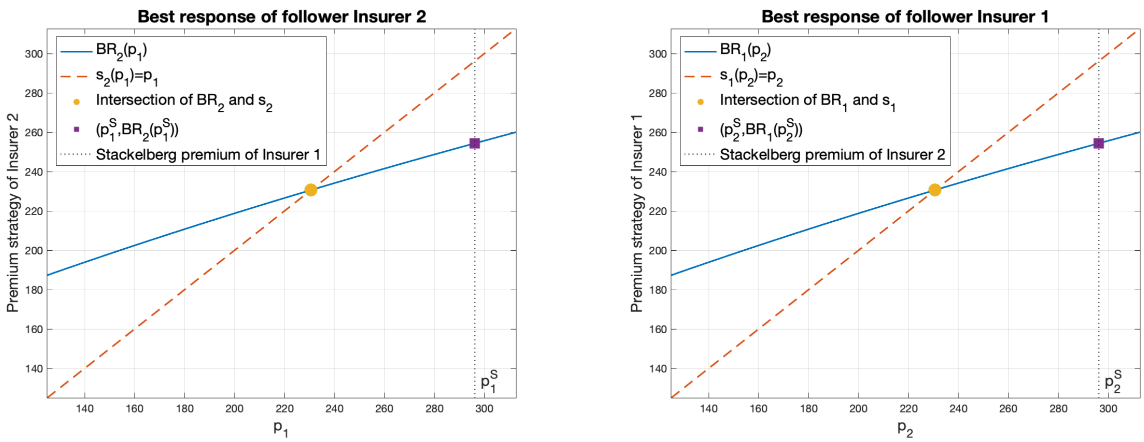
	Nash game		Stackelberg game		Stackelberg game	
	Insurer 1	Insurer 2	Insurer 1**(L)**	Insurer 2	Insurer 1	Insurer 2**(L)**
Indifference premiums	125.06	125.06	125.06	125.06	125.06	125.06
Equilibrium premiums	230.65	230.65	296.11	254.36	254.36	296.11
Expected policies	2,500	2,500	1,865	3,215	3,215	1,865
Expected difference in policies (%)	0	0	−25.38	28.59	28.59	−25.38
Expected surplus	2,303,757	2,303,757	2,340,235	2,461,494	2,461,494	2,340,235
Expected difference in surplus (%)	15.19	15.19	17.01	23.07	23.07	17.01
Log-negative objective at equilibrium	-8,812	-8,812	-8,878	-9,228	-9,228	-8,878

Table 4: Equilibrium outcomes in the symmetric, two-insurer market. In the Nash game, both insurers choose premium simultaneously. In a Stackelberg game, the leader, indicated by (L), is the insurer who discloses the premium to the competitor and the competitor responds to it. The differences are calculated as [(equilibrium value)-(initial value)]/(initial value).



(a) **(b)**

Figure 3: Conditions for the leader's SE premium in the symmetric insurance duopoly. The leader of the Stackelberg game is Insurer 1 in Fig. (a) and Insurer 2 in Fig. (b). The left y-axis depicts the sufficient condition in (17) over the leader's feasible premium region. The right y-axis depicts the value of the leader's second-order condition. The blue-circle and orange-square marker denote the sufficient condition in (17) and the SOC, respectively, evaluated at the leader's SE premium.



(a) **(b)**

Figure 4: Premium strategies for the follower Insurer 2, Fig. (a), and the follower Insurer 1, Fig. (b), in the symmetric insurance duopoly. The blue-solid line depicts the best response, whereas the red-dashed line depicts the constant premium strategy. The circle marker denotes the intersection of the two strategies. The square marker denotes the best response of the follower to the leader's SE premium.

The matrix-form game in Table 5 is obtained by substituting the log-negative objective functions at equilibrium points given in Table 4, into Table 2. In each cell of Table 5, the first element of the payoff vector is associated with Insurer 1, and the second element is associated with Insurer 2. The payoff profile at (D_1, D_2) is derived from the Nash game, the profile at (D_1, ND_2) is from the Stackelberg game with Insurer 1 as the leader, and the last non-zero payoff profile at (ND_1, D_2) is from the Stackelberg game with Insurer 2 as the leader.

Table 4 allows us to easily verify that the conditions in (21) and (22) are satisfied for both insurers. Thus, Theorem 3 guarantees that the two NE decision profiles are (D_1, ND_2) and (ND_1, D_2), which result in the two Stackelberg games.[3]

It is unsurprising that both insurers prefer a game with a sequential-move information structure. Besides Theorem 3, which proves this result in terms of utility, Table 4 shows that both the leader and the follower in a Stackelberg game expect a greater increase in surplus compared to the Nash game. However, we cannot be conclusive about the existence of an insurer who prefers being the leader to being the follower and to playing Nash, and the competitor who prefers being the follower to being the leader and to playing Nash, i.e., the existence of a natural Stackelberg situation as defined by Albaek (1990). In our case, there is no agreement on the assignment of roles since both insurers perform better as the follower than as the leader.

	D_2	ND_2
D_1	$(-8812, -8812)$	$(-8878, -9228)$
ND_1	$(-9228, -8878)$	$(0, 0)$

Table 5: Decision game in matrix form with payoff vectors for the symmetric market. In the payoff vectors, the first element is associated with Insurer 1 and the second element with Insurer 2.

[3] We can derive the same conclusion only by looking at the payoffs in Table 5. Specifically, if Insurer 2 chooses D_2, the best response from Insurer 1 is ND_1. Subsequently, the best response from Insurer 2 to ND_1 is D_2. Thus, (ND_1, D_2) is an equilibrium profile. A similar argument applies for the equilibrium profile (D_1, ND_2).

5.2 Non-symmetric insurance duopoly

We turn our attention to the non-symmetric case for the two-insurer market. Based on insurers' initial size of surplus and number of policies, we assume Insurer 1 possesses greater market power than the competitor. The market power of insurers is reflected in the price-sensitivity parameter and their capacity to preserve policyholders. Specifically, when insurers charge $r = 20\%$ above the competitor's premium, Insurer 1 expects to maintain $R_1 = 70\%$ of the current policyholders, whereas Insurer 2 expects to maintain $R_2 = 67\%$. Thus, the insurers' price-sensitivity parameter is derived from Eq. (5) by solving the equation $\exp(-0.2a_i) = R_i$. All the model parameters of the non-symmetric insurance duopoly are summarized in Table 6.

Parameter	Notation	Insurer 1	Insurer 2
Current surplus	$w_{i,0}$	2,000,000	1,300,000
Current number of policies	$n_{i,0}$	2,500	1,800
Expense rate	e_i	0.07	0.07
Expected individual claim size	μ_{X_i}	100	100
Risk-aversion parameter	λ_i	0.004	0.004
Holdings (%) of previous policies	R_i	70	67
Price-sensitivity parameter	a_i	1.7834	2.0024

Table 6: Basic model parameters in the non-symmetric, two-insurer market.

Theorems 2 and 1 provide the equilibrium premium profiles in the Nash and Stackelberg games, respectively. Table 7 presents the NE and SE outcomes of the non-symmetric, two-insurer market. Regarding the leader's SE premium, Figure 5 verifies that the second-order condition as well as the more restrictive sufficient condition in (17) are satisfied. Particularly, the latter is satisfied on the leader's feasible premium region.

The lower the price-sensitivity parameter, the higher the tolerance of an insurer to premium increases. Thus, Insurer 1, in the simultaneous-move game, charges a higher premium than the competitor. As a consequence, the expected number of Insurer 1's policies at the end of the period shows a decrease of approximately 5.5%. Insurer 2, being the cheaper provider, grows the initial policy numbers by 6.33%. Since both insurers' NE premiums are above the indifference premiums, they realize a growth in the surplus, with Insurer 2's increase being about 2.5% higher than Insurer 1's.

As the Stackelberg leader, Insurer 1's premium exceeds the follower's by 46.21. Insurer 1's conservative premium strategy results in an expected loss of 29.11% of the current policyholders. On the other hand, the follower, Insurer 2, expects an increase of 38.24% in the number of policies at the end of the period. Since both insurers' SE premium is above the indifference premium, they expect to realize a growth in the surplus. Due to the sharp difference in the expected number of policies, Insurer 2's growth in surplus surpasses the competitor's by 9.53%.

A similar pattern in the SE outcomes is observed when Insurer 2 takes over the role of the leader. That is, the follower insurer sets a lower premium than the leader in order to gain competition advantage.This translates to a greater increase in surplus for the follower compared to the leader. For both Stackelberg games, Figure 6 illustrates the best response and constant premium strategy of the follower over the leader's feasible premium region. We notice that the follower, in the SE, is cheaper than the leader.

The difference in the insurers' price-sensitivity parameters affects the SE outcomes. Particularly, the leader Insurer 2's premium is less than the leader Insurer 1's premium, whereas the follower Insurer 1's premium is larger than the follower Insurer 2's. As a result, the leader Insurer 2 is expecting to lose a smaller proportion of policies, whereas the follower Insurer 1 to gain a smaller proportion of policies, compared to their counterparts of the Stackelberg game with Insurer 1 at the leader's role. Thus, the growth in surplus is greater for the leader Insurer 2 compared to the leader Insurer 1, whereas it is lower for the follower Insurer 1 than the follower Insurer 2.

It is worth mentioning that both SE premium profiles are well above the NE premium, and the insurers' performance, in terms of surplus growth, is better in the sequential-move games compared to the simultaneous-move game.

	Nash game		Stackelberg game		Stackelberg game	
	Insurer 1	Insurer 2	Insurer 1(**L**)	Insurer 2	Insurer 1	Insurer 2(**L**)
Indifference premiums	125.06	125.06	125.06	125.06	125.06	125.06
Equilibrium premiums	226.36	219.42	285.73	239.52	247.91	277.75
Expected policies	2,363	1,914	1,772	2,488	3,028	1,415
Expected difference in policies (%)	−5.49	6.33	−29.11	38.24	21.12	−21.42
Expected surplus	2,277,678	1,512,574	2,306,118	1,622,878	2,416,519	1,533,834
Expected difference in surplus (%)	13.88	16.35	15.31	24.84	20.83	17.99
Log-negative objective at equilibrium	-8,742	-5,767	-8,797	-6,063	-9,111	-5,813

Table 7: Equilibrium outcomes in the non-symmetric, two-insurer market. In the Nash game, both insurers choose premium simultaneously. In a Stackelberg game, the leader, indicated by (L), is the insurer who discloses the premium to the competitor and the competitor responds to it. The differences are calculated as [(equilibrium value)-(initial value)]/(initial value).

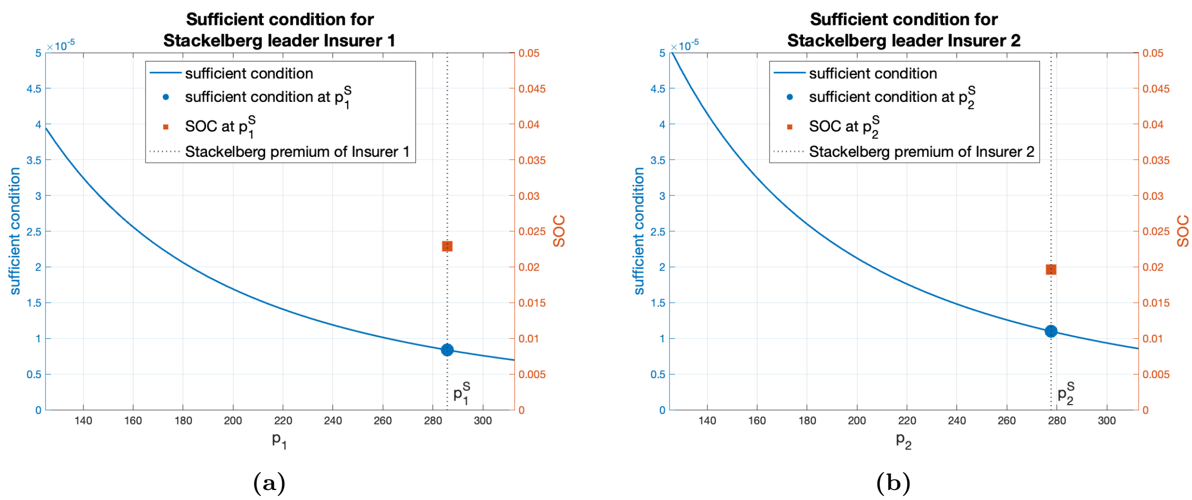
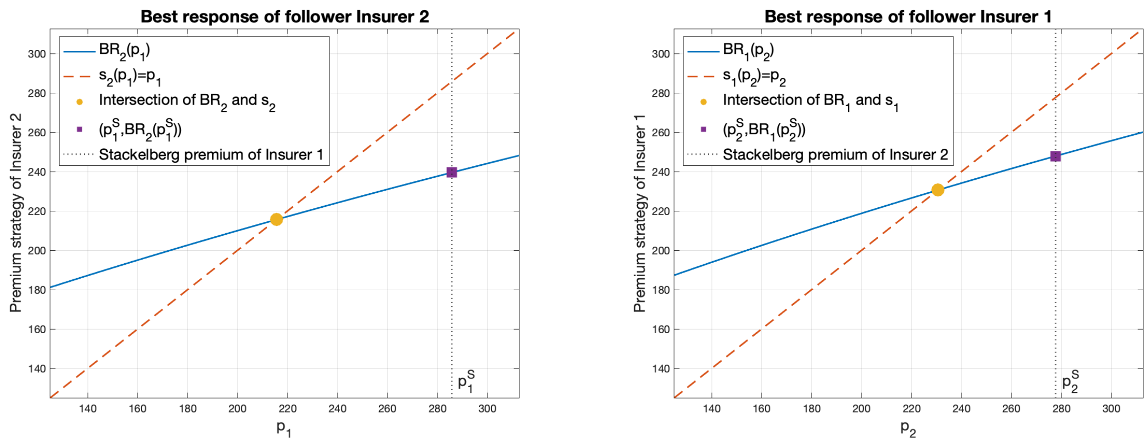


Figure 5: Conditions for the leader's SE premium in the non-symmetric insurance duopoly. The leader of the Stackelberg game is Insurer 1 in Fig. (a) and Insurer 2 in Fig. (b). The left y-axis depicts the sufficient condition in (17) over the leader's feasible premium region. The right y-axis depicts the value of the leader's second-order condition. The blue-circle and orange-square marker denote the sufficient condition in (17) and the SOC, respectively, evaluated at the leader's SE premium.

Similar to the symmetric case, we use the equilibrium values of the log-negative objective functions in Table 7 to construct the matrix-form decision game presented in Table 8. An inspection of the payoffs in the table leads to the NE decision profiles of (D_1, ND_2) and (ND_1, D_2).[4] This is an anticipated result since the equilibrium outcomes in Table 7 satisfy the two conditions, (22) and (21), in Theorem 3. The

[4]We observe that Insurer 1's best response to D_2 is ND_1, and Insurer 2's best response to ND_1 is D_2. Now, Insurer 1's best response to ND_2 is D_1, and Insurer 2's best response to D_1 is ND_2. Thus, the intersection of best responses are the decision profiles (D_1, ND_2) and (ND_1, D_2).



(a) **(b)**

Figure 6: Premium strategies for the follower Insurer 2, Fig. (a), and the follower Insurer 1, Fig. (b), in the non-symmetric insurance duopoly. The blue-solid line depicts the best response, whereas the red-dashed line depicts the constant premium strategy. The circle marker denotes the intersection of the two strategies. The square marker denotes the best response of the follower to the leader's SE premium.

fact that both insurers prefer a sequential-move information structure can also be justified by the growth in their surplus size. We see in Table 7 that both insurers, whether as leader or follower, are expected to realize a larger increase in surplus compared to the Nash game. However, a natural Stackelberg situation is absent in the non-symmetric insurance duopoly since both insurers perform better as followers than as leaders.

	D_2	ND_2
D_1	$(-8742, -5767)$	$(-8797, -6063)$
ND_1	$(-9111, -5813)$	$(0, 0)$

Table 8: Decision game in matrix form with payoff vectors for the non-symmetric, two-insurer market. In the payoff vectors, the first element is associated with Insurer 1 and the second element with Insurer 2.

5.3 Parameter-sensitivity analysis

This section investigates trends in the equilibrium results with respect to model parameters. Each time, we vary only one parameter while keeping the rest at the benchmark values presented in Table 3 or Table 6, depending on the market's symmetry.

Our focus is on changes in the equilibrium premiums and expected surplus as the price-sensitivity and risk-aversion parameters of Insurer 1 vary within an interval. The choice of Insurer 1 is irrelevant for the symmetric insurance duopoly. However, we use Insurer 1's parameters due to the insurer's market dominance in the non-symmetric case. We segment our analysis based on the symmetry between the two insurers. Specifically, Figure 7 pertains to the symmetric insurance game, whereas Figure 8 pertains to the non-symmetric insurance game.

Each diagram in Figures 7 and 8 consists of two y-axes. The left y-axis depicts the equilibrium premium, while the right y-axis depicts the expected surplus. Additionally, the diagrams are organized into three columns according to the information structure, i.e., whether the game is a simultaneous-move or sequential-move game. Specifically, from left to right, the columns show the sensitivity analysis for the Nash game, the Stackelberg game with Insurer 1 as the leader, and the Stackelberg game with Insurer 1 as the follower. The last segmentation is regarding the parameter under investigation. The top-row diagrams are devoted to the price sensitivity of Insurer 1, whereas the bottom-row diagrams are devoted to Insurer 1's risk aversion.

Distinctive patterns emerge in the sensitivity analysis concerning Insurer 1's price-sensitivity parameter. In Figures 7 and 8, the NE and SE premiums of Insurer 1 are declining in a_1 and drift Insurer 2's equilibrium premiums downwards as well. However, Insurer 1's rate of decrease in the equilibrium premiums is sharper than Insurer 2's. This is a well-anticipated trend, given the connection between the price-sensitivity parameter and the insurers' ability to maintain policyholders. That is, higher values of a_1 indicate that Insurer 1 is penalized by losing a greater proportion of policyholders. Thus, Insurer 1 decreases the premium in order to mitigate losses in exposure volume. As a consequence of the premium decreases, both insurers' expected surplus declines as well.

In the symmetric and non-symmetric insurance game, Insurer 1's NE premium displays a steeper increase with respect to λ_1 compared to Insurer 2's NE premium. Being considerably cheaper, Insurer 2 expects an exponential increase in the end-of-period surplus in contrast to Insurer 1's moderate increase. However, when Insurer 1's risk aversion is below the benchmark value of 0.004, the insurers' NE premiums are close to each other, resulting in negligible changes in the expected surplus.

When Insurer 1 discloses the premium first, both insurers' SE premium slope downwards for low values of λ_1 and upwards as λ_1 increases. The slope, as well as the size of the leader's SE premium, is higher than the follower's. Therefore, the follower's expected surplus considerably decreases over the range of λ_1 associated with decreasing SE premiums and increasing otherwise. On the other hand, the fluctuation in the leader's expected surplus is not very significant. Similar patterns of insurers' SE premium and expected surplus are observed when Insurer 2 is the leader. The difference is that now the downward and upward slopes of both insurers' SE premium are steep, and noticeably, the follower Insurer 1's premium surpasses the leader's premium over large values of λ_1. Thus, Insurer 1's expected surplus drops as λ_1 increases. These trends are similar in the symmetric and non-symmetric insurance market.

6 Conclusion

This paper studies a stochastic insurance duopoly. Insurers are considered exponential utility maximizers. An insurer-specific exponential demand function is used to describe the intensity of the Poisson-distributed number of policyholders. We prove the existence of Stackelberg and Nash equilibrium premiums. In the Stackelberg equilibrium, we show that the follower's log-negative objective function deteriorates when the leader implements an aggressive premium strategy (i.e., a low premium rate).

We further challenge the pre-defined information structure by constructing a decision game in which an insurer chooses whether to disclose the premium to the competitor or not. The insurers' decisions result in either a simultaneous-move or sequential-move information structure, with the associated payoff inherited from the Nash or Stackelberg insurance game. Our model provides conditions under which the duopolists prefer participating in a sequential-move game rather over a simultaneous-move game.

The optimal market premiums are determined by the pre-allocation of roles such as the Nash player, Stackelberg leader, and Stackelberg follower. We believe insurers should endogenously select their roles based on their utility preferences. As a direction for future research, we propose exploring a multi-period framework to investigate how market conditions and internal insurer characteristics influence role decisions over time. Additionally, it would be valuable to examine the impact of non-negligible delays between the premium disclosures of competing insurers. Specifically, it is worth analyzing whether it is more advantageous (in terms of utility) for an insurer to withhold premium disclosure and, if so, how long the response should be delayed to achieve optimal equilibrium results.

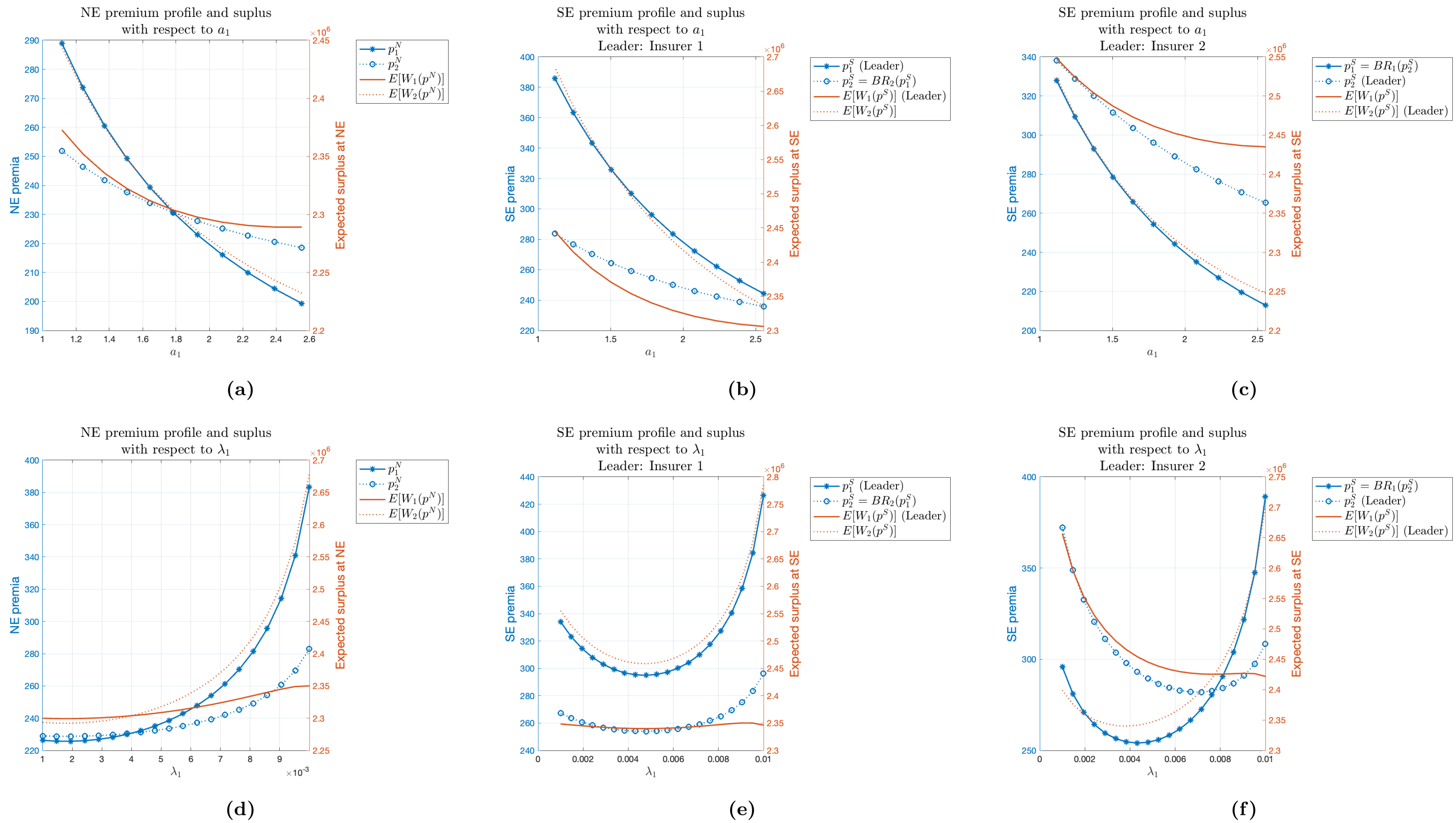


Figure 7: Parameter-sensitivity analysis in the *symmetric*, two-insurer market. The left y-axis represents the equilibrium premium, and the right y-axis represents the expected surplus evaluated at the equilibrium profile. Blue-colored lines with markers are associated with the left-side y-axis, and orange-colored lines with the right-side y-axis. Figs. (a), (b), and (c) present changes in equilibrium outcomes with respect to Insurer 1's price sensitivity. Figs. (d), (e), and (f) present changes in equilibrium outcomes with respect to Insurer 1's risk aversion. Figs. (a) and (d) are associated with the Nash game. Figs. (b) and (e) are associated with the Stackelberg game, in which Insurer 1 is the leader. Figs. (c) and (f) are associated with the Stackelberg game, in which Insurer 1 is the follower.

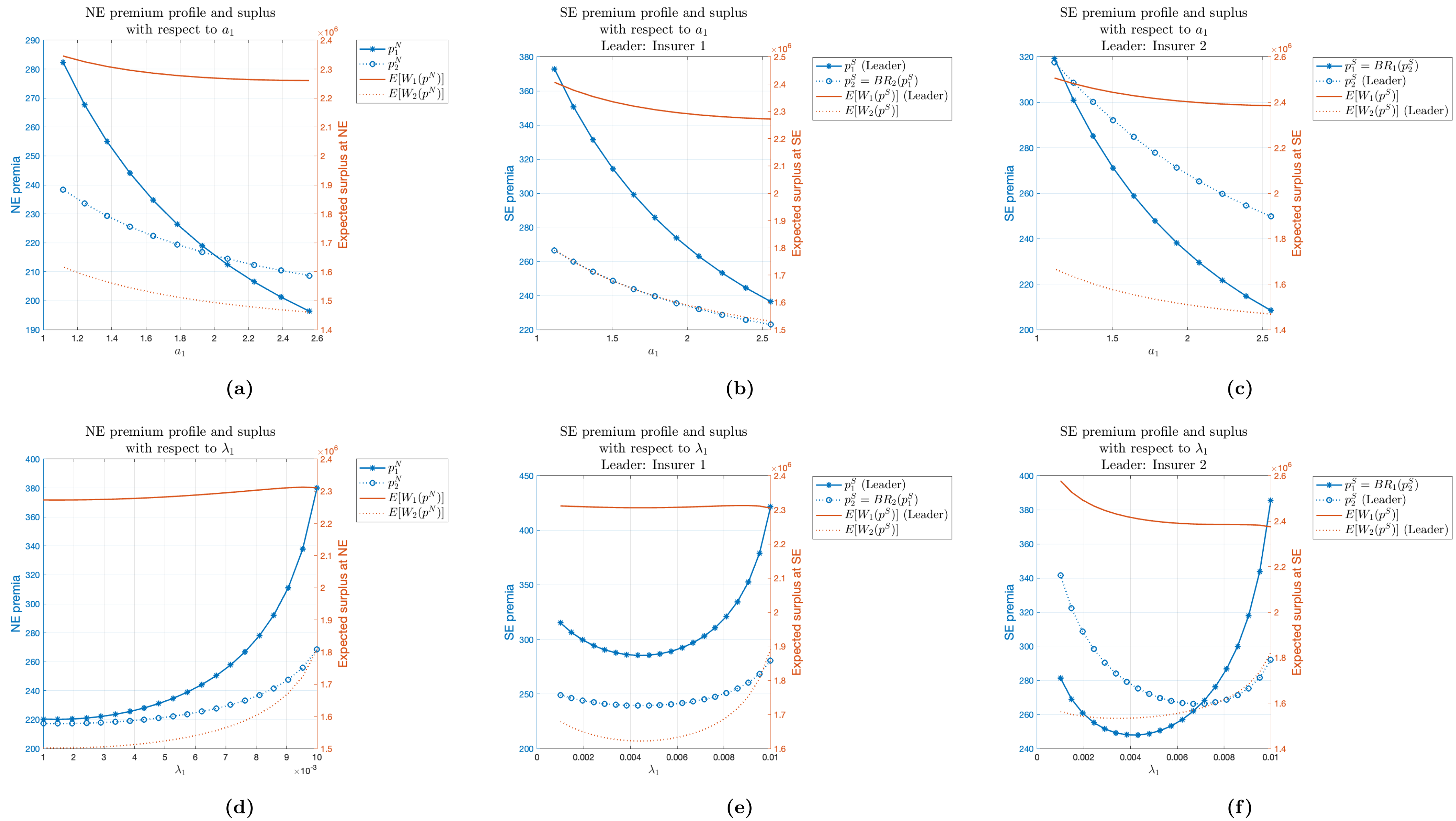


Figure 8: Parameter-sensitivity analysis in the *non-symmetric*, two-insurer market. The left y-axis represents the equilibrium premium, and the right y-axis represents the expected surplus evaluated at the equilibrium profile. Blue-colored lines with markers are associated with the left-side y-axis, and orange-colored lines with the right-side y-axis. Figs. (a), (b), and (c) present changes in equilibrium outcomes with respect to Insurer 1's price sensitivity. Figs. (d), (e), and (f) present changes in equilibrium outcomes with respect to Insurer 1's risk aversion. Figs. (a) and (d) are associated with the Nash game. Figs. (b) and (e) are associated with the Stackelberg game, in which Insurer 1 is the leader. Figs. (c) and (f) are associated with the Stackelberg game, in which Insurer 1 is the follower.

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A Proofs

A.1 Proof of Proposition 1

Insurer i's objective function is equal to the expected utility of the surplus realized at the end of the period, namely, $o_i(p_i, p_{-i}) = \mathbb{E}\left[U_i\left(W_i\left(N_i(p_i, p_{-i})\right)\right)\right]$. Since the surplus depends on the realized number of policies underwritten during the period, we use the law of iterated expectations to derive the objective function. That is,

$$
\begin{aligned}
o_i(p_i, p_{-i}) &= \mathbb{E}\left[-e^{-\lambda_i W_i}\right] = \mathbb{E}\left[-\exp\left(-\lambda_i\left[w_{i,0} + (1-e_i)\left(p_i N_i - \sum_{k=1}^{N_i} X_{i,k}\right)\right]\right)\right] \\
&= -e^{-\lambda_i w_{i,0}}\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\lambda_i(1-e_i)p_i N_i\right)\exp\left(\lambda_i(1-e_i)\sum_{k=1}^{N_i} X_{i,k}\right)\Big| N_i\right]\right] \\
&= -e^{-\lambda_i w_{i,0}}\mathbb{E}\left[\exp\left(-\lambda_i(1-e_i)p_i N_i\right)\mathbb{E}\left[\exp\left(\lambda_i(1-e_i)\sum_{k=1}^{N_i} X_{i,k}\right)\Big| N_i\right]\right] \\
&= -e^{-\lambda_i w_{i,0}}\mathbb{E}\left[\exp\left(-\lambda_i(1-e_i)p_i N_i\right)\mathbb{E}^{N_i}\left[\exp\left(\lambda_i(1-e_i)X_i\right)\right]\right].
\end{aligned}
$$

The last equality is due to Assumption (A2), which states that $\left(X_{i,k}\right)_k$ and N_i are independent random variables, and Assumption (A1), which states that $\left(X_{i,k}\right)_k$ are independent and identically distributed as the generic random variable X_i. From the condition in (8), we recognize the MGF of X_i evaluated at $\lambda_i(1-e_i)$ and, hence, we obtain

$$
\begin{aligned}
o_i(p_i, p_{-i}) &= -e^{-\lambda_i w_{i,0}}\mathbb{E}\left[\exp\left(-\lambda_i(1-e_i)p_i N_i\right)M_{X_i}^{N_i}\left(\lambda_i(1-e_i)\right)\right] \\
&= -e^{-\lambda_i w_{i,0}}\mathbb{E}\left[\left(e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\right)^{N_i}\right] \\
&= -e^{-\lambda_i w_{i,0}}P_{N_i}\left(e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\right).
\end{aligned}
$$

In the last equality, P_{N_i} stands for the probability generating function (PGF) of N_i evaluated at $\exp(-\lambda_i(1-e_i)p_i M_{X_i}(\lambda_i(1-e_i))$. Its existence is guaranteed by the condition in (9), which is equivalent to writing

$$
e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right) \leq 1. \tag{A.1}
$$

Taking into account the fact that N_i is Poisson distributed, as given in (2), we derive the expression for Insurer i's objective function in (10).

A.2 Proof of Lemmata 1 and 2

We start with the proof of Lemma 2, and will subsequently prove Lemma 1. Since $p_i \mapsto c_i(p_i, p_j)$ is a twice-continuously differentiable function, we use the first- and second-order conditions to justify our argument. Specifically, the first-order condition is equal to

$$\begin{aligned}
\frac{\partial c_i(p_i,p_j)}{\partial p_i} &= 0, \\
\frac{\partial}{\partial p_i}\left[-\lambda_i w_{i,0} + f_i(p_i,p_j) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right)\right] &= 0 \\
\frac{\partial f_i(p_i,p_j)}{\partial p_i} n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right) & \\
-\lambda_i(1-e_i) f_i(p_i,p_j) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= 0 \\
-\frac{a_i}{p_j} f_i(p_i,p_j) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right) & \\
-\lambda_i(1-e_i) f_i(p_i,p_j) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= 0 \\
\left[a_i + \lambda_i(1-e_i)p_j\right] e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= a_i,
\end{aligned}$$

where we have used the fact that $f_i(p_i, p_j)n_{i,0}$ is always positive. Continuing to solve with respect to p_i yields

$$\begin{aligned}
e^{-\lambda_i(1-e_i)p_i} &= \left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) M_{X_i}^{-1}\left(\lambda_i(1-e_i)\right), \text{ or} \\
-\lambda_i(1-e_i)p_i &= \ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right),
\end{aligned}$$

and, hence, we find the unique solution of

$$\begin{aligned}
p_i^*(p_j) &= -\frac{1}{\lambda_i(1-e_i)}\left[\ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right)\right] \\
&= -\frac{1}{\lambda_i(1-e_i)}\ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) + \frac{1}{\lambda_i(1-e_i)}\ln M_{X_i}\left(\lambda_i(1-e_i)\right) \\
&= -\frac{1}{\lambda_i(1-e_i)}\ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j}\right) + p_i^L.
\end{aligned}$$

Since the argument of the natural logarithm is less than one, we verify that $p_i^*(p_j) > p_i^L$. The second-order condition verifies that the critical point $p_i^*(p_j)$ is a minimum, specifically,

$$\begin{aligned}
\left.\frac{\partial^2 c_i(p_i,p_j)}{\partial p_i^2}\right|_{p_i=p_i^*(p_j)} &= \left(\frac{a_i}{p_j}\right)^2 f_i(p_i,p_j) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right) \\
&\quad +2\lambda_i(1-e_i)\frac{a_i}{p_j} f_i(p_i,p_j) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) \\
&\quad \left. + \left[\lambda_i(1-e_i)\right]^2 f_i(p_i,p_j) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)\right|_{p_i=p_i^*(p_j)} \\
&= \left(\frac{a_i}{p_j}\right)^2 f_i(p_i,p_j) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right) \\
&\quad +\lambda_i(1-e_i)\frac{a_i}{p_j} f_i(p_i,p_j) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) \\
&\quad \left. +\lambda_i(1-e_i)\frac{a_i}{p_j} f_i(p_i,p_j) n_{i,0}\right|_{p_i=p_i^*(p_j)} \\
&= \left.\lambda_i(1-e_i)\frac{a_i}{p_j} f_i(p_i,p_j) n_{i,0}\right|_{p_i=p_i^*(p_j)} > 0,
\end{aligned}$$

where the second and third equalities have been derived using the first-order condition. This concludes the proof of Lemma 2.

We continue with the proof of Lemma 1. Insurer i's log-negative objective function is twice continuously differentiable with respect to p_i. This allows us to use the characterization of quasi-convexity presented in Greenberg and Pierskalla (1971). Based on this characterization, the function $p_i \mapsto c_i(p_i, p_j)$ is quasi-convex if $c_i(p_i^y, p_j) \leq c_i(p_i^x, p_j)$ implies $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$.

Lemma 2 guarantees that the function $p_i \mapsto c_i(p_i, p_j)$ has a unique minimum denoted by $p_i^*(p_j)$. We can easily verify that $c_i(p_i, p_j)$ is decreasing for $p_i \in [p_i^L, p_i^*(p_j)]$ and increasing for $p_i > p_i^*(p_j)$. That is, $\frac{\partial c_i(p_i,p_j)}{\partial p_i} \leq 0$ can be written as $[a_i + \lambda_i(1-e_i)p_j]\, e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) \geq a_i$. We can rewrite this as follows:

$$
\begin{aligned}
e^{-\lambda_i(1-e_i)p_i} &\geq \left(\frac{a_i}{a_i + \lambda_i(1-e_i)p_j}\right) \frac{1}{M_{X_i}\left(\lambda_i(1-e_i)\right)}, \\
-\lambda_i(1-e_i)p_i &\geq \ln\left(\frac{a_i}{a_i + \lambda_i(1-e_i)p_j}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right),
\end{aligned}
$$

and thus,

$$
p_i \leq -\frac{1}{\lambda_i(1-e_i)}\left[\ln\left(\frac{a_i}{a_i + \lambda_i(1-e_i)p_j}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right)\right] = p_i^*(p_j).
$$

Moreover, we can similarly derive that $\partial c_i(p_i, p_j)/\partial p_i \geq 0$ implies $p_i \geq p_i^*(p_j)$.

Let p_i^x and p_i^y be two premium rates for Insurer i, and assume $c_i(p_i^y, p_j) \leq c_i(p_i^x, p_j)$. We want to prove that $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$. To do so, we investigate the four possible choices of p_i^x and p_i^y.

(i) Let $p_i^x,\ p_i^y \in [p_i^L, p_i^*(p_j)]$. Since $p_i \mapsto c_i(p_i, p_j)$ is decreasing on this interval and $c_i(p_i^y, p_j) \leq c_i(p_i^x, p_j)$, we derive that $\partial c_i(p_i^x, p_j)/\partial p_i \leq 0$ and $p_i^y \geq p_i^x$. Thus, $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$.

(ii) Let p_i^x, p_i^y be greater than $p_i^*(p_j)$. Since $p_i \mapsto c_i(p_i, p_j)$ is increasing on this interval and $c_i(p_i^y, p_j) \leq c_i(p_i^x, p_j)$, we derive that $\partial c_i(p_i^x, p_j)/\partial p_i \geq 0$ and $p_i^y \leq p_i^x$. Thus, $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$.

(iii) Let $p_i^L \leq p_i^x \leq p_i^*(p_j) < p_i^y$. Since $p_i \mapsto c_i(p_i, p_j)$ is decreasing on $[p_i^L, p_i^*(p_j)]$, it holds that $\partial c_i(p_i^x, p_j)/\partial p_i \leq 0$. Thus, $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$.

(iv) Let $p_i^L \leq p_i^y \leq p_i^*(p_j) < p_i^x$. Since $p_i \mapsto c_i(p_i, p_j)$ is increasing for values greater than $p_i^*(p_j)$, it holds that $\partial c_i(p_i^x, p_j)/\partial p_i \geq 0$. Thus, $(\partial c_i(p_i^x, p_j)/\partial p_i)(p_i^y - p_i^x) \leq 0$.

This concludes the proof.

A.3 Proof of Theorem 1

The backward induction argument characterizes the SE of the insurance game $\mathcal{G}^S$. We begin with the follower Insurer j's optimization problem. Let p_i be a premium selection of the leader in $\mathcal{P}_i$. The best response of Insurer j to p_i is the solution to the minimization problem presented in (13). Lemma 2 proves that there exists a unique minimum for the follower which is greater than the indifference premium. That is, the best response of Insurer j to p_i, according to Eq. (12), is equal to

$$
BR_j(p_i) = -\frac{1}{\lambda_j(1-e_j)}\left[\ln\left(\frac{a_j}{a_j + \lambda_j(1-e_j)p_i}\right) - \ln M_{X_j}\left(\lambda_j(1-e_j)\right)\right],
$$

which is differentiable, and satisfies

$$
BR_j(p_i) > p_j^L. \tag{A.2}
$$

The next step in the backward induction is the premium selection made by Insurer i. As a rational player, the leader Insurer i anticipates Insurer j to best respond to any p_i by choosing $BR_j(p_i)$. We have found that BR_j is a continuous function of p_i. By definition, $p_i \mapsto c_i(p_i, p_j)$ is continuous on $\mathcal{P}_i$

for any p_j, and $\mathcal{P}_i$ is assumed a non-empty, compact subset of the real numbers. Thus, Weierstrass' theorem guarantees that the minimization problem in (14) has a solution.

Before solving the FOC, we provide some relationships for the leader's demand function. Particularly, Insurer i's relative change in the number of policies is written as:

$$f_i(p_i, BR_j(p_i)) = \exp\left(-a_i \frac{p_i - BR_j(p_i)}{BR_j(p_i)}\right),$$

and its derivative with respect to p_i is equal to

$$\frac{df_i(p_i, BR_j(p_i))}{dp_i} = -a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)} f_i(p_i, BR_j(p_i)). \tag{A.3}$$

The assumption of positive price elasticity of demand implies for the leader's demand function that it is decreasing in p_i. On closer inspection of Eq. (A.3), we derive the condition

$$BR_j(p_i) - p_i BR_j'(p_i) > 0, \tag{A.4}$$

in order for $df_i(p_i, BR_j(p_i))/dp_i$ to be negative and consistent with the relationship in (6).

Insurer i's optimal premium is the solution to the first-order condition given by:

$$\begin{aligned}
\frac{dc_i(p_i, BR_j(p_i))}{dp_i} &= 0\\
\frac{d}{dp_i}\left[-\lambda_i w_{i,0} + f_i(p_i, BR_j(p_i)) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)\right] &= 0\\
\frac{df_i(p_i, BR_j(p_i))}{dp_i} n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)&\\
-\lambda_i(1-e_i) f_i(p_i, BR_j(p_i)) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= 0\\
-a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)} f_i(p_i, BR_j(p_i)) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)&\\
-\lambda_i(1-e_i) f_i(p_i, BR_j(p_i)) n_{i,0} e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= 0,
\end{aligned} \tag{A.5}$$

and because $f_i(p_i, BR_j(p_i)) n_{i,0}$ is always positive, we finally obtain

$$\begin{aligned}
a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)&\\
+\lambda_i(1-e_i) e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) &= 0.
\end{aligned} \tag{A.6}$$

Let p_i^S denote the critical point that satisfies the FOC in Eq. (A.6). We notice that $c_i(p_i^L, BR_j(p_i^L)) = -\lambda_i w_{i,0}$, $\lim_{p_i \to +\infty} c_i(p_i, BR_j(p_i)) = -\lambda_i w_{i,0}$ as well as $c_i(p_i, BR_j(p_i)) < -\lambda_i w_{i,0}$ for all $p_i > p_i^L$. The limit of c_i is derived by considering that both $\exp(-\lambda_i(1-e_i)p_i)$ and $f_i(p_i, BR_j(p_i))$ vanish as p_i goes to infinity. Thus, we deduce that p_i^S is greater than the indifference premium, i.e., $p_i^L < p_i^S$, which yields the strict inequalities:

$$0 < e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}\left(\lambda_i(1-e_i)\right) < 1. \tag{A.7}$$

We check the second-order condition for the leader Insurer i to guarantee that p_i^S is a minimum.

$$\begin{aligned}
&\left.\frac{d^2 c_i(p_i, BR_j(p_i))}{dp_i^2}\right|_{p_i = p_i^S}\\
&= \frac{d}{dp_i}\left(-a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)}\right) f_i(p_i, BR_j(p_i)) n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)
\end{aligned}$$

$$+\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)^2 f_i(p_i,BR_j(p_i))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right)$$
$$+2\lambda_i(1-e_i)\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)$$
$$+\left[\lambda_i(1-e_i)\right]^2 f_i(p_i,BR_j(p_i))n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\Big|_{p_i=p_i^S}.$$

Using the FOC in Eq. (A.6), the second term is zero, whereas the third and fourth terms can be simplified; we obtain

$$\left.\frac{d^2c_i(p_i,BR_j(p_i))}{dp_i^2}\right|_{p_i=p_i^S}$$
$$=\frac{d}{dp_i}\left(-a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right)$$
$$+\lambda_i(1-e_i)\left[a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}f_i(p_i,BR_j(p_i))n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\right.$$
$$\left.\left.+\lambda_i(1-e_i)f_i(p_i,BR_j(p_i))n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\right]\right|_{p_i=p_i^S}$$
$$=\frac{d}{dp_i}\left(-a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right)$$
$$+\lambda_i(1-e_i)\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}\Big|_{p_i=p_i^S},$$

where the last equality is derived by simplifying the term in the square brackets in the penultimate equality using the FOC in (A.6). Considering that the quantity in Eq. (A.4) is nonzero, we can apply the FOC in (A.6) to the first term and obtain

$$\left.\frac{d^2c_i(p_i,BR_j(p_i))}{dp_i^2}\right|_{p_i=p_i^S}$$
$$=\frac{d}{dp_i}\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)\left(\frac{\lambda_i(1-e_i)f_i(p_i,BR_j(p_i))n_{i,0}}{a_i\frac{BR_j(p_i)-p_i\frac{dBR_j(p_i)}{dp_i}}{BR_j^2(p_i)}}\right)e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)$$
$$+\lambda_i(1-e_i)\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}\Big|_{p_i=p_i^S}.$$

We can reduce the positive, second term (recall the inequality in (A.4)) by multiplying with the quantity in (A.7). Then, we find

$$\left.\frac{d^2c_i(p_i,BR_j(p_i))}{dp_i^2}\right|_{p_i=p_i^S}$$
$$>\frac{d}{dp_i}\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)\left(\frac{\lambda_i(1-e_i)f_i(p_i,BR_j(p_i))n_{i,0}}{a_i\frac{BR_j(p_i)-p_i\frac{dBR_j(p_i)}{dp_i}}{BR_j^2(p_i)}}\right)e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)$$
$$+\lambda_i(1-e_i)\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)\Big|_{p_i=p_i^S}$$
$$=\left[\frac{d}{dp_i}\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)f_i(p_i,BR_j(p_i))+\left(a_i\frac{BR_j(p_i)-p_iBR_j'(p_i)}{BR_j^2(p_i)}\right)^2 f_i(p_i,BR_j(p_i))\right]$$
$$\times\left.\frac{\lambda_i(1-e_i)n_{i,0}e^{-\lambda_i(1-e_i)p_i}M_{X_i}\left(\lambda_i(1-e_i)\right)}{a_i\frac{BR_j(p_i)-p_i\frac{dBR_j(p_i)}{dp_i}}{BR_j^2(p_i)}}\right|_{p_i=p_i^S}.$$

From (A.4) we have that the denominator is positive. From (A.3), we can derive for Insurer i's relative change in the number of policies that

$$\frac{d^2 f_i(p_i, BR_j(p_i))}{dp_i^2} = -\frac{d}{dp_i}\left(a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)}\right) f_i(p_i, BR_j(p_i)) + \left(a_i \frac{BR_j(p_i) - p_i BR_j'(p_i)}{BR_j^2(p_i)}\right)^2 f_i(p_i, BR_j(p_i)).$$

From the condition in (17), we can conclude that term in the square brackets is non-negative and, hence, $d^2 c_i(p_i, BR_j(p_i))/dp_i^2 > 0$ at $p_i = p_i^S$.

A.4 Proof of Proposition 2

Differentiating Eq. (15) with respect to p_i provides the rate of change in the follower's best response with respect to the leader's premium. It is straightforward to obtain

$$BR_j'(p_i) = \frac{1}{a_j + \lambda_j(1 - e_j)p_i}, \tag{A.8}$$

which is positive for any $p_i \in \mathcal{P}_i$. This indicates that Insurer i, by choosing greater values for p_i, forces Insurer j to increase its premium rate.

Given the leader's premium p_i, the optimal value for the follower's log-negative objective function is equal to $c_j\left(BR_j(p_i), p_i\right)$. Consequently, the leader's strategic incentive is reflected in the changes in $c_j\left(BR_j(p_i), p_i\right)$ with respect to p_i. That is,

$$\begin{aligned} &\frac{dc_j\left(BR_j(p_i), p_i\right)}{dp_i} \\ =\ &\frac{df_j(BR_j(p_i), p_i)}{dp_i} n_{j,0}\left(e^{-\lambda_j(1-e_j)BR_j(p_i)} M_{X_j}\left(\lambda_j(1-e_j)\right) - 1\right) \\ &-\lambda_j(1-e_j)BR_j'(p_i) f_j(BR_j(p_i), p_i) n_{j,0} e^{-\lambda_j(1-e_j)BR_j(p_i)} M_{X_j}\left(\lambda_j(1-e_j)\right) \\ =\ &-a_j \frac{p_i BR_j'(p_i) - BR_j(p_i)}{p_i^2} f_j(BR_j(p_i), p_i) n_{j,0}\left(e^{-\lambda_j(1-e_j)BR_j(p_i)} M_{X_j}\left(\lambda_j(1-e_j)\right) - 1\right) \\ &-\lambda_j(1-e_j)BR_j'(p_i) f_j(BR_j(p_i), p_i) n_{j,0} e^{-\lambda_j(1-e_j)BR_j(p_i)} M_{X_j}\left(\lambda_j(1-e_j)\right). \end{aligned}$$

According to the sign of the quantities in Eqs. (9), (A.4) and (A.8), we conclude $\frac{dc_j(BR_j(p_i), p_i)}{dp_i} < 0$, for all $p_i \in \mathcal{P}_i$. In terms of utilities, the follower's performance diminishes when the leader Insurer i follows aggressive strategies by charging a low premium rate.

A.5 Proof of Proposition 3

In both cases, it is sufficient to look at the monotonicity of the leader's SE premium with respect to either its own or the follower's price-sensitivity parameter. To see this, recall Proposition 2 which proves that the best response of the follower is an increasing function of the leader's premium. Letting a denote either a_i or a_j, the chain rule yields:

$$\frac{dBR_j(p_i^S)}{da} = BR_j'(p_i^S)\frac{dp_i^S}{da},$$

which, in conjunction with the positiveness of (A.8), leads to:

$$\text{sign}\left(\frac{dBR_j(p_i^S)}{da}\right) = \text{sign}\left(\frac{dp_i^S}{da}\right). \tag{A.9}$$

The price-sensitivity parameters and the leader's SE premium satisfy the first-order condition in

(16). We denote the left-hand side in (16) as $FOC_i(a, p_i^S)$ where a takes the value of a_i or a_j. That is,

$$FOC_i(a, p_i^S) = a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} \left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) + \lambda_i(1-e_i)e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)), \tag{A.10}$$

where a_j appears only in $BR_j(p_i^S)$ as defined in (15).

Next, we need the sign of the relative change in FOC_i with respect to the leader's premium at the solution points (a, p_i^S), where a equals a_i or a_j. That is,

$$\begin{aligned}\frac{\partial FOC_i(a, p_i^S)}{\partial p_i} &= -a_i \frac{2BR_j(p_i^S)BR_j'(p_i^S)\left(BR_j(p_i^S) - p_i^S BR_j'(p_i^S)\right) + p_i^S BR_j''(p_i^S)BR_j^2(p_i^S)}{BR_j^4(p_i^S)} \\ &\quad \times \left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) \\ &\quad - \lambda_i(1-e_i)a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) \\ &\quad - [\lambda_i(1-e_i)]^2 e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)),\end{aligned}$$

and using the fact that $FOC_i(a, p_i^S) = 0$, we can simplify the expression as

$$\begin{aligned}\frac{\partial FOC_i(a, p_i^S)}{\partial p_i} &= -a_i \frac{2BR_j(p_i^S)BR_j'(p_i^S)\left(BR_j(p_i^S) - p_i^S BR_j'(p_i^S)\right) + p_i^S BR_j''(p_i^S)BR_j^2(p_i^S)}{BR_j^4(p_i^S)} \\ &\quad \times \left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) - \lambda_i(1-e_i)a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}.\end{aligned} \tag{A.11}$$

Since p^S is the SE premium profile, the second-order condition for minimization requires

$$\frac{d^2 c_i(p_i^S, BR_j(p_i^S))}{dp_i^2} > 0. \tag{A.12}$$

Differentiating (A.5) with respect to p_i yields

$$\begin{aligned}&\frac{d^2 c_i(p_i^S, BR_j(p_i^S))}{dp_i^2} \\ &= \frac{d}{dp_i}\left(-a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}\right) f_i(p_i^S, BR_j(p_i^S))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) \\ &+ \left(a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}\right)^2 f_i(p_i^S, BR_j(p_i^S))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) \\ &+ 2\lambda_i(1-e_i)a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} f_i(p_i^S, BR_j(p_i^S))n_{i,0}e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) \\ &+ [\lambda_i(1-e_i)]^2 f_i(p_i^S, BR_j(p_i^S))n_{i,0}e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)),\end{aligned}$$

and using the fact that $FOC_i(a, p_i^S) = 0$, we can simplify the expression as

$$\begin{aligned}&\frac{d^2 c_i(p_i^S, BR_j(p_i^S))}{dp_i^2} \\ &= \frac{d}{dp_i}\left(-a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}\right) f_i(p_i^S, BR_j(p_i^S))n_{i,0}\left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}(\lambda_i(1-e_i)) - 1\right) \\ &+ \lambda_i(1-e_i)a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} f_i(p_i^S, BR_j(p_i^S))n_{i,0} \\ &= \Bigg[a_i \frac{2BR_j(p_i^S)BR_j'(p_i^S)\left(BR_j(p_i^S) - p_i^S BR_j'(p_i^S)\right) + p_i^S BR_j''(p_i^S)BR_j^2(p_i^S)}{BR_j^4(p_i^S)}\end{aligned}$$

$$\times \left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1 \right) + \lambda_i(1-e_i)a_i \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} \Bigg] f_i(p_i^S, BR_j(p_i^S)) n_{i,0}. \quad \text{(A.13)}$$

The positiveness of $f_i(p_i^S, BR_j(p_i^S))n_{i,0}$ and the second-order condition in (A.12) lead the term in brackets in Eq. (A.13) to be positive. This further implies for Eq. (A.11) that

$$\frac{\partial FOC_i(a, p_i^S)}{\partial p_i} < 0. \quad \text{(A.14)}$$

We have that (a_i, p_i^S) and (a_j, p_i^S) are two solution points to $FOC_i(a, p_i) = 0$. Moreover, $\partial FOC_i(a, p_i^S)/\partial p_i$, by the inequality in (A.14), is irreversible for a equal to a_i or a_j. Then, the implicit function theorem leads to

$$\frac{dp_i^S}{da} = -\frac{\frac{\partial FOC_i(a, p_i^S)}{\partial a}}{\frac{\partial FOC_i(a, p_i^S)}{\partial p_i}},$$

for a equal to a_i or a_j. Note that the denominator is always negative. Therefore,

$$\text{sign}\left(\frac{dp_i^S}{da}\right) = \text{sign}\left(\frac{\partial FOC_i(a, p_i^S)}{\partial a}\right). \quad \text{(A.15)}$$

The two cases for the price-sensitivity parameter a are presented separately.

1. We consider the leader's price-sensitivity parameter a_i. An inspection of (A.10) verifies that $a_i \mapsto FOC_i(a_i, p_i^S)$ is linear and, hence, the relative change of $FOC_i(a_i, p_i^S)$ with respect to a_i is equal to

$$\frac{\partial FOC_i(a_i, p_i^S)}{\partial a_i} = \frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)} \left(e^{-\lambda_i(1-e_i)p_i^S} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1 \right) \leq 0. \quad \text{(A.16)}$$

The last inequality results from the conditions in (A.1) and (A.4). Thus, (A.16) in conjunction with the relationships in (A.15) and (A.9) yield the leader's SE premium and the follower's best response at SE are decreasing in a_i.

2. We consider the follower's price-sensitivity parameter a_j, which appears in FOC_i only in the best response $BR_j(p_i^S)$. Recalling Eqs. (15) and (A.8), we can derive the following auxiliary partial derivatives:

$$\frac{\partial BR_j(p_i^S)}{\partial a_j} = -\frac{p_i^S}{a_j} BR_j'(p_i^S), \quad \text{(A.17)}$$

$$\frac{\partial^2 BR_j(p_i^S)}{\partial a_j \partial p_i} = -\frac{1}{a_j + \lambda_j(1-e_j)p_i^S} BR_j'(p_i^S), \quad \text{(A.18)}$$

and using Eqs. (A.17) and (A.18) we further obtain

$$\begin{aligned}
\frac{\partial}{\partial a_j}\left(\frac{BR_j(p_i^S) - p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}\right) &= \frac{1}{BR_j^4(p_i^S)} \Bigg[\frac{p_i^S}{a_j + \lambda_j(1-e_j)p_i^S} BR_j^2(p_i^S) BR_j'(p_i^S) \\
&\quad + \frac{p_i^S}{a_j} BR_j(p_i^S) BR_j'(p_i^S) \left(BR_j(p_i^S) - 2p_i^S BR_j'(p_i^S) \right) \Bigg] \\
&\geq \frac{1}{BR_j^4(p_i^S)} \Bigg[-\frac{(p_i^S)^2}{a_j} BR_j(p_i^S) \left(BR_j'(p_i^S) \right)^2 \\
&\quad + \frac{p_i^S}{a_j + \lambda_j(1-e_j)p_i^S} BR_j^2(p_i^S) BR_j'(p_i^S) \Bigg] \\
&\overset{\text{(A.8)}}{=} \frac{1}{BR_j^4(p_i^S)} \Bigg[-\frac{(p_i^S)^2}{a_j(a_j + \lambda_j(1-e_j)p_i^S)^2} BR_j(p_i^S)
\end{aligned}$$

$$+\frac{p_i^S}{(a_j+\lambda_j(1-e_j)p_i^S)^2}BR_j^2(p_i^S)\Big]$$
$$= \frac{1}{BR_j^4(p_i^S)}\frac{p_i^S BR_j(p_i^S)}{a_j(a_j+\lambda_j(1-e_j)p_i^S)^2}\left(a_j BR_j(p_i^S)-p_i^S\right), \tag{A.19}$$

where the inequality arises from (A.4) which implies $BR_j(p_i^S) - 2p_i^S BR_j'(p_i^S) > -p_i^S BR_j'(p_i^S)$.

From Eq. (A.10), the relative change in $FOC_i(a_j, p_i^S)$ with respect to a_j is equal to

$$\frac{\partial FOC_i(a_j,p_i^S)}{\partial a_j} = a_i\left(e^{-\lambda_i(1-e_i)p_i^S}M_{X_i}\left(\lambda_i(1-e_i)\right)-1\right)\frac{\partial}{\partial a_j}\left(\frac{BR_j(p_i^S)-p_i^S BR_j'(p_i^S)}{BR_j^2(p_i^S)}\right).$$

When the condition in (18) is met, the derivative in (A.19) is positive. Considering the constraint for the indifference premium in (A.1), we get $\frac{\partial FOC_i(a_j,p_i^S)}{\partial a_j} \le 0$, and when applied to the relationships in (A.15) and (A.9), we conclude that the leader's SE premium and the follower's best response at SE are decreasing in a_j.

A.6 Proof of Theorem 2

For each $i \in \mathcal{I}$, the strategy space $\mathcal{P}_i$ is, by assumption, a nonempty, compact, and convex set. By definition, the log-negative objective function c_i is continuous in p. Based on Lemma 1, c_i is quasi-convex in p_i. All these are the conditions of Theorem 1.2 in Fudenberg and Tirole (1991) (the theorem is based on Debreu, 1952; Glicksberg, 1952; Fan, 1952), which guarantee the existence of a (pure-strategy) Nash equilibrium in the insurance duopoly $\mathcal{G}^N$.

Let the premium vector p^N denote the NE. Lemma 2 proves that, for each $i \in \mathcal{I}$, the equilibrium premium p_i^N is completely characterized by (20).

A.7 Proof of Proposition 4

Without loss of generality, we consider Insurer i's price-sensitivity parameter. The first-order condition to Insurer i's minimization problem in (19) results in expressing p_i^N explicitly in terms of a_i. To see that, the best response of Insurer i to p_j^N is derived in a similar way as the best response of Insurer j in the proof of Theorem 1. Namely, we obtain from $\frac{\partial c_i(p_i,p_j^N)}{\partial p_i} = 0$ that

$$p_i^N = -\frac{1}{\lambda_i(1-e_i)}\left[\ln\left(\frac{a_i}{a_i+\lambda_i(1-e_i)p_j^N}\right) - \ln M_{X_i}\left(\lambda_i(1-e_i)\right)\right].$$

It is straightforward to see that:

$$\frac{dp_i^N}{da_i} = -\frac{p_j^N}{a_i(a_i+\lambda_i(1-e_i)p_j^N)} < 0,$$

i.e., Insurer i's NE premium is decreasing in a_i. From Eq. (A.8), we know that Insurer j's best response is an increasing function of p_i. The chain rule yields

$$\frac{dp_j^N}{da_i} = \frac{dBR_j(p_i^N)}{dp_i}\frac{dp_i^N}{da_i},$$

where the equality is due to the definition of NE, namely, the intersection of best responses. It is evident from the above relationship that the NE premiums of insurers i and j display the same monotonicity with respect to a_i. Hence, Insurer j's NE premium is decreasing in a_i.

A.8 Proof of Theorem 3

Assume that Insurer j makes the decision D_j. Then, Insurer i's best response is based on the ranking of the payoffs $\pi_i(D_i, D_j)$ and $\pi_i(ND_i, D_j)$.

Based on Table 2, the payoffs of the decision game $\mathcal{G}$ are inherited from the resulting insurance games $\mathcal{G}^S$ and $\mathcal{G}^N$. Therefore, Insurer i will make the decision ND_i if $\pi_i(ND_i, D_j) \leq \pi_i(D_i, D_j)$, i.e., $c_i(BR_i(p_j^S), p_j^S) \leq c_i(p_i^N, p_j^N)$. From the characterization of NE in (20), we can see that the NE is the intersection of insurers' best responses. Thus, the last inequality is equivalent to $c_i(BR_i(p_j^S), p_j^S) \leq c_i(BR_i(p_j^N), p_j^N)$.

Now, according to Lemma 2 (see also the inequality in (A.2)), the best response of the follower insurer is always strictly greater than the indifference premium and hence, in conjunction with the condition in (9), we have $\exp(-\lambda_i(1-e_i)BR_i\left(p_j^S\right))M_{X_i}\left(\lambda_i(1-e_i)\right) - 1 < 0$.

Further, in Proposition 2, we show that the best response of an insurer is an increasing function of the competitor's premium. From condition (21), we get that $BR_i\left(p_j^N\right) < BR_i\left(p_j^S\right)$, and since $\lambda_i > 0$, $0 < 1 - e_i < 1$ and $M_{X_i}(\lambda_i(1-e_i)) \geq 1$, we obtain

$$0 > e^{-\lambda_i(1-e_i)BR_i\left(p_j^N\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1 > e^{-\lambda_i(1-e_i)BR_i\left(p_j^S\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1,$$

i.e.,

$$0 < \frac{e^{-\lambda_i(1-e_i)BR_i\left(p_j^N\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1}{e^{-\lambda_i(1-e_i)BR_i\left(p_j^S\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1} < 1. \tag{A.20}$$

Now, from condition (22), the definition of NE, i.e., $BR_i\left(p_j^N\right) = p_i^N$ and the definition of Insurer i's relative change in number of policies in (3), we get

$$\frac{f_i\left(BR_i\left(p_j^S\right), p_j^S\right)}{f_i\left(BR_i\left(p_j^N\right), p_j^N\right)} = \exp\left(-a_i\left(\frac{BR_i\left(p_j^S\right)}{p_j^S} - \frac{BR_i\left(p_j^N\right)}{p_j^N}\right)\right) > 1. \tag{A.21}$$

Combining the inequalities in (A.20) and (A.21), we see that the inequality in (A.22) is satisfied

$$\frac{f_i\left(BR_i\left(p_j^S\right), p_j^S\right)}{f_i\left(BR_i\left(p_j^N\right), p_j^N\right)} \geq \frac{e^{-\lambda_i(1-e_i)BR_i\left(p_j^N\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1}{e^{-\lambda_i(1-e_i)BR_i\left(p_j^S\right)}M_{X_i}\left(\lambda_i(1-e_i)\right) - 1}. \tag{A.22}$$

That is, ND_i is Insurer i's best response to Insurer j's D_j. In other words, from inequality (A.22) and the definition of the log-negative objective function, (11), the $c_i(BR_i(p_j^S), p_j^S) \leq c_i(BR_i(p_j^N), p_j^N)$ holds.

Now, assume that Insurer i decides ND_i. Since $\pi_j\left(D_j, ND_i\right) = c_j\left(p_j^S, BR_i\left(p_j^S\right)\right) < 0 = \pi_j\left(ND_j, ND_i\right)$, the best response of Insurer j to ND_i is D_j (notice that the log-negative objective function as defined in (11) is always negative). Therefore, we conclude that no insurer can benefit by deviating from (ND_i, D_j), making it a NE.

Moreover, from the arguments above, (D_i, D_j) and (ND_i, ND_j) are not NE. Under similar arguments, we show that (D_i, ND_j) is the other NE of the decision game.

Online supplementary material: Appendix

"Competitive insurance pricing in a duopoly"

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Abstract

A single-period stochastic insurance duopoly is formulated to examine the pre-assignment of roles to the insurance game's players. This paper considers two information structures. In the first structure, one insurer assumes the role of the Stackelberg leader by setting the premium first, while the competitor, acting as the Stackelberg follower, responds after observing the leader's premium. In the second structure, both insurers act as Nash players, setting premiums simultaneously without considering the competitor's premium. This paper shows the existence of Stackelberg and Nash equilibria in these settings and identifies which information structure leads to superior utility when the decision to disclose the premium to the competitor is endogenous. A decision game is developed to determine the conditions under which both insurers prefer sequential over simultaneous premium setting in terms of utility.

Keywords: Risk Management; Game theory; Stackelberg equilibrium; Nash equilibrium; Insurance duopoly.

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Market dynamics: Panjer class for N_i

In this section, we follow the parameterisations for the Panjer distributions presented in Table 1 of Fackler (2011), particularly those using the expected value as a parameter. Specifically, the Poisson distribution has a single parameterisation using the expectation as the only parameter, whereas the negative binomial has two parameters, the number of successes and the expected number of failures.[6]

In our analysis, we assume the number of policies underwritten by Insurer i in the current period, N_i, belongs to the Panjer $(a, b, 0)$ class, and $\mathbb{E}[N_i(p_i, p_{-i})] = f_i(p_i, p_{-i})n_{i,0} > 0$. Following the parameterisation in Fackler (2011), the PGF of N_i is given by

$$P_{N_i}(t) = \mathbb{E}\left[t^{N_i(p_i,p_{-i})}\right] = \left[1 - \frac{\mathbb{E}[N_i(p_i, p_{-i})]}{\bar{\alpha}_i}(t-1)\right]^{-\bar{\alpha}_i}, \tag{B.1}$$

in which if $\bar{\alpha}_i \in (0, +\infty)$ we obtain the negative binomial$(\bar{\alpha}_i, \mathbb{E}[N_i(p_i, p_{-i})])$, whereas if $\bar{\alpha}_i \to +\infty$ we obtain the Poisson$(\mathbb{E}[N_i(p_i, p_{-i})])$ (i.e., the case studied in the main text).

In the proof of Proposition 1, Insurer i's objective function is equal to

$$o_i(p_i, p_{-i}) = -e^{-\lambda_i w_{i,0}} P_{N_i}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right)\right),$$

and based on Eq. (B.1), we derive

$$\bar{o}_i(p_i, p_{-i}) = -\,e^{-\lambda_i w_{i,0}}\left[1 - \frac{f_i(p_i, p_{-i})n_{i,0}}{\bar{\alpha}_i}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)\right]^{-\bar{\alpha}_i}. \tag{B.2}$$

Taking the logarithm of the additive inverse of the objective function in Eq. (B.2) yields the counterpart of Insurer i's log-negative objective function in Eq. (11). Therefore, it is defined as

$$\bar{c}_i(p_i, p_{-i}) = -\lambda_i w_{i,0} - \bar{\alpha}_i \ln\left[1 - \frac{f_i(p_i, p_{-i})n_{i,0}}{\bar{\alpha}_i}\left(e^{-\lambda_i(1-e_i)p_i} M_{X_i}\left(\lambda_i(1-e_i)\right) - 1\right)\right]. \tag{B.3}$$

From Eqs. (11) and (B.3), the relationship between the log-negative objective function under the Panjer class and under the Poisson distribution is given by

$$\bar{c}_i(p_i, p_{-i}) = -\lambda_i w_{i,0} - \bar{\alpha}_i \ln\left[1 - \frac{c_i(p_i, p_{-i}) + \lambda_i w_{i,0}}{\bar{\alpha}_i}\right]. \tag{B.4}$$

Remark 1 *Fackler (2011) shows that the Panjer $(a, b, 0)$ class reduces to the Poisson distribution when $\bar{\alpha}_i \to +\infty$. Considering the relationship in Eq. (B.4), we also obtain that $\lim_{\bar{\alpha}_i \to +\infty} \bar{c}_i(p_i, p_{-i}) = c_i(p_i, p_{-i})$. Indeed,*

$$\begin{aligned}\lim_{\bar{\alpha}_i \to +\infty} \bar{c}_i(p_i, p_{-i}) &= -\lambda_i w_{i,0} - \lim_{\bar{\alpha}_i \to +\infty} \ln\left[1 - \frac{c_i(p_i, p_{-i}) + \lambda_i w_{i,0}}{\bar{\alpha}_i}\right]^{\bar{\alpha}_i} = -\lambda_i w_{i,0} - \ln e^{-c_i(p_i,p_{-i}) - \lambda_i w_{i,0}} \\ &= c_i(p_i, p_{-i}).\end{aligned}$$

Remark 2 *The log-negative objective functions c_i and $\bar{c}_i$ display the same monotonicity with respect to p_i. That is, $\partial c_i/\partial p_i$ and $\partial \bar{c}_i/\partial p_i$ have the same sign. To see that, we take the derivative of $\bar{c}_i$, given in*

[6] Under these parameterisations, the probability mass functions evaluated at x are given by $\frac{\bar{n}^x}{x!}e^{-\bar{n}}$ and $\binom{k+x-1}{x}\left(\frac{k}{k+\bar{n}}\right)^k\left(\frac{\bar{n}}{k+\bar{n}}\right)^x$ for the Poisson and negative binomial distributions, respectively, in which k is the number of successes, π is the probability of success and $\bar{n} = k(1-\pi)/\pi$ is the expected number of failures.

Eq. (B.4)*, with respect to* p_i *and get*

$$\frac{\partial \bar{c}_i(p_i,p_{-i})}{\partial p_i} = -\frac{\bar{\alpha}_i}{1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}}\left(-\frac{1}{\bar{\alpha}_i}\frac{\partial c_i(p_i,p_{-i})}{\partial p_i}\right) = \frac{1}{1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}}\frac{\partial c_i(p_i,p_{-i})}{\partial p_i}. \quad \text{(B.5)}$$

From Eq. (11)*, Ineq.* (A.1) *and the fact that* $f_i(p_i,p_{-i})n_{i,0}$ *is positive, we have* $c_i(p_i,p_{-i})+\lambda_i w_{i,0} \le 0$ *and, hence, the denominator in* (B.5) *is positive, since* $\bar{\alpha}_i > 0$ *as well. Therefore,* c_i *and* $\bar{c}_i$ *have the same unique extremum and, according to Lemma 2, it is denoted by* $p_i^*(p_j)$*. The fact that it is a minimum can be verified by the SOC, i.e.,*

$$\begin{aligned}\frac{\partial^2 \bar{c}_i(p_i,p_{-i})}{\partial p_i^2} &= \frac{\frac{\partial^2 c_i(p_i,p_{-i})}{\partial p_i^2}\left(1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}\right) - \frac{\partial c_i(p_i,p_{-i})}{\partial p_i}\left(-\frac{1}{\bar{\alpha}_i}\frac{\partial c_i(p_i,p_{-i})}{\partial p_i}\right)}{\left(1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}\right)^2} \\ &= \frac{\frac{1}{\bar{\alpha}_i}\left(\frac{\partial c_i(p_i,p_{-i})}{\partial p_i}\right)^2}{\left(1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}\right)^2} + \frac{1}{1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}}\frac{\partial^2 c_i(p_i,p_{-i})}{\partial p_i^2}\end{aligned}$$

evaluated at $p_i^*(p_j)$ *yields*

$$\left.\frac{\partial^2 \bar{c}_i(p_i,p_{-i})}{\partial p_i^2}\right|_{p_i=p_i^*(p_j)} = \frac{1}{1-\frac{c_i(p_i,p_{-i})+\lambda_i w_{i,0}}{\bar{\alpha}_i}}\left.\frac{\partial^2 c_i(p_i,p_{-i})}{\partial p_i^2}\right|_{p_i=p_i^*(p_j)} > 0,$$

since $\left.\frac{\partial c_i(p_i,p_{-i})}{\partial p_i}\right|_{p_i=p_i^*(p_j)} = 0$ *and* $\left.\frac{\partial^2 c_i(p_i,p_{-i})}{\partial p_i^2}\right|_{p_i=p_i^*(p_j)} > 0$ *(from FOC and SOC, respectively, as shown in the proof of Lemma 2).*

Remark 3 *The proof of Lemma 1 uses the monotonicity of* c_i*. Remark 2 shows that* c_i *and* $\bar{c}_i$ *display the same monotonicity with respect to* p_i *and have the same global minimum point. Hence, the arguments used in the lemma yield the conclusion that* $p_i \mapsto \bar{c}_i(p_i,p_{-i})$ *is quasi-convex.*

Remark 4 *Due to the relationship of* c_i *and* $\bar{c}_i$ *in Eq.* (B.4)*, the arguments presented in Theorems 1, 2 and 3 also apply in the case of the Panjer* $(a,b,0)$ *class for the policyholder number* N_i*.*

In the Stackelberg game, the best response of the follower Insurer j*,* $\bar{BR}_j(p_i)$*, is given by* $\partial\bar{c}_j(p_j,p_i)/\partial p_j = 0$*. Remark 2 and Eq.* (B.5) *imply that* $\partial\bar{c}_j(p_j,p_i)/\partial p_j = 0$ *if* $\partial c_j(p_j,p_i)/\partial p_j = 0$ *and, hence,* $\bar{BR}_j(p_i) = BR_j(p_i)$ *given in Theorem 1 and Eq.* (15)*. Similarly,* $d\bar{c}_i(p_i,\bar{BR}_j(p_i))/dp_i = 0$ *if* $dc_i(p_i,BR_j(p_i))/dp_i = 0$ *implies that the equilibrium premium for the Stackelberg leader Insurer* i*,* $\bar{p}_i^S$ *is the same as its counterpart* p_i^S *characterized in Theorem 1 and Eq.* (16)*.*

The arguments in Remark 2 show that Theorem 2 provides the NE premium profile in the case of the Panjer $(a,b,0)$ *class as well.*

Finally, the statement of Theorem 3 also applies to the case of N_i *belonging to the Panjer* $(a,b,0)$ *class. The concluding argument in the theorem's proof is derived by Ineq.* (A.22) *that implies* $c_i(BR_i(p_j^S),p_j^S) \le c_i(BR_i(p_j^N),p_j^N)$*. The last inequality yields* $\bar{c}_i(\bar{BR}_i(\bar{p}_j^S),\bar{p}_j^S) \le \bar{c}_i(\bar{BR}_i(\bar{p}_j^N),\bar{p}_j^N)$ *since* $\bar{c}_i$ *is increasing with respect to* c_i *based on Eq.* (B.4)*.*