Optimal reinsurance design under the moment-based premium principle: a representative reinsurer's perspective

Tim J. Boonen[1], Wenjun Jiang[2], and Yiying Zhang[3]

[1] *Department of Statistics and Actuarial Science, The University of Hong Kong, Hong Kong, P.R. China. E-mail: tjboonen@hku.hk*

[2] *Department of Mathematics and Statistics, University of Calgary, Calgary, AB, T2N 1N4, Canada. E-mail: wenjun.jiang@ucalgary.ca*

[3] *Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, P.R. China. E-mail: zhangyy3@sustech.edu.cn*

Abstract

This paper investigates the optimal reinsurance problem between one insurer and multiple reinsurers, where each reinsurer prices the contract based on the first two moments of the ceded loss, and the insurer aims to minimize a distortion risk measure. We provide a representative reinsurer's perspective to solve the problem; the representative reinsurer's premium principle admits an analytical form and possesses the properties of monotonicity and convexity. This allows us to use a convex programming approach to numerically solve the main problem. If all the reinsurers apply the same safety loading factor for the first moment of the ceded loss in their premium principles, the representative reinsurer's premium principle also relies only on the first two moments of the ceded loss. This significantly reduces the complexity of the original problem, allowing us to use a quadratic programming approach to find the solution.

Keywords: Optimal reinsurance, multiple reinsurers, distortion risk measure, moment-based premium principle.

JEL code: C60, G22

1 Introduction

Reinsurance provides insurance companies with the means to mitigate catastrophic losses and enhance their underwriting capacities. A reinsurance contract consists of two key components - the indemnity function I and the reinsurance premium $\pi(I)$. The indemnity function determines the reinsurance company's payout when a loss occurs, while the reinsurance premium is the compensation paid by the insurance company prior to the contract's activation. The determination of the optimal indemnity function is the primary focus of optimal reinsurance contracting problems, with the premium being closely tied to it. The pioneering works of Borch (1960) and Arrow (1974) applied the expected premium principle and verified the optimality of the excess-of-loss function when the insurance buyer aims to either minimize the variance of the retained loss or maximize the expected utility of her end-of-period wealth. Since then, numerous developments have been observed during the past few decades, which extend the classical results along a variety of directions. To name but a few, Bernard et al. (2015) extended the Arrow's model within the rank-dependent utility (RDU) framework and found that the insurer demands not only insurance for the losses exceeding certain deductible level but also full insurance against small losses. Xu et al. (2019) complemented the study of Bernard et al. (2015) by imposing exogenously the incentive compatibility condition on indemnity functions, which mitigates the potential *ex post* moral hazard issue. Chi and Wei (2020) extended Arrow's model by including a dependent background risk and derived some qualitative properties for the optimal contract. Cao et al. (2024) considered a general convex premium principle and proposed an efficient algorithm to compute the optimal indemnity function under the exponential utility function.

It is well known that the premium principle is one of the key elements that affect the optimal reinsurance contract. In most literature, the optimal reinsurance problems are studied under specific premium principles. To name but a few, Young (1999) and Cheung (2010) studied the optimal reinsurance problems under the Wang's premium principle within the expected utility maximization and Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) minimization frameworks respectively. By investigating as many as seventeen premium principles, Tan et al. (2009) studied the optimal parameters in the excess-of-loss and quota-share functions under the VaR and TVaR preferences. Instead of applying specific premium principles, Chi (2012) proposed the use of a class of premium principles, where only partial properties (e.g., law invariance, risk loading and preserving convex order) of the premium principles are known, in the study of optimal reinsurance problems. In many recent papers on Bowley reinsurance (e.g., Boonen and Zhang, 2022; Boonen and Ghossoub, 2023), the economic premium principle is applied, where a pricing kernel needs to be selected by the reinsurer in response to the contract selected by the insurer. For more premium principles, as well as their applications in the optimal reinsurance contracting, we refer the interested readers to Young (2014) and Cai and Chi (2020).

In this paper we consider a moment-based premium principle, where different safety loading factors are applied to different moments of the ceded loss. The moment-based premium principle is a natural extension of the premium principles built on the first-order moment of the ceded loss, such as the actuarial-value premium principle (Golubin, 2006), which includes the commonly used expected value premium principle as a special case. Moreover, the moment-based premium principle is gaining popularity in the recent studies on dynamic optimal (re)insurance models, such as Lin et al. (2023); Cao et al. (2023a,b). The moment-based premium principle satisfies all the properties listed in Chi (2012), including being law invariant, having risk loading, and preserving the convex order. Unlike the classical mean-variance premium principle, the moment-based premium principle, as will be seen later, preserves the increasing convex order.

A prevailing setting in the realm of optimal reinsurance contracting is that the insurer is doing business with only one reinsurer. In practice, the difference in the pricing strategies of reinsurers, which may partially attribute to the heterogeneity of the beliefs regarding the loss distribution or the preferences and the competition in the market, gives the insurer options to do business simultaneously with more than one reinsurer. Chi and Meng (2014) studied the optimal reinsurance that minimizes the insurer's total risk exposure under VaR or TVaR in the presence of two reinsurers in the market. An extensive generalization work was done by Boonen and Ghossoub (2021), where the optimal reinsurance contracts get fully characterized by considering multiple reinsurers, distortion risk measures, premium budgets, and heterogeneous beliefs. We refer the readers to, for instance, Malamud et al. (2016) and Zhu et al. (2023) for more recent advances in optimal reinsurance with multiple reinsurers under either the expected utility setting or the game-theoretic setting. A pertinent problem is the optimal reinsurance problem with one reinsurer and multiple insurers, see, e.g., Bernard et al. (2020) for recent advancements.

A common way to tackle the optimal reinsurance problem with multiple reinsurers is by characterizing a representative reinsurer. This representative reinsurer represents all reinsurers jointly, and uses a premium principle that the insurer uses to determine the optimal aggregate indemnity. The aggregate indemnity is further allocated to the individual reinsurers, each using their original premium principles. This allows us to solve the optimal reinsurance problem in two stages: the insurer first selects an aggregate indemnity and corresponding aggregate premium, and then the aggregate reinsurance contract is further allocated to the individual reinsurers. In this way, the original problem can be reduced to the problem between one insurer and one reinsurer. Once the optimal contract with the representative reinsurer is identified, distributing the indemnities and premiums among the individual reinsurers is another risk sharing problem. See, for example, Boonen et al. (2016), and Boonen and Ghossoub (2019, 2021) for the introduction and application of this method.

In comparison to the existing literature, our paper's contribution is three-fold. Firstly, we derive full analytical solutions for the optimal reinsurance problem with multiple reinsurers under the moment-based premium principles. Secondly, we provide a representative reinsurer's perspective, and derive the closed form of the representative reinsurer's premium principle. We demonstrate the monotonicity and convexity properties for this premium principle. Under certain circumstances, the representative reinsurer's premium principle is shown to be explicit, and also moment-based. Lastly, efficient numerical approaches, like convex programming and quadratic programming, are applied to solve the optimal reinsurance problem. This paper contributes to the thin literature on computational (re)insurance contracting, in which the optimal contracts are approximated numerically when the closed-form solutions are not easily derivable. While the closed-form solutions are of the main interest in most literature, where the roles played by different economic factors are more straightforwardly observable, there is a growing trend of using numerical methods to tackle the optimal (re)insurance problems in the recent literature. The representative works on this track are, to name but a few, Asimit et al. (2017, 2018), which studied the optimal insurance/risk sharing problems in the presence of model uncertainty or other economic and solvency constraints. While our primary focus is on second-order moment-based premium principles, the results can be seamlessly extended to cases where higher-order moment-based premium principles are applied.

The remainder of this paper is structured as follows. Section 2 provides a quick review of some preliminaries. Section 3 sets up the problem. Section 4 solves the main problem by novelly combining the theoretical and numerical approaches, where the former simplifies the main problem to a tractable form via employing the notion of representative reinsurer, and the latter provides an efficient manner to solve the simplified problem. Section 5 studies in detail a special case where all the safety loading factors for the first-order moment are equal. Section 6 studies a classical setting with only expected value premium principles as a special case. Section 7 concludes the paper and outlines potential avenues for future research.

2 Preliminaries for distortion risk measures

Throughout the paper, we use the notations $(x)_+ = \max\{x, 0\}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. The indicator function $\mathbb{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.

Distortion risk measures stem from the dual utility theory of Yaari (Yaari, 1987), and provide ways to evaluate uncertain losses or gains under distorted probability measures. A distortion risk measure of a bounded random variable Z on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is given by:

$$\rho_g(Z) = \int_0^\infty g(\mathbf{P}(Z > z))dz + \int_{-\infty}^0 [g(\mathbf{P}(Z > z)) - 1]dz, \tag{2.1}$$

where g is called the distortion function that is increasing on $[0, 1]$ and satisfies $g(0) = 0$ and $g(1) = 1$. In this paper, we only focus on the non-negative variables, which conventionally represent losses in risk management and actuarial science. For this point, the distortion risk measures only involve the first term of (2.1). Basically, distortion risk measures satisfy the following properties (see Chapter 2 of Denuit et al. (2006)):

- *Translation invariance*: $\rho_g(Z+c) = \rho_g(Z)+c$ for any constant $c \in \mathbf{R}$ and all random variables Z.

- *Comonotonic additivity*: $\rho_g(Z_1 + Z_2) = \rho_g(Z_1) + \rho_g(Z_2)$ for any pair of comonotonic random variables Z_1 and Z_2.[1]

As will be shown in the next section, these two properties will play crucial roles in the subsequent analysis.

If the distortion function g is concave, then the distortion risk measure ρ_g is coherent in the sense of Artzner et al. (1999). To keep generality, we do not assume the concavity of the distortion function g. Many well-known risk measures being used in the banking industry are special cases of distortion risk measures, such as VaR and TVaR. In insurance industry, the proportional hazards (PH) transform and Wang transform (Wang, 1995, 2000), which have been widely applied in risk-adjusted ratemaking and pricing, are also within the family of distortion risk measures. We present below the definitions of some of the above-mentioned risk measures, which will be used in the later sections.

Definition 2.1. *The VaR of a random variable Z at probability level $\alpha \in (0,1)$ is given by*

$$VaR_\alpha(Z) = F_Z^{-1}(\alpha).$$

The distortion function of VaR is given by $g_V(t) = \mathbb{1}_{t>1-\alpha}(t)$, $t \in [0,1]$.

Definition 2.2. *The TVaR of a random variable Z at probability level $\alpha \in (0,1)$ is given by*

$$TVaR_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(Z)ds.$$

The distortion function of TVaR is given by $g_T(t) = 1 \wedge \frac{t}{1-\alpha}$, $t \in [0,1]$.

Definition 2.3. *Given some exogenous index $r \in (0,1]$, the non-negative random variable Y is said to be the PH transform of a non-negative random variable Z if $\mathbf{P}(Y > z) = \big(\mathbf{P}(Z > z)\big)^r$, and the resulting expectation becomes $\mathbf{E}[Y] = \int_0^\infty \mathbf{P}(Y > z)dz = \int_0^\infty \big(\mathbf{P}(Z > z)\big)^r dz$. The distortion function of PH transform is $g_{PH}(t) = t^r$, $t \in [0,1]$.*

3 Insurer's problem

We focus on a one-period economy where there are one insurer and n reinsurers in the market. The set of reinsurers is indexed as $\{1, \ldots, n\}$. The insurer is faced with a non-negative random loss X, which is built on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and can only be realized at the end of the period. We assume that $M := \operatorname{ess\,sup} X < \infty$ and X admits a probability density function $f_X(x)$ on $(0, M]$. The cumulative distribution function and survival function of X are given by $F_X(x)$ and $S_X(x) := 1 - F_X(x)$, respectively.

The insurer is now seeking a reinsurance contract (I_i, π_i) from reinsurers, where I_i is known as the indemnity function which maps the loss to indemnity and π_i is the premium paid to reinsurer

[1] The random variables Z_1 and Z_2 are said to be comonotonic if and only if $(Z_1, Z_2) =_d (F_{Z_1}^{-1}(U), F_{Z_2}^{-1}(U))$, where $=_d$ stands for equality in distribution and $F_Z^{-1}(\alpha) := \inf\{z : F_Z(z) \geq \alpha\}$ for $\alpha \in (0,1)$.

$i \in \{1, \ldots, n\}$. To mitigate the potential *ex post* moral hazard issue (Huberman et al., 1983; Xu et al., 2019), we restrict the indemnity function to the following class

$$\mathcal{I} = \left\{ I : [0, M] \mapsto [0, M] \;\middle|\; I(0) = 0, 0 \le I(x_2) - I(x_1) \le x_2 - x_1 \text{ for any } 0 \le x_1 \le x_2 \right\}. \tag{3.1}$$

It is noteworthy that if $I \in \mathcal{I}$ the retained loss borne by the insurer is increasing and continuous with respect to the total loss, and the increment of payment does not exceed the increment of loss. These would disincentivize the insurer from manipulating her loss. The class $\mathcal{I}$ is rich enough and includes many commonly used indemnity functions, such as the excess-of-loss function $I(x) = (x - d)_+$ for some $d \ge 0$, the quota-share function $I(x) = c \cdot x$ for some $c \in [0, 1]$, the limited excess-of-loss function $I(x) = d_2 \wedge (x - d_1)_+$ for some $d_1, d_2 \ge 0$ and so on. Furthermore, it is straightforward that any $I \in \mathcal{I}$ is 1-Lipschitz continuous and admits the integral representation

$$I(x) = \int_0^x I'(t)dt,$$

where I' is called the *marginal indemnity function*. $I(\cdot)$ and $I'(\cdot)$ are mutually determined. Hence, seeking the optimal indemnity function is equivalent to seeking its marginal indemnity function, and vice versa.

We write $\overrightarrow{I} = (I_1, I_2, \ldots, I_n)$ as the set of indemnity functions that the insurer purchases from the reinsurers, and $\overrightarrow{I} \in \overrightarrow{\mathcal{I}}$, where

$$\overrightarrow{\mathcal{I}} = \left\{ (I_1, I_2, \ldots, I_n) : \; I_i \in \mathcal{I} \text{ for all } i \in \{1, \ldots, n\} \quad \text{and} \quad \sum_{k=1}^{n} I_k \in \mathcal{I} \right\} \tag{3.2}$$

such that the *ex post* moral hazard issues can be minimized. We assume that the i-th reinsurer adopts the second-moment premium principle

$$\pi_i(I_i) = (1 + \theta_i)\mathbf{E}[I_i(X)] + \frac{\gamma_i}{2}\mathbf{E}[I_i(X)^2], \tag{3.3}$$

where $\theta_i, \gamma_i \ge 0$ are the safety loading factors picked by the i-th reinsurer. Such premium principle starts to enjoy its popularity in the study of dynamic optimal reinsurance (Lin et al., 2023; Cao et al., 2023a,b). Also, they can be seen as special case of the premium principle given by $\mathbf{E}[h_i(I_i(X))]$ for a convex, increasing function h_i with $h_i(0) = 0$ that exceeds the identity function. This class is studied by Ghossoub et al. (2023), and our focus is on the function $h_i(x) = \psi_i(x) := (1+\theta_i)x + \frac{\gamma_i}{2}x^2$. The function ψ_i is an increasing convex function, and $\pi_i(I_i) = \mathbf{E}[\psi_i(I_i(X))]$. Thus, π_i can preserve the increasing convex (or stop-loss) order of $I_i(X)$.[2] Moreover, we note that $\pi_i(I_i)$ can also be written as

$$\pi_i(I_i) = \psi_i(\mathbf{E}[I_i(X)]) + \frac{\gamma_i}{2}\mathrm{Var}[I_i(X)],$$

which can be regarded as a transformed variance premium principle.[3]

[2]Given two indemnity functions $I_i, \hat{I}_i \in \mathcal{I}$, $\pi_i(I_i) \le \pi_i(\hat{I}_i)$ if $I_i(X) \le_{icx} \hat{I}_i(X)$ where $\le_{icx}$ denotes the increasing convex (or stop-loss) order. For more details, see Shaked and Shanthikumar (2007).

[3]Without the variance part, $\pi_i(I_i) = \psi_i(\mathbf{E}[I_i(X)])$ gives the premium based on the actuarial value of the indemnity, where ϕ_i satisfies $\phi_i(0) = 0$ and $\phi_i'(x) \ge 1$ for $x \ge 0$. Such premium principle can be found in, for example, Golubin (2006); Boonen and Jiang (2022).

The premium principle in (3.3) clearly satisfies the following two properties (Ghossoub et al., 2023):

- *Monotonicity*: If $I_i(X) \le \hat{I}_i(X)$ a.s., then $\pi_i(I_i) \le \pi_i(\hat{I}_i)$;
- *Convexity:* If $I_i, \hat{I}_i \in \mathcal{I}$ and $\lambda \in [0,1]$, then $\pi_i(\lambda I_i + (1-\lambda)\hat{I}_i) \le \lambda \pi_i(I_i) + (1-\lambda)\pi_i(\hat{I}_i)$.

With the reinsurance contract, the insurer's end-of-period loss becomes $X - \sum_{k=1}^n I_k(X) + \sum_{k=1}^n \pi_k(I_k)$. Throughout the paper, we assume that the insurer's preference is dictated by the distortion risk measure ρ_g.

Problem 1.

$$\min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}} \; L(\overrightarrow{I}),$$

where $L(\overrightarrow{I}) := \rho_g\left(X - \sum_{k=1}^n I_k(X) + \sum_{k=1}^n \pi_k(I_k)\right)$.

We next show the existence and uniqueness of the solution to Problem 1.

Theorem 3.1. *There exists a solution to Problem 1. Furthermore, if* $\gamma_i > 0$ *for all* $i \in \{1, \ldots, n\}$*, the solution to Problem 1 is unique in the sense that* $\overrightarrow{I_1}(X) = \overrightarrow{I_2}(X)$ $\mathbf{P}$*-a.s. if both* $\overrightarrow{I_1}$ *and* $\overrightarrow{I_2}$ *solve Problem 1.*

Proof. We write $\overrightarrow{I_j} = \{I_{jk}\}_{k=1}^n$ for $j \in \mathbb{Z}^+$. For the set $\mathcal{I}^n := \mathcal{I} \times \cdots \times \mathcal{I}$, define the metric

$$D(\overrightarrow{I_1}, \overrightarrow{I_2}) = \max_{x \in [0,M]} ||\overrightarrow{I_1}(x) - \overrightarrow{I_2}(x)||_1, \tag{3.4}$$

where $||\cdot||_1$ denotes the first-order norm. By the translation invariance and comonotonic additivity of the distortion risk measure, we have

$$\rho_g\left(X - \sum_{k=1}^n I_k(X) + \sum_{k=1}^n \pi_k(I_k)\right) = \rho_g(X) - \rho_g\left(\sum_{k=1}^n I_k(X)\right) + \sum_{k=1}^n \pi_k(I_k).$$

Note that

$$\begin{aligned}
\operatorname{ess\,sup} \left| \sum_{k=1}^n I_{1k}(X) - \sum_{k=1}^n I_{2k}(X) \right| &\le \operatorname{ess\,sup} \sum_{k=1}^n |I_{1k}(X) - I_{2k}(X)| \\
&= \max_{x \in [0,M]} \sum_{k=1}^n |I_{1k}(x) - I_{2k}(x)| \\
&= \max_{x \in [0,M]} ||\overrightarrow{I_1}(x) - \overrightarrow{I_2}(x)||_1 = D(\overrightarrow{I_1}, \overrightarrow{I_2}),
\end{aligned}$$

and Theorem 1 of Wang et al. (2020) shows the uniform sup-continuity of distortion riskmetrics, which include distortion risk measures as special cases. Thus, $\rho_g(\sum_{k=1}^n I_k(X))$ is continuous in $\overrightarrow{I}$ under the metric (3.4). Since the expectation is a special distortion risk measure, it can be proved similarly that $\sum_{k=1}^n \pi_k(I_k)$ is also continuous in $\overrightarrow{I}$ under the metric (3.4). This leads to the continuity of $L(\overrightarrow{I})$ with respect to $\overrightarrow{I}$. Thus, the minimum of $L(\overrightarrow{I})$ is attainable if the set $\overrightarrow{\mathcal{I}}$ is compact. The following results are direct based on the definition of $\mathcal{I}$:

(i). The functions in $\mathcal{I}$ are equicontinuous (due to their 1-Lipschitz continuity).

(ii). The functions in $\mathcal{I}$ are uniformly bounded (by M).

(iii). The 1-Lipschitz continuity is preserved under the uniform convergence.

Then, as per Arzelà-Ascoli Theorem, the set $\mathcal{I}$ is sequentially compact, or equivalently, compact. By Tychonoff's theorem, the product space $\mathcal{I}^n$ is compact. Since $\overrightarrow{\mathcal{I}}$ is a closed subset of $\mathcal{I}^n$, we get the compactness of $\overrightarrow{\mathcal{I}}$. Hence, the the solution to Problem 1 exists.

If both $\overrightarrow{I_1}$ and $\overrightarrow{I_2}$ solve Problem 2 and $\overrightarrow{I_1}(X)$ is not equal to $\overrightarrow{I_2}(X)$ $\mathbf{P}$-a.s., then let $\overrightarrow{I_3} = \lambda\overrightarrow{I_1} + (1-\lambda)\overrightarrow{I_2}$ for arbitrary $\lambda \in (0,1)$. We obtain

$$
\begin{aligned}
L(\overrightarrow{I_3}) =& \rho_g(X - \sum_{k=1}^{n} I_{3k}(X) + \sum_{k=1}^{n} \pi(\tilde{I}_{3k})) \\
=& \rho_g(X) - \sum_{k=1}^{n} \rho_g(I_{3k}(X)) + \sum_{k=1}^{n}(1+\theta_k)\mathbf{E}[I_{3k}(X)] + \sum_{k=1}^{n} \frac{\gamma_k}{2}\mathbf{E}[I_{3k}(X)^2] \\
=& \rho_g(X) - \sum_{k=1}^{n} \big(\lambda\rho_g(I_{1k}(X)) + (1-\lambda)\rho_g(I_{2k}(X)\big) \\
&+ \sum_{k=1}^{n}(1+\theta_k)\big(\lambda\mathbf{E}[I_{1k}(X)] + (1-\lambda)\mathbf{E}[I_{2k}(X)]\big) \\
&+ \sum_{k=1}^{n} \frac{\gamma_k}{2}\mathbf{E}[(\lambda I_{1k}(X) + (1-\lambda)I_{2k}(X))^2] \\
\leq& \rho_g(X) - \left(\lambda\sum_{k=1}^{n} \rho_g(I_{1k}(X)) + (1-\lambda)\sum_{k=1}^{n} \rho_g(I_{2k}(X)\right) \\
&+ \lambda\sum_{k=1}^{n}(1+\theta_k)\mathbf{E}[I_{1k}(X)] + (1-\lambda)\sum_{k=1}^{n}(1+\theta_k)\mathbf{E}[I_{2k}(X)] \\
&+ \lambda\sum_{k=1}^{n} \frac{\gamma_k}{2}\mathbf{E}[I_{1k}(X)^2] + (1-\lambda)\sum_{k=1}^{n} \frac{\gamma_k}{2}\mathbf{E}[I_{2k}(X)^2] \\
=& \lambda L(\overrightarrow{I_1}) + (1-\lambda)L(\overrightarrow{I_2}),
\end{aligned}
$$

where the equality is achieved only when $\overrightarrow{I_1}(X) = \overrightarrow{I_2}(X)$ $\mathbf{P}$-a.s.. This contradicts with that $\overrightarrow{I_1}$ and $\overrightarrow{I_2}$ solve Problem 2. Thus, the uniqueness of the solution is shown, which completes the proof. □

We close this section by remarking on two aspects of Theorem 3.1.

(i). Theorem 3.1 uses essentially only the convexity, translation invariance and sup-continuity of the distortion risk measure, as well as the convexity and sup-continuity of the second-order-moment-based premium principle, to prove the existence and uniqueness of the solution to our problem. The result still holds for the cases of which the risk measures and premium principles possess the same properties.

(ii). The optimal reinsurance contracting problem between one insurer and multiple reinsurers is also studied by Boonen and Ghossoub (2021) under belief heterogeneity on loss distributions, where distortion-risk-measure-based premium principles are applied. Based on their Theorem 3.1, it generally cannot be guaranteed that the optimal way of sharing the risk among the multiple agents is unique. Note that the expectation is a special case of the distortion risk measure, and by Theorem 3.1 the uniqueness of the solution to our problem cannot be guaranteed either if all the reinsurers price the contracts by using only the first moments of indemnities. However, when all the reinsurers incorporate the second moments of indemnities into the pricing, the solution becomes unique.

4 Representative reinsurer

This section will provide our main results, and characterize the representative reinsurer in our general model setting. To do so, we first define

$$\overrightarrow{\mathcal{I}}(I) := \left\{ \overrightarrow{I} : \ \overrightarrow{I} \in \overrightarrow{\mathcal{I}} \text{ and } \sum_{k=1}^{n} I_k(x) = I(x) \text{ for all } x \in [0, M] \right\},$$

for $I \in \mathcal{I}$. By construction, it holds that $\overrightarrow{\mathcal{I}} = \{\overrightarrow{\mathcal{I}}(I) : I \in \mathcal{I}\}$, and so Problem 1 is equivalent to solving

$$\min_{I \in \mathcal{I}} \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} L(\overrightarrow{I}).$$

Then, this can be further simplified to

$$\begin{aligned}
\min_{I \in \mathcal{I}} \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} L(\overrightarrow{I}) &= \min_{I \in \mathcal{I}} \left\{ \rho_g \left(X - I(X) + \left\{ \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^{n} \pi_k(I_k) \right\} \right) \right\} \\
&= \min_{I \in \mathcal{I}} \left\{ \rho_g(X - I(X)) + \left\{ \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^{n} \pi_k(I_k) \right\} \right\} \\
&= \min_{I \in \mathcal{I}} \left\{ \rho_g(X) - \rho_g(I(X)) + \left\{ \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^{n} \pi_k(I_k) \right\} \right\},
\end{aligned}$$

where the first equality follows from monotonicity of ρ_g (see Boonen and Ghossoub, 2021), the second equality follows from translation invariance of ρ_g, and the last equality follows from comonotonitic additivity of ρ_g. To proceed, we first solve the inner minimization problem, which is

$$\min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^{n} \pi_k(I_k(X)), \tag{4.1}$$

where $I \in \mathcal{I}$. In the remaining of this section, we assume without loss of generality that

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_n. \tag{4.2}$$

The next theorem presents the full analytical solution to Problem (4.1).

Theorem 4.1. *If $\theta_i \geq 0$ and $\gamma_i > 0$ for all $i \in \{1, 2, \ldots, n\}$, the optimal indemnity functions $(I_1^*, I_2^*, \ldots, I_n^*)$ that solve Problem (4.1) are given by*

$$I_i^*(x) = \begin{cases} \dfrac{\lambda(x) - (1+\theta_i)}{\gamma_i}, & \text{if } 1+\theta_i \le \lambda(x), \\ 0, & \text{if } 1+\theta_i > \lambda(x), \end{cases} \tag{4.3}$$

for any $x \in [0, M]$, where $\lambda(x)$ solves the equation

$$\sum_{\{k:\ 1+\theta_k \le \lambda(x)\}} \frac{\lambda(x) - (1+\theta_k)}{\gamma_k} = I(x). \tag{4.4}$$

Proof. We define

$$\mathcal{I}_0(\bar{I}) := \left\{ I : [0, M] \mapsto [0, M] \mid 0 \le I(x) \le \bar{I}(x) \text{ for all } x \in [0, M] \right\}.$$

Note that

$$\begin{aligned} \sum_{k=1}^n \pi_k(I_k) &= \sum_{k=1}^n \left\{ (1+\theta_k)\mathbf{E}[I_k(X)] + \frac{\gamma_k}{2}\mathbf{E}[I_k(X)^2] \right\} \\ &= \sum_{k=1}^n \left\{ \frac{\gamma_k}{2}\mathbf{E}\left[\left(I_k(X) + \frac{1+\theta_k}{\gamma_k} \right)^2 \right] - \frac{(1+\theta_k)^2}{2\gamma_k} \right\} \\ &= \mathbf{E}\left[\sum_{k=1}^n \frac{\gamma_k}{2} \left(I_k(X) + \frac{1+\theta_k}{2} \right)^2 \right] - \sum_{k=1}^n \frac{(1+\theta_k)^2}{2\gamma_i}. \end{aligned}$$

Instead of solving Problem (4.1) directly, we first consider at the following problem:

$$\begin{aligned} \min_{I_1, \ldots, I_n \in \mathcal{I}_0(I)} \ & \int_0^M \left(\sum_{k=1}^n \frac{\gamma_i}{2} \left(I_k(x) + \frac{1+\theta_k}{\gamma_k} \right)^2 \right) dF_X(x) \\ \text{s.t. } & \sum_{k=1}^n I_k(x) \ge I(x) \text{ for all } x \in [0, M]. \end{aligned} \tag{4.5}$$

Since the objective function of (4.5) is increasing in $I_1, \ldots, I_n$, it is straightforward to see that the optimal $(I_1^*, \ldots, I_n^*)$ for (4.5) will always make the constraint binding. Similar to the proof of Corollary 1 in Malamud et al. (2016), solving Problem (4.5) is equivalent to solving

$$\min_{I_1, \ldots, I_n \in \mathcal{I}_0} \ L(I_1, \ldots, I_n) := \sum_{i=1}^n \frac{\gamma_i}{2} \left(I_i(x) + \frac{1+\theta_i}{\gamma_i} \right)^2 - \lambda(x) \left(\sum_{i=1}^n I_i(x) - I(x) \right) \tag{4.6}$$

for any fixed $x \in [0, M]$, where $\lambda(x) \geq 0$ is the Lagrangian multiplier. The first-order condition $\frac{\partial L}{\partial I_k} = 0$ yields

$$\gamma_k I_k(x) + (1 + \theta_k) = \lambda(x),$$

and, thus,

$$I_k^*(x) = 0 \vee \left(\frac{\lambda(x) - (1 + \theta_k)}{\gamma_k} \right). \tag{4.7}$$

Here the Lagrangian multiplier $\lambda(x)$ satisfies the equation

$$\sum_{k=1}^{n} \left\{ 0 \vee \left(\frac{\lambda(x) - (1 + \theta_k)}{\gamma_k} \right) \right\} = I(x), \tag{4.8}$$

which also implies that $I_k^*(x) \leq I(x)$ for $k = 1, 2, \ldots, n$. Under the assumption (4.2) and the condition $\gamma_i > 0$ for all $i \in \{1, \ldots, n\}$, if $\lambda(x) \in [1 + \theta_j, 1 + \theta_{j+1})$, (4.8) reduces to

$$\sum_{k=1}^{j} \frac{\lambda(x) - (1 + \theta_k)}{\gamma_k} = I(x) \implies \lambda(x) = I(x) \left(\sum_{k=1}^{j} \gamma_k^{-1} \right)^{-1} + \left(\sum_{k=1}^{j} \frac{1 + \theta_k}{\gamma_k} \right) \left(\sum_{k=1}^{j} \gamma_k^{-1} \right)^{-1},$$

from which one has

$$I_k^*(x) = \frac{\lambda(x) - (1 + \theta_k)}{\gamma_k} \text{ for } k \leq j \quad \text{and} \quad I_k^*(x) = 0 \text{ for } k > j. \tag{4.9}$$

As such,

$$I_k^{*\prime}(x) = \frac{\gamma_k^{-1}}{\sum_{s=1}^{j} \gamma_s^{-1}} \in (0, 1) \text{ for } k \leq j \quad \text{and} \quad I_k^{*\prime}(x) = 0 \text{ for } k > j,$$

which implies that the optimal indemnity functions characterized via (4.7) are within $\mathcal{I}$ and thus also solve the problem (4.1). The proof is complete. □

For $x \in [0, M]$ we define

$$H_k^*(x) = 1 + \theta_k + \gamma_k I_k^*(x) \quad \text{and} \quad \mathcal{K}^*(x) = \underset{1 \leq k \leq n}{\arg\min} \, \{H_k^*(x)\}.$$

Then, (4.9) implies that if $\lambda(x) \in [1 + \theta_j, 1 + \theta_{j+1})$ we have

$$H_1^*(x) = H_2^*(x) = \cdots = H_j^*(x) = \lambda(x) < \min_{k>j} \{1 + \theta_k\} = \min_{k>j} \{H_k^*(x)\},$$

and

$$\mathcal{K}^*(x) = \{k : 1 + \theta_k \leq \lambda(x)\} = \{1, 2, \ldots, j\}.$$

The implication of Theorem 4.1 is as follows: under the second-order moment-based premium principle, the safety loading factor for the first-order moment determines whether or not a reinsurer participates into the risk sharing, while the loading factor for the second-order moment determines the proportion of loss ought to be borne by the reinsurer.

Remark 4.1. *Theorem 4.1 is closely related to the results in Borch (1962) (see also Section 8 of Gerber and Pafumi, 1998), where the following utility-based Pareto-optimal risk exchange model is studied:*

$$\max_{X_1,\dots,X_n,\ \sum_{k=1}^n X_k = X} \sum_{k=1}^n \mathbf{E}[u_k(W_k - X_k)], \tag{4.10}$$

where u_k, W_k and X_k are the utility function, initial wealth, and allocated risk of the kth individual, respectively. Under the quadratic utility function: $u_i(x) = x - b_i x^2$ for $x \le \frac{1}{2b_i}$ and $b_i > 0$, Borch (1962) showed that the optimal allocated risk for the ith individual is of the simple form $X_i^* = c_i X + d_i$ where $c_i \in [0,1]$ and $d_i \in \mathbf{R}$ are interpreted as a proportionality factor and a side payment, respectively.

Note that under the quadratic utility function the problem (4.10) reduces to

$$\min_{X_1,\dots,X_n,\ \sum_{k=1}^n X_k = X} \left\{ \sum_{k=1}^n (1 - 2b_k W_k)\mathbf{E}[X_k] + b_k \mathbf{E}[X_k^2] \right\}, \tag{4.11}$$

which is the same as the problem (4.1). Therefore, Theorem 4.1 gives the solution to (4.11) by restricting $d_i = 0$ for $i = 1, 2, \dots, n$ (or no side payments are allowed): $X_i^* = I_i^*(X)$, where $I_i^{*\prime}(x) = \frac{b_i^{-1}}{\sum_{k \in \mathcal{K}^*(x)} b_k^{-1}}$ if $i \in \mathcal{K}^*(x)$ and $I_i^{*\prime}(x) = 0$ otherwise.

Our model formulation bears some similarity with the sup-convolution problem studied by Acciaio (2006) (see Section 4.1 therein) in optimal risk allocation:

$$\sup_{X_1,\dots,X_n,\ \sum_{k=1}^n X_k = X} \sum_{k=1}^n V_k(-X_k), \tag{4.12}$$

where all the agents are endowed with monetary utility functionals V_i, $i \in \{1, \dots, n\}$.[4] *However, the sup-convolution problem (4.12) differs from our problem in several ways. First, the exogenous risk loadings θ_i in our optimal insurance setting lead to insurer-specific weights of the expectation, and weights do not appear in (4.12). Second, second-moment-based premium principles are not translation invariant unless $\theta_i = \gamma_i = 0$. Third, comonotonicity is not a common assumption for feasible risk allocations, whereas it is common for reinsurance contracts (Huberman et al., 1983; Xu et al., 2019).*

With the characterization of the optimal indemnity functions that solve Problem (4.1), we can also derive the optimal value for the objective function in (4.1). The following notations will be useful in stating the next result:

$$\kappa_1(x) = \sum_{j=0}^{n-1} \left\{ \sum_{k=1}^{j+1} w_k(j)(1+\theta_k) \mathbb{1}_{[x_j, x_{j+1})}(x) \right\},$$

$$\kappa_2(x) = \sum_{j=0}^{n-1} \left\{ \left(\sum_{k=1}^{j+1} \gamma_k^{-1} \right)^{-1} \mathbb{1}_{[x_j, x_{j+1})}(x) \right\},$$

[4] A mapping V_i is called a monetary utility functional if it is translation invariant, concave and monotonic (Acciaio, 2006).

where $w_k(j) = \frac{\gamma_k^{-1}}{\sum_{s=1}^{j+1} \gamma_s^{-1}}$, and

$$x_0 = 0, \quad x_n = M,$$
$$x_j = \inf\left\{ x \in [0, M] : I(x) \geq \sum_{i=1}^{j} \frac{\theta_{j+1} - \theta_i}{\gamma_i} \right\}, \text{ for } j = 1, 2, \ldots, n-1,$$

where $\inf \emptyset$ is defined as the right-end point M by convention.

Theorem 4.2. *Let $\gamma_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. The optimal value for the objective function in Problem (4.1) is*

$$\pi(I) = \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^{n} \pi_k(I_k(X)) = \int_0^M \left\{ \kappa_1(x) + \kappa_2(x) I(x) \right\} S_X(x) dI(x). \tag{4.13}$$

Moreover, it holds that $\mathcal{K}^*(z) = \{1, \ldots, m\}$, where $m \in \{1, \ldots, n\}$ is such that $z \in [x_{m-1}, x_m)$.

Proof. If $x_1 := \inf\left\{ x \in [0, M] : I(x) \geq \frac{\theta_2 - \theta_1}{\gamma_1} \right\} > 0$, then we have

$$1 + \theta_1 + \gamma_1 I(x_1) = 1 + \theta_2.$$

In other words, for $x \in [0, x_1)$

$$H_1^*(x) = 1 + \theta_1 + \gamma_1 I_1^*(x) < \min\{1 + \theta_2, \ldots, 1 + \theta_n\} \leq \min\{H_2^*(x), \ldots, H_n^*(x)\}.$$

Therefore, $\mathcal{K}^*(x) = \{1\}$ for $x \in [0, x_1)$. Hence, by Theorem 4.1, $I_1^*(x) = I(x)$ and $I_i^*(x) = 0$ for $i \neq 1$ for $x \in [0, x_1)$.

Since $H_1^*(x_1) = 1 + \theta_1 + \gamma_1 I_1^*(x_1) = 1 + \theta_2 = H_2^*(x_1)$, by the proof of Theorem 4.1, we have $H_1^*(x) = H_2^*(x)$ for $x \in [x_1, M]$. If $x_2 > x_1$, then it holds that

$$\frac{\theta_3 - \theta_2}{\gamma_2} + \frac{\theta_3 - \theta_1}{\gamma_1} > \frac{\theta_2 - \theta_1}{\gamma_1},$$

which is equivalent to $\theta_3 > \theta_2$. Then we have

$$H_1^*(x_1) = H_2^*(x_1) < \min\{1 + \theta_3, \ldots, 1 + \theta_n\} \leq \min\{H_3^*(x_1), \ldots, H_n^*(x_1)\}.$$

By the continuity of $H_i^*(x)$, there exists some $\epsilon > 0$ such that $H_1^*(x) = H_2^*(x) < \min\{H_3^*(x), \ldots, H_n^*(x)\}$ for $x \in [x_1, x_1 + \epsilon)$. As such, by Theorem 4.1, we have

$$I_1^{*\prime}(x) = \frac{\gamma_1^{-1}}{\gamma_1^{-1} + \gamma_2^{-1}} I'(x) \quad \text{and} \quad I_2^{*\prime}(x) = \frac{\gamma_2^{-1}}{\gamma_1^{-1} + \gamma_2^{-1}} I'(x)$$

for $x \in [x_1, x_1 + \epsilon)$. Note that

$$\begin{aligned} H_2^*(x_2) =& 1 + \theta_2 + \gamma_2 I_2^*(x_2) \\ =& 1 + \theta_2 + \gamma_2 \left(\int_0^{x_1} I_2^{*\prime}(x) dx + \int_{x_1}^{x_2} I_2^{*\prime}(x) dx \right) \end{aligned}$$

$$
\begin{aligned}
&=1+\theta_2+\gamma_2\cdot\frac{\gamma_2^{-1}}{\gamma_1^{-1}+\gamma_2^{-1}}(I(x_2)-I(x_1))\\
&=1+\theta_2+\frac{1}{\gamma_1^{-1}+\gamma_2^{-1}}\left(\frac{\theta_3-\theta_2}{\gamma_2}+\frac{\theta_3-\theta_1}{\gamma_1}-\frac{\theta_2-\theta_1}{\gamma_1}\right)\\
&=1+\theta_3.
\end{aligned}
$$

Hence, for $x\in[x_1,x_2)$,

$$
H_2^*(x)=1+\theta_2+\gamma_2 I_2^*(x)<1+\theta_3\le\min\left\{H_3^*(x),\dots,H_n^*(x)\right\}.
$$

This shows that $\mathcal{K}^*(x)=\{1,2\}$ for $x\in[x_1,x_2)$.

By mathematical induction, we can show that $\mathcal{K}^*(x)=\{1,2,\dots,j+1\}$ for $x\in[x_j,x_{j+1})$. Furthermore, when $x\in[x_j,x_{j+1})$, we have $H_{k_1}^*(x)=H_{k_2}^*(x)$ for $k_1,k_2\in\mathcal{K}^*(x)$, which leads to

$$
\begin{aligned}
&1+\theta_1+\gamma_1 I_1^*(x)=1+\theta_k+\gamma_k I_k^*(x)\\
\Longrightarrow\ &I_k^*(x)=\frac{\theta_1-\theta_k}{\gamma_k}+\frac{\gamma_1}{\gamma_k}I_1^*(x)
\end{aligned}
$$

if $k\in\mathcal{K}^*(x)$. As such, when $x\in[x_j,x_{j+1})$ one has

$$
\begin{aligned}
\sum_{k\in\mathcal{K}^*(x)} I_k^*(x)=I(x)\ &\Longrightarrow\ \sum_{k=2}^{j+1}\frac{\theta_1-\theta_k}{\gamma_k}+\sum_{k=2}^{j+1}\frac{\gamma_1}{\gamma_k}I_1^*(x)+I_1^*(x)=I(x)\\
&\Longrightarrow\ I_1^*(x)=\frac{I(x)}{1+\sum_{k=2}^{j+1}\gamma_1\gamma_k^{-1}}+\frac{\sum_{k=2}^{j+1}\frac{\theta_1-\theta_k}{\gamma_k}}{1+\sum_{k=2}^{j+1}\gamma_1\gamma_k^{-1}}\\
&\Longrightarrow\ 1+\theta_1+\gamma_1 I_1^*(x)=\frac{I(x)}{\sum_{k=1}^{j+1}\gamma_k^{-1}}+\frac{\sum_{k=1}^{j+1}(1+\theta_k)\gamma_k^{-1}}{\sum_{k=1}^{j+1}\gamma_k^{-1}}\\
&\Longrightarrow\ H_1^*(x)=\kappa_1(x)+\kappa_2(x)I(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pi(I)&=\int_0^M\left(\sum_{k=1}^n H_k^*(x)I_k^{*\prime}(x)\right)S_X(x)dx\\
&=\int_0^M\left(\sum_{k\in\mathcal{K}^*(x)} H_k^*(x)I_k^{*\prime}(x)\right)dx\\
&=\int_0^M H_1^*(x)\sum_{k\in\mathcal{K}^*(x)} I_k^{*\prime}(x)S_X(x)dx\\
&=\int_0^M H_1^*(x)I'(x)S_X(x)dx\\
&=\int_0^M\left\{(\kappa_1(x)+\kappa_2(x)I(x))I'(x)\right\}S_X(x)dx.
\end{aligned}
$$

This ends the proof. □

Remark 4.2. *The two recent works on using the representative reinsurer method to tackle the optimal reinsurance problem between one insurer and multiple reinsurers are Boonen et al. (2016) and Boonen and Ghossoub (2019), where the distortion premium principles are applied by the reinsurers for reinsurance pricing. In both articles, given the total ceded loss $I(x)$ the rule of loss sharing among the participating reinsurers might not be unique, which is one of the main distinctions that differs Theorem 4.1 from the existing results. Moreover, Theorem 4.2 shows that the representative reinsurer's premium principle is non-linear in the indemnity function I, while in both Boonen et al. (2016) and Boonen and Ghossoub (2019) the representative reinsurer's premium principles are linear in I.*

Theorem 4.2 provides an explicit formulation of the set $\mathcal{K}^*(z)$ that does not depend on the optimal indemnities $(I_1^*, \ldots, I_n^*)$, and thus the indemnity functions solving Problem (4.1), as stated in (4.3), are in closed form.

Collecting the results in Theorems 4.1 and 4.2, it is found that solving Problem 1 boils down to solving

$$\min_{I \in \mathcal{I}} \ \rho_g(X - I(X) + \pi(I)), \tag{4.14}$$

where $\pi(I)$ is given by Theorem 4.2. Here $(I, \pi(I))$ can be understood as the contract offered by a *representative reinsurer* in the market.

Note that when $x \in [x_{j-1}, x_j)$, we have

$$\begin{aligned}
&\sum_{i=1}^{j-1} \frac{\theta_j - \theta_i}{\gamma_i} \le I(x) < \sum_{i=1}^{j} \frac{\theta_{j+1} - \theta_i}{\gamma_i}, \text{ or} \\
&\sum_{i=1}^{j-1} \frac{(1+\theta_j) - (1+\theta_i)}{\gamma_i} \le I(x) < \sum_{i=1}^{j} \frac{(1+\theta_{j+1}) - (1+\theta_i)}{\gamma_i}.
\end{aligned}$$

Note that for $x \in [x_{j-1}, x_j)$, we have $\kappa_1(x) = \sum_{i=1}^{j} w_i(j)(1+\theta_i) = \kappa_2(x) \sum_{i=1}^{j} \frac{1+\theta_i}{\gamma_i}$, where $\kappa_2(x) = (\sum_{i=1}^{j} \gamma_i^{-1})^{-1}$. The above inequality can be rewritten as

$$\begin{aligned}
&(1+\theta_j)\left(\sum_{i=1}^{j-1} \gamma_i^{-1}\right) - \sum_{i=1}^{j-1} \frac{1+\theta_i}{\gamma_i} \le I(x) < (1+\theta_{j+1})\left(\sum_{i=1}^{j} \gamma_i^{-1}\right) - \sum_{i=1}^{j} \frac{1+\theta_i}{\gamma_i}, \text{ or} \\
&1 + \theta_j \le \kappa_1(x) + \kappa_2(x) I(x) < 1 + \theta_{j+1}.
\end{aligned}$$

Hence, we have

$$(1+\theta_j) \int_{x_{j-1}}^{x_j} S_X(x) dI(x) \le \int_{x_{j-1}}^{x_j} \{\kappa_1(x) + \kappa_2(x) I(x)\} S_X(x) dI(x) < (1+\theta_{j+1}) \int_{x_{j-1}}^{x_j} S_X(x) dI(x).$$

Now define

$$\tilde{\theta}_j = \frac{\int_{x_{j-1}}^{x_j} \{\kappa_1(x) + \kappa_2(x) I(x)\} S_X(x) dI(x)}{\int_{x_{j-1}}^{x_j} S_X(x) dI(x)} - 1, \tag{4.15}$$

where apparently $\tilde{\theta}_j \in [\theta_j, \theta_{j+1})$, the equation (4.13) becomes

$$\pi(I) = \int_0^M \sum_{i=1}^{n} (1+\tilde{\theta}_i) \mathbb{1}_{[x_{i-1}, x_i)}(x) S_X(x) dI(x) = \sum_{i=1}^{n} (1+\tilde{\theta}_i) \int_{x_{i-1}}^{x_i} S_{I(X)}(x) dx,$$

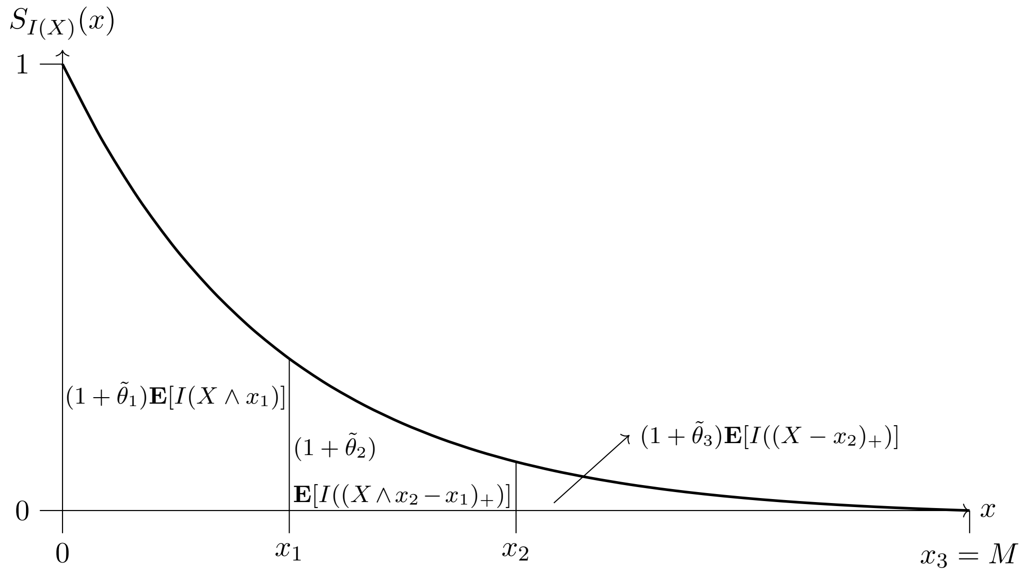


Figure 1: Illustration of the construction of $\pi(I)$ with $n=3$, which is composed of the sum of three elements: $\pi(I)=(1+\tilde{\theta}_1)\mathbf{E}[I(X\wedge x_1)]+(1+\tilde{\theta}_2)\mathbf{E}[I((X\wedge x_2-x_1)_+)]+(1+\tilde{\theta}_3)\mathbf{E}[I((X-x_2)_+)]$.

where the second equation is due to Zhuang et al. (2016). Compared with the classical expected-value premium principle $(1+\theta)\mathbf{E}[I(X)]=(1+\theta)\int_0^M S_{I(X)}(x)dx$, the representative reinsurer is adjusting the loading for different layers of coverage. In Figure 1, we illustrate the "piecewise safety loading" structure of the representative premium principle π.

In general, the premium principle $\pi(I)$ is no longer of the same form as (3.3). However, we can show that $\pi(I)$ is a monotonic and convex premium principle, and this is shown in the following theorem.

Theorem 4.3. *If $\gamma_i>0$ for all $i\in\{1,2,\dots,n\}$, the premium principle $\pi(I)$ in (4.13) is monotonic and convex.*

Proof. For the monotonicity of $\pi(I)$, we need to utilize the Lagrangian perspective provided in the appendix. For $\tilde{I},\hat{I}\in\mathcal{I}$, which satisfies $\tilde{I}(x)\le\hat{I}(x)$ for all $x\in[0,M]$, the optimal $\tilde{I}_i$ and $\hat{I}_i$, where $i\in\{1,\dots,n\}$, for the problems $\min_{\vec{I}\in\vec{\mathcal{I}}(\tilde{I})}\sum_{k=1}^n\pi_k(I_k(X))$ and $\min_{\vec{I}\in\vec{\mathcal{I}}(\hat{I})}\sum_{k=1}^n\pi_k(I_k(X))$ are given by (4.7) with $\lambda(x)$ replaced by $\tilde{\lambda}(x)$ and $\hat{\lambda}(x)$, where $\tilde{\lambda}(x)$ and $\hat{\lambda}(x)$ respectively satisfy the equations

$$\sum_{k=1}^n\left\{0\vee\left(\frac{\tilde{\lambda}(x)-(1+\theta_k)}{\gamma_k}\right)\right\}=\tilde{I}(x)$$

and

$$\sum_{k=1}^n\left\{0\vee\left(\frac{\hat{\lambda}(x)-(1+\theta_k)}{\gamma_k}\right)\right\}=\hat{I}(x).$$

Apparently, $\tilde{\lambda}(x) \le \hat{\lambda}(x)$ for $x \in [0, M]$ due to the monotonicity of the left-hand sides of the above two equations. Thus, we have $\hat{I}_i^*(x) \ge \tilde{I}_i^*(x)$ for $x \in [0, M]$, which naturally yields

$$\sum_{k=1}^{n} \pi_k(\hat{I}_k^*) = \pi(\hat{I}) \ge \pi(\tilde{I}) = \sum_{k=1}^{n} \pi_k(\tilde{I}_k^*).$$

We next show the convexity of π. Let $\tilde{I}, \hat{I} \in \mathcal{I}$ and $\lambda \in [0, 1]$. Define

$$\mathcal{S} = \left\{ \overrightarrow{I} : \overrightarrow{I} = \lambda \overrightarrow{I_1} + (1-\lambda)\overrightarrow{I_2},\ \overrightarrow{I_1} \in \overrightarrow{\mathcal{I}}(\tilde{I}),\ \overrightarrow{I_2} \in \overrightarrow{\mathcal{I}}(\hat{I}) \right\}.$$

It is straightforward that $\mathcal{S} \subseteq \overrightarrow{\mathcal{I}}(\lambda \tilde{I} + (1-\lambda)\hat{I})$. Hence, we have

$$\begin{aligned}
\pi(\lambda \tilde{I} + (1-\lambda)\hat{I}) &= \min_{\overrightarrow{I} \in \overrightarrow{\mathcal{I}}(\lambda \tilde{I} + (1-\lambda)\hat{I})} \sum_{k=1}^{n} \pi_k(I_k(X)) \\
&\le \min_{\overrightarrow{I} \in \mathcal{S}} \sum_{k=1}^{n} \pi_k(I_k(X)) \\
&= \min_{\overrightarrow{I_1} \in \overrightarrow{\mathcal{I}}(\tilde{I}), \overrightarrow{I_2} \in \overrightarrow{\mathcal{I}}(\hat{I})} \sum_{k=1}^{n} \pi_k(\lambda I_{1k}(X) + (1-\lambda) I_{2k}(X)) \\
&\le \min_{\overrightarrow{I_1} \in \overrightarrow{\mathcal{I}}(\tilde{I}), \overrightarrow{I_2} \in \overrightarrow{\mathcal{I}}(\hat{I})} \left\{ \lambda \sum_{k=1}^{n} \pi_k(I_{1k}(X)) + (1-\lambda) \sum_{k=1}^{n} \pi_k(I_{2k}(X)) \right\} \\
&= \lambda \left\{ \min_{\overrightarrow{I_1} \in \overrightarrow{\mathcal{I}}(\tilde{I})} \sum_{k=1}^{n} \pi_k(I_{1k}(X)) \right\} + (1-\lambda) \left\{ \min_{\overrightarrow{I_2} \in \overrightarrow{\mathcal{I}}(\hat{I})} \sum_{k=1}^{n} \pi_k(I_{2k}(X)) \right\} \\
&= \lambda \pi(\tilde{I}) + (1-\lambda)\pi(\hat{I}).
\end{aligned}$$

The proof is complete. □

Remark 4.3. *The second-moment premium principle in* (3.3) *can be extended to include higher-order moments. We show in Appendix* A *that the main results of this section can be transformed to allow for a finitely many higher moments. However, the results become more difficult to understand. We will continue this paper with the second-moment premium principle, also because we will show in Section* 4.1 *that it yields an efficient algorithm to solve numerically.*

4.1 A numerical way to solve Problem (4.14)

Due to the complexity of the premium principle $\pi(I)$ as shown in Theorem 4.2, we solve the problem (4.14) using a numerical approach. The application of numerical methods in optimal (re)insurance design is however rather scarce in the literature. We refer to Asimit et al. (2017, 2018) for constructive examples.

By using the translation invariance and comonotonic additivity properties of distortion risk measures, the objective function of (4.14) can be written as

$$\rho_g(X - I(X) + \pi(I)) = \rho_g(X) - \rho_g(I(X)) + \pi(I).$$

The positive homogeneity[5] and translation invariance properties of distortion risk measures imply that $\rho_g(I(X))$ is linear in I. Combining with Theorem 4.3, the problem

$$\min_{I\in\mathcal{I}} \; -\rho_g(I(X)) + \pi(I) \tag{4.16}$$

is a convex optimization problem.

We discretize Problem (4.16) first. Let $\{t_1, t_2, \ldots, t_m\}$ be exogenous and distinct loss data points sorted in increasing order. We denote $p_j = \mathbf{P}(X = t_j)$, where

$$p_j = F_X\left(\frac{t_j + t_{j+1}}{2}\right) - F_X\left(\frac{t_{j-1} + t_j}{2}\right),$$

where $t_0 = -\infty$ and $t_{m+1} = M$. Let $y = I(x)$, or more specifically $y_j = I(t_j)$ for $j \in \{1, 2, \ldots, m\}$, the discrete form of $\rho_g(I(X))$ is

$$\rho_g(I(X)) = \int_0^M g(S_X(x))dI(x) \approx \sum_{j=1}^{m} g(S_X(t_j))(y_j - y_{j-1}),$$

where $y_0 = 0$. For the premium principle $\pi(I)$, we note from Theorem 4.2 that $\kappa_1(x)$ and $\kappa_2(x)$ are both step functions and $I(x)$-dependent. In numerical optimization, it is more convenient to define: for $j \in \{0, 1, \ldots, n-1\}$

$$\phi_1(y) = \left(\sum_{k=1}^{j+1} \frac{1+\theta_k}{\gamma_k}\right)\left(\sum_{k=1}^{j+1} \gamma_k^{-1}\right)^{-1} \quad \text{if} \quad \sum_{i=1}^{j} \frac{\theta_{j+1} - \theta_i}{\gamma_i} \le y < \sum_{i=1}^{j+1} \frac{\theta_{j+2} - \theta_i}{\gamma_i}$$

and

$$\phi_2(y) = \left(\sum_{k=1}^{j+1} \gamma_k^{-1}\right)^{-1} \quad \text{if} \quad \sum_{i=1}^{j} \frac{\theta_{j+1} - \theta_i}{\gamma_i} \le y < \sum_{i=1}^{j+1} \frac{\theta_{j+2} - \theta_i}{\gamma_i}.$$

It can be easily verified that $\phi_1(y) = \phi_1(I(x)) = \kappa_1(x)$ and $\phi_2(y) = \phi_2(I(x)) = \kappa_2(x)$ for $x \in [0, M)$. The discrete form of $\pi(I)$ is then

$$\pi(I) \approx \sum_{j=1}^{m} \left(\phi_1(y_j) + \phi_2(y_j)y_j\right) S_X(t_j)(y_j - y_{j-1}).$$

Since $I \in \mathcal{I}$, following Asimit et al. (2017), the constraints below are placed on $y_1, \ldots, y_m$:

$$\begin{cases} \mathbf{0} \le \boldsymbol{y} \le \boldsymbol{t}, \\ \mathbf{0} \le A\boldsymbol{y} \le A\boldsymbol{t}, \end{cases} \tag{4.17}$$

where $\mathbf{0} = [0, \ldots, 0]^T$, $\boldsymbol{y} = [y_1, \ldots, y_m]^T$, $\boldsymbol{t} = [t_1, \ldots, t_m]^T$, and A is a $(m-1) \times m$ matrix

$$A = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

[5] A risk measure ρ is called positive homogeneous if $\rho(c \cdot Z) = c \cdot \rho(Z)$ for all $c > 0$ and all random variables Z.

To this end, the discrete form of the problem (4.16) is

$$\begin{aligned}\min_{y_1,\dots,y_m} \; & \sum_{j=1}^{m} \Big\{ \big(\phi_1(y_j)+\phi_2(y_j)y_j\big)\, S_X(t_j) - g(S_X(t_j)) \Big\} (y_j - y_{j-1}) \\ & \text{s.t. the constraint (4.17).}\end{aligned} \tag{4.18}$$

The problem (4.18) can be solved by using the readily available function "fmincon" or the "CVX" solver of MATLAB.

Example 4.1 (Proportional hazard transform)**.** *We assume that the loss variable follows a gamma-type distribution, i.e.*

$$F_X(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} \displaystyle\int_0^x t^{\alpha-1} e^{-\frac{t}{\beta}}, & x \in (0, 10^6], \\ 1, & x \in (10^6, \infty], \end{cases}$$

with $\alpha = 10$ *and* $\beta = 30$*. We further assume that there are three reinsurers in the market, who may apply different safety loading factors on either the first moment or second moment of the indemnity function under the second-moment-based premium principle. We assume that the insurer's preference is dictated by the proportional hazard (PH) transform, whose distortion function is* $g(t) = g_{PH}(t) = t^a$ *for* $a \in (0,1)$*. To discretize the optimization problem, we consider the 200 discrete points that are equally spaced between* 1 *and* 800*.*[6]

Figures 2 and 3 display the optimal indemnity functions that solve (4.14) under different safety loading factors. All the indemnity functions provide no coverage when the loss size is small, partial coverage for the excess amount when the loss size is medium, and full coverage for the excess amount when the loss size is large. One direct observation is that the indemnity function in demand is more sensitive to the change of safety loading of the second-order moment of the ceded loss, and this is also verified by checking the aggregate premiums as shown in Table 1 for seven different parameter settings of the insurers.

Besides the optimal contract that the insurer seeks from the representative reinsurer, we also provide the optimal indemnities shared among the three individual reinsurers, as shown in Figure 4. One straightforward finding is that the insurer cedes out her loss in tranches, which are determined by both safety loading factors in the second-order-moment-based premium principles. As shown in all the plots in Figure 4, smaller losses will be covered by the reinsurer who uses smaller safety loading factor for the first-order moment of the ceded loss, while the large losses will be mainly covered by the reinsurer who offers smaller safety loading factor for the second-order moment of the ceded loss.

In Table 1 we summarize the aggregate premiums, and we find that the aggregate premium decreases if one of the parameters θ_i *and* γ_i *increases. Increasing such a parameter leads to more expensive insurance contract, and the insurer will in response retain more risk and cede less to the*

[6] Note that under the setting of this example, $\mathbf{P}(X > 800) \approx 7.26 \times 10^{-5}$.

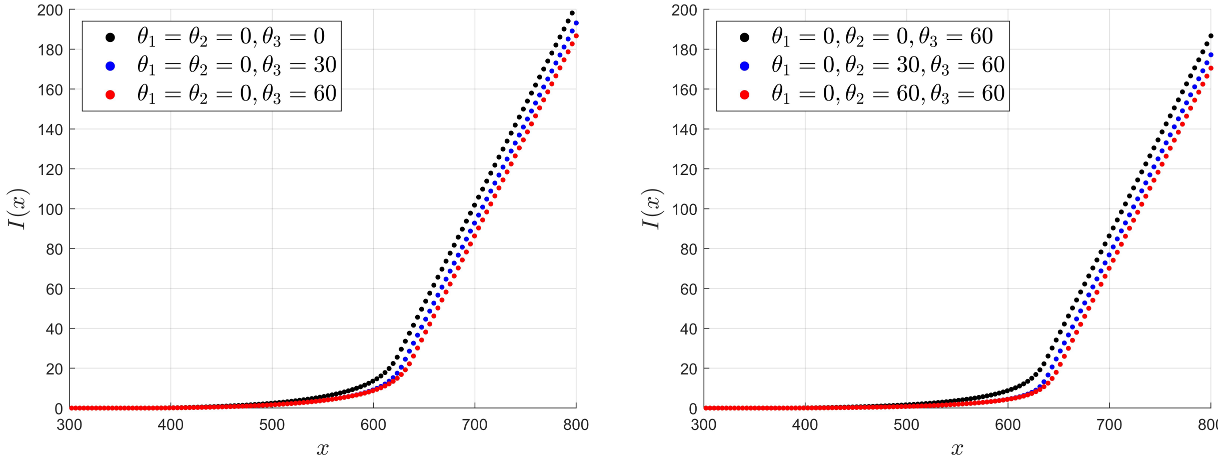


Figure 2: Optimal indemnity functions under the setting $\gamma_1 = \gamma_2 = \gamma_3 = 3$. We display Cases 1, 2, and 3 in the left figure, and Cases 3, 4, and 5 in the right figure.

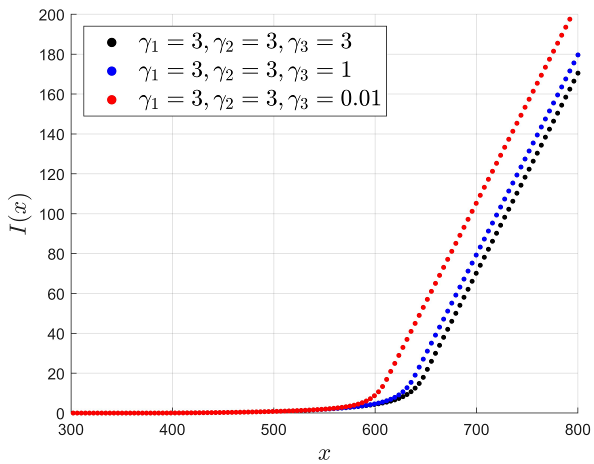


Figure 3: Optimal indemnity functions under the setting $\theta_1 = 0, \theta_2 = \theta_3 = 60$ (Cases 5, 6, and 7).

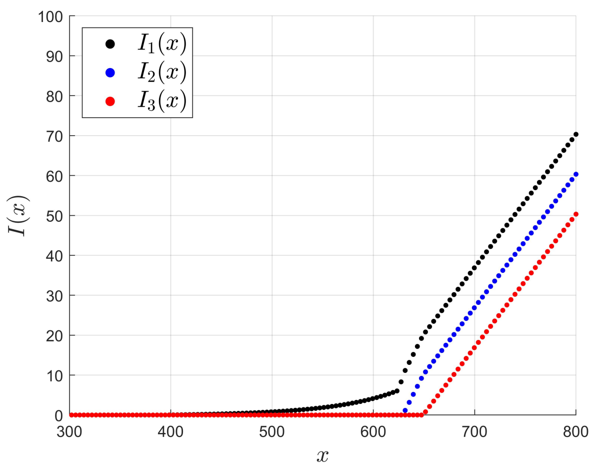
collective of reinsurers. This lower indemnity will have a larger impact on the aggregate premium than the increase in the safety loading factors.

As shown by Figure 4(c), when γ_3 is closer to 0, almost all the marginal loss will be borne by the third insurer after she starts to share the loss. This also corresponds to a limiting case of Theorem 4.1, where $I_3^{*\prime}(x) = \frac{\gamma_3^{-1}}{\sum_{s=1}^3 \gamma_s^{-1}} \to 1$ and $I_k^{*\prime}(x) = \frac{\gamma_k^{-1}}{\sum_{s=1}^3 \gamma_s^{-1}} \to 0$ for $k = 1, 2$ when $\gamma_3 \to 0$.

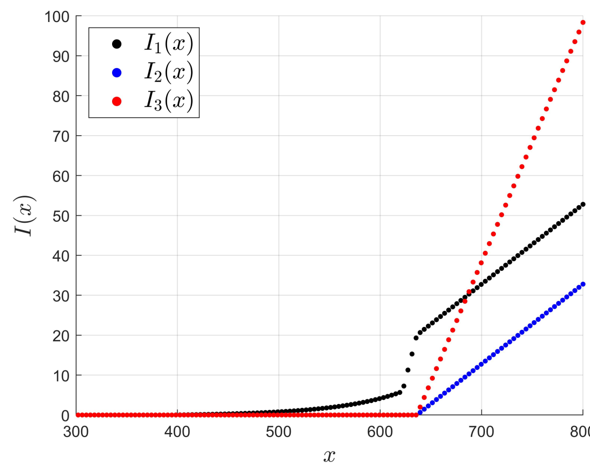
We close this section by remarking on the applicability of numerical methods for our problem. As shown in this section, the explicit form and convexity of $\pi(I)$ are crucial for the implementation of numerical method. The explicit form of $\pi(I)$ allows us to easily discretize the problem, and the convexity of $\pi(I)$ ensures the convexity of the whole optimization problem, which facilitates the use of many readily available packages. The closed-form of $\pi(I)$ also exists when all the agents are endowed with distortion risk measures (Boonen and Ghossoub, 2019, 2021). However, for

Case number	θ_1	θ_2	θ_3	γ_1	γ_2	γ_3	$\pi(I)$
1	0	0	0	3	3	3	12.43
2	0	0	30	3	3	3	11.81
3	0	0	60	3	3	3	11.34
4	0	30	60	3	3	3	10.65
5	0	60	60	3	3	3	10.27
6	0	60	60	3	3	1	10.76
7	0	60	60	3	3	0.01	13.83

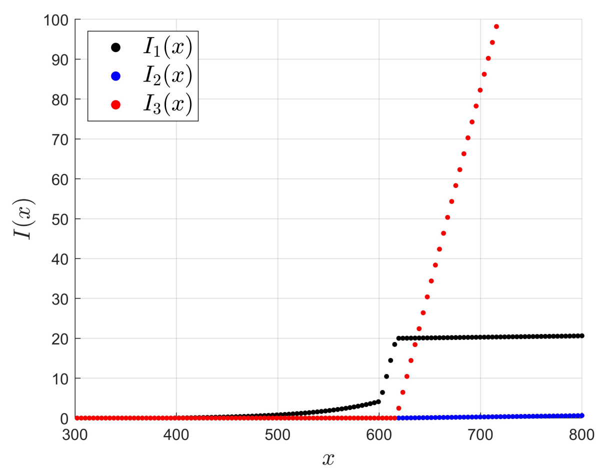
Table 1: The premiums charged by the representative reinsurer under different safety loading factors.



(a) Case 4.



(b) Case 6.



(c) Case 7.

Figure 4: Optimal indemnity functions with different reinsurers.

risk measures and utility functions that are not positive homogeneous, translation invariant or comonotonic additive, the closed-form of $\pi(I)$ is often not obtainable, which becomes the main obstacle for applying the numerical method.

5 Equal safety loading factor

In this section, we study a special case of Problem 1 where $\theta_i = \theta$ for all $i \in \{1, 2, \ldots, n\}$. Here, we assume that the safety loading factor for the first moment of indemnity is set to cover all the expenses and profits that would vary linearly with respect to the expected payoff or pure premium (see Werner and Modlin (2010) for the calculations of a variety of ratios that are commonly used in industrial practice), and a common safety loading factor is used in a competitive market. In some popular premium principles, such as the variance and standard-deviation premium principles, the safety loading factor for the first moment of indemnity is set directly to zero (Young, 2014). In such a case, we have

$$H_1^*(0) = H_2^*(0) = \cdots = H_n^*(0) = 1 + \theta.$$

Following the proof of Theorem 4.1, we have $H_1^*(x) = H_2^*(x) = \cdots = H_n^*(x) = H^*(x)$, or $\mathcal{K}^*(x) = \{1, 2, \ldots, n\}$, for all $x \in [0, M]$. Then (4.3) becomes

$$I_i^{*\prime}(x) = \frac{\gamma_i^{-1}}{\sum_{k=1}^n \gamma_k^{-1}} I'(x), \text{ for } x \in [0, M], \tag{5.1}$$

which implies that for any $i \in \{1, 2, \ldots, n\}$

$$I_i^*(x) = \frac{\gamma_i^{-1}}{\sum_{k=1}^n \gamma_k^{-1}} I(x). \tag{5.2}$$

If a common safety loading factor θ is applied by all the reinsurers, i.e. $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, then it can be easily calculated that

$$x_j = \inf\{x \in [0, M] : \ I(x) \geq 0\} = 0, \text{ for } j = 1, 2, \ldots, n-1.$$

Then

$$\kappa_1(x) = 1 + \theta, \quad \kappa_2(x) = \left(\sum_{k=1}^n \gamma_k^{-1} \right)^{-1}, \text{ for } x \in [0, M],$$

which implies that

$$\begin{aligned} \pi(I) &= \int_0^M (1 + \theta + (\sum_{k=1}^n \gamma_k^{-1})^{-1} I(x)) S_X(x) dI(x) \\ &= (1+\theta)\mathbf{E}[I(X)] + \frac{1}{2(\sum_{k=1}^n \gamma_k^{-1})} \mathbf{E}[I(X)^2]. \end{aligned}$$

Thus, in summary, we obtain the following corollary of Theorems 4.1 and 4.2.

Corollary 5.1. *If $\theta_i = \theta$ and $\gamma_i > 0$ for all $i \in \{1, 2, \ldots, n\}$, then $I_i^{*\prime}(x) = \frac{\gamma_i^{-1}}{\sum_{k=1}^n \gamma_k^{-1}} I'(x)$, and so $I_i^*(X) = \frac{\gamma_i^{-1}}{\sum_{k=1}^n \gamma_k^{-1}} I(X)$. Moreover, it then holds that $\pi(I) = (1+\theta)\mathbf{E}[I(X)] + \frac{1}{2(\sum_{k=1}^n \gamma_k^{-1})} \mathbf{E}[I(X)^2]$.*

From Corollary 5.1, the remaining problem to be solved is given as follows:

Problem 2.

$$\min_{I\in\mathcal{I}} \ L(I) := \rho_g(X - I(X) + \pi(I)),$$

where $\pi(I) = (1+\theta)\mathbf{E}[I(X)] + w^*\mathbf{E}[I(X)^2]$ *and* $w^* = \frac{1}{2(\sum_{k=1}^n \gamma_k^{-1})}$.

5.1 An implicit solution to Problem 2

By using the translation invariance and comonotonic additivity properties of distortion risk measures, the objective function of Problem 2 can be written as

$$\rho_g(X - I(X) + \pi(I)) = \rho_g(X) - \rho_g(I(X)) + (1+\theta)\mathbf{E}[I(X)] + w^*\mathbf{E}[I(X)^2].$$

Thus, solving Problem 2 is equivalent to solving

$$\min_{I\in\mathcal{I}} \ -\rho_g(I(X)) + (1+\theta)\mathbf{E}[I(X)] + w^*\mathbf{E}[I(X)^2]. \tag{5.3}$$

By applying the calculus of variations, we obtain the full characterization of the solution to Problem 5.3.

Theorem 5.1. *The optimal indemnity function to Problem* (5.3) *is given by* $I^*(x) = \int_0^x I^{*\prime}(t)dt$, *where*

$$I^{*\prime}(x) = \mathbb{1}_{\{x:H(x)<0\}}(x) + \xi(x)\mathbb{1}_{\{x:H(x)=0\}}(x), \tag{5.4}$$

where

$$H(x) = -g(S_X(x)) + (1+\theta)S_X(x) + 2w^* \int_x^M I^*(t)dF_X(t), \tag{5.5}$$

and $\xi(x) \in [0,1]$ *is implicitly determined by* $H(x) = 0$. *Furthermore, if* $\theta > 0$ *and* $S_X(0) > \sup\{t \in (0,1] : (1+\theta)x \le g(t)\}$, *then* I^* *contains a deductible level* $d^* > 0$.

Proof. Let I^* denote the optimal indemnity function to Problem 5.3. For any $I \in \mathcal{I}$ and $\epsilon \in [0,1]$, we have $I^*_\epsilon = (1-\epsilon)I^* + \epsilon I \in \mathcal{I}$, by convexity of the set $\mathcal{I}$.

Let the objective function of (5.3) be $L(I)$, then it is straightforward to verify that the function $L(I^*_\epsilon)$ is a quadratic function of ϵ. Given that I^* is the solution to (5.3), we have

$$\begin{aligned} &\frac{\partial L(I^*_\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} \ge 0 \\ \Longrightarrow \ &-\rho_g(I(X)) + (1+\theta)\mathbf{E}[I(X)] + 2w^*\mathbf{E}[I(X)I^*(X)] \\ &\ge -\rho_g(I^*(X)) + (1+\theta)\mathbf{E}[I^*(X)] + 2w^*\mathbf{E}[I^*(X)^2]. \end{aligned}$$

Therefore, I^* solves Problem (5.3) if and only if it also solves

$$\min_{I\in\mathcal{I}} \ -\rho_g(I(X)) + (1+\theta)\mathbf{E}[I(X)] + 2w^*\mathbf{E}[I(X)I^*(X)]. \tag{5.6}$$

By using the Fubini's theorem, we can get

$$-\rho_g(I(X)) + (1+\theta)\mathbf{E}[I(X)] + 2w^*\mathbf{E}[I(X)I^*(X)]$$

$$
\begin{aligned}
&= -\int_0^M g(S_X(x))dI(x) + (1+\theta)\int_0^M S_X(x)dI(x) + 2w^*\int_0^M I^*(x)\left(\int_0^x I'(t)dt\right)dF_X(x) \\
&= -\int_0^M g(S_X(x))dI(x) + (1+\theta)\int_0^M S_X(x)dI(x) + 2w^*\int_0^M \left(\int_x^M I^*(t)dF_X(t)\right)dI(x) \\
&= \int_0^M H(x)dI(x) = \int_0^M H(x)I'(x)dx, \qquad (5.7)
\end{aligned}
$$

with $H(x)$ defined in (5.5). It is apparent that the optimal I that can minimize (5.7) should be of the form $I^*(x) = \int_0^x I^{*\prime}(t)dt$ with

$$
I^{*\prime}(x) = \mathbb{1}_{\{x:H(x)<0\}}(x) + \xi(x)\mathbb{1}_{\{x:H(x)=0\}}(x).
$$

If $\theta > 0$ and $S_X(0) > \sup\{t \in (0,1] : (1+\theta)t \le g(t)\}$, then

$$
H(0) = -g(S_X(0)) + (1+\theta)S_X(0) + 2w^*\int_0^M I^*(t)dF_X(t) > 0.
$$

Now define $d^* := \inf\{x \in [0,M] : H(x) \le 0\}$, then we have $H(x) > 0$ for $x \in [0, d^*)$. As such, I^* contains a deductible level d^*. □

If the safety loading factor $\theta > 0$, then the condition $S_X(0) > \sup\{t \in (0,1] : (1+\theta)t \le g(t)\}$ can be easily fulfilled if X does not possess point mass at 0.

Though Theorem 5.1 seems to fully characterize the solution to Problem (5.3), the solution is not explicit as the right-hand side of (5.4) also depends on I^* itself. Such issue is well-known in the literature (e.g., Chi and Zhuang, 2020; Ghossoub et al., 2022; Birghila et al., 2023). One can find the explicit solutions under some additional assumptions on either the distortion risk measure or the distribution of X, or identify the semi-explicit (or parametric) solutions by resorting to an optimization over a partitioned domain (see, e.g., Chi and Tan, 2021; Ghossoub et al., 2022). In the following corollary, we present the parametric solution to Problem (5.3) when the distortion risk measure is TVaR.[7]

Corollary 5.2. *If $\rho_g = TVaR_\alpha$, then the solution to Problem (5.3) is given by*

$$
I^*(x) = (x \wedge L - d)_+ \qquad (5.8)
$$

for some d and L that satisfy $0 \le d \le L \le M$.

Proof. By inserting the distortion function of TVaR to (5.5), we have

$$
H(x) = \begin{cases} -1 + (1+\theta)S_X(x) + 2w^*\int_x^M I^*(t)dF_X(t), & x \le \mathrm{VaR}_\alpha(X), \\ \left((1+\theta) - \dfrac{1}{1-\alpha}\right)S_X(x) + 2w^*\int_x^M I^*(t)dF_X(t), & x > \mathrm{VaR}_\alpha(X). \end{cases}
$$

[7]We leave other types of distortion functions as future research.

We define $d := \inf\{x \in [0, M] : H(x) \le 0\}$ with $\inf \emptyset := M$, and show that $H(x)$ can up-cross[8] x-axis at most once and never down-cross x-axis when $x \in (d, M]$. It is apparent that $H(x)$ is decreasing over $[0, \mathrm{VaR}_\alpha(X)]$, thus the up-crossing point of H, if exists, must be located within $(d \vee \mathrm{VaR}_\alpha(X), M]$. If H does not have an up-crossing point, then $H(x) \le 0$ on $(d, M]$. As per Theorem 5.1, $I^*(x) = (x - d)_+$ (which is a special case of (5.8)) is a solution to Problem (5.3). If H has an up-crossing point, which we denote by L, then by the mean value theorem, there exists a point $x_1 > L$ such that $H'(x_1) > 0$, or equivalently

$$H'(x_1) = -\left((1+\theta) - \frac{1}{1-\alpha} + 2w^* I^*(x_1)\right) f_X(x) > 0.$$

If H down-crosses the x-axis at some point $\tilde{d} \in (L, M]$, then again by the mean value theorem, there exists a point $x_2 \in (x_1, \tilde{d})$ such that $H'(x_2) < 0$, or equivalently

$$H'(x_2) = -\left((1+\theta) - \frac{1}{1-\alpha} + 2w^* I^*(x_2)\right) f_X(x) < 0.$$

As such, we have $I^*(x_1) < I^*(x_2)$. However, note that $H(x) > 0$ when $x \in [x_1, x_2]$, which, as per Theorem 5.1 leads to $I^{*\prime}(x) = 0$ when $x \in [x_1, x_2]$, or equivalently $I^*(x_1) = I^*(x_2)$. This gives rise to the contradiction. Therefore, if L is the up-crossing point of H, we have $H(x) > 0$ when $x \in (L, M]$. Then, as per Theorem 5.1, we get $I^*(x) = (x \wedge L - d)_+$. The proof is complete. $\square$

In the next section, we propose the use of numeric quadratic programming approach to approximate the solutions to Problem (5.3).

5.2 A numerical solution to Problem 2

We follow the same idea as presented in Section 4.1 to discretize Problem (5.3). First, we have empirically

$$\mathbf{E}[I(X)] \approx \sum_{j=1}^{m} p_j y_j, \quad \mathbf{E}[I(X)^2] \approx \sum_{j=1}^{m} p_j y_j^2,$$

where $\{y_j\}$ and $\{p_i\}$ are defined in Section 4.1.

Using integration by parts, we have

$$\rho_g(I(X)) = \int_0^M g(S_X(x))dI(x) = \int_0^M I(x)d[1 - g(S_X(x))].$$

Thus, the empirical approximation of $\rho_g(I(X))$ is (see also Section 3.2 of Asimit et al., 2017)

$$\rho_g(I(X)) \approx \sum_{j=1}^{m} \pi_j y_j,$$

where $\pi_1 = 1 - g(1 - p_1)$, and $\pi_j = g\left(1 - \sum_{k=1}^{j-1} p_k\right) - g\left(1 - \sum_{k=1}^{j} p_k\right)$ for $j = 2, \ldots, m$.

[8] A function h up-crosses (resp. down-crosses) the x-axis at the point x_0 if there exists a $\epsilon > 0$ such that $h(x) \le 0$ when $x \in (x_0 - \epsilon, x_0]$ and $h(x) > 0$ when $x \in (x_0, x_0 + \epsilon)$ (resp. $h(x) \ge 0$ when $x \in (x_0 - \epsilon, x_0]$ and $h(x) < 0$ when $x \in (x_0, x_0 + \epsilon)$).

Let $\boldsymbol{p} = [p_1, \ldots, p_m]^T$ and $\boldsymbol{\pi} = [\pi_1, \ldots, \pi_m]^T$, the matrix form of Problem (5.3) is therefore given by

$$\begin{aligned} &\min_{\boldsymbol{y}} \ \frac{1}{2}\boldsymbol{y}^T H \boldsymbol{y} + \boldsymbol{v}^T \boldsymbol{y} \\ &\text{s.t. } B\boldsymbol{y} \leq b, \end{aligned} \tag{5.9}$$

where $H = 2w^* \times \text{diag}(\boldsymbol{p})$, $\boldsymbol{v} = (1+\theta) \times \boldsymbol{p} - \boldsymbol{\pi}$, and $B\boldsymbol{y} \leq b$ is a linear constraint on $\boldsymbol{y}$ that combines the constraints from (4.17).

Problem (5.9) can be easily solved using standard quadratic programming algorithms (e.g., "quadprog" or "mpcactivesolver" functions from MATLAB).

Example 5.1. *Following Example 4.1, we assume that the loss variable follows a gamma-type distribution with* $\alpha = 10$ *and* $\beta = 30$. *Moreover, we consider the following setting. The constant safety loading factor is* $\theta = 0.5$. *The insurer minimizes the Tail Value-at-Risk (TVaR) with confidence level parameter* $\alpha = 0.99$. *The optimal indemnities for the loss data* $\{1, 2, \ldots, 800\}$ *are displayed in Figures 5 and 6.*

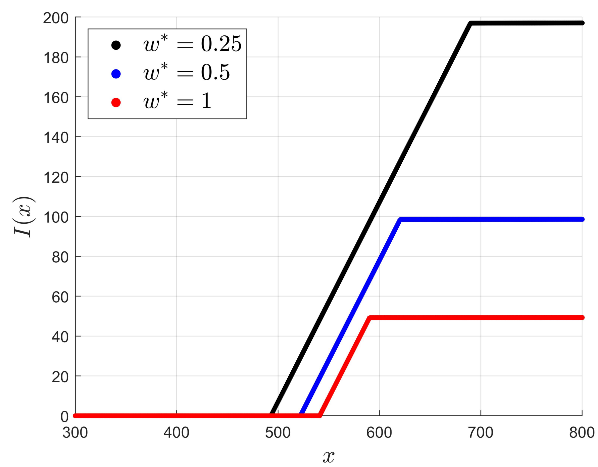


Figure 5: The optimal indemnities for Problem (5.9) when $\rho_g = TVaR_{0.99}$.

We find that all the optimal aggregate indemnity functions are of the limited stop-loss form. This is an interesting finding, as the optimal indemnity function with TVaR under the expected value premium principles is well-known to be of a stop-loss form (Cai et al., 2008; Cheung, 2010). Here, because of the second-order moment of the ceded loss in the premium principle, covering large losses is relatively more expensive, and this leads to a cap on the optimal indemnity function. This cap is larger when the second-order moment has a smaller weight w^* *(see Figure 5). Moreover, from Figure 6 we see that for a larger confidence level parameter* α, *the deductible increases as a consequence of TVaR only considering losses deeper in the right tail, making reinsurance for smaller losses too costly. As a consequence of putting weight in the right tail, the insurer selects an indemnity with a larger cap.*

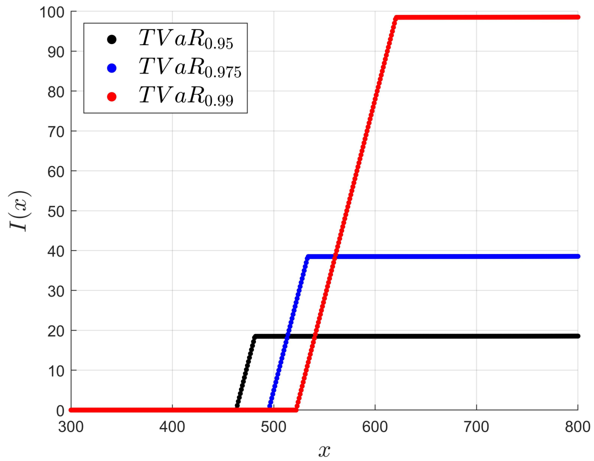


Figure 6: The optimal indemnities for Problem (5.9) when $w^* = 0.5$ under different confidence level parameters.

We close this section by comparing the computational complexities of solving Problems (4.14) and 2. Note that Problem 2 is a quadratic programming problem, whose objective function is continuously differentiable, facilitating the use of some derivative-based optimization method, such as the gradient descent method. However, the objective function of the problem (4.14) is generally not smooth, which lowers the efficiency of some conventional optimization method.

6 Expected value premium principles

In this section, we study the case in which $\gamma_i = 0$ for all i. Then, all the reinsurers use expected value premium principles. Note that this case is not treated in Theorems 4.1 and 4.2, and we will show that this is an easy case to solve and will yield closed-form solutions. For the ease of presentation, we define $\underline{\theta} = \min\{\theta_1, \ldots, \theta_n\}$.

Proposition 6.1. *If $\gamma_i = 0$ for all $i \in \{1, 2, \ldots, n\}$, then $\pi(I) = (1 + \underline{\theta})\mathbf{E}[I(X)]$. Moreover, $I_i(X) = 0$ if $i \notin \arg\min\{\theta_j : j = 1, 2, \ldots, n\}$.*

Proof. Similar to the proof of Theorem 4.1, we have

$$\sum_{k=1}^{n} \pi_k(I_k(X)) = \int_0^M \left(\sum_{k=1}^{n} (1 + \theta_k) I'_k(x) \right) S_X(x) dx.$$

Since

$$\begin{aligned}
\sum_{k=1}^{n} (1 + \theta_k) I'_k(x) &= \sum_{\{k:\theta_k > \underline{\theta}\}} (1 + \theta_k) I'_k(x) + \sum_{\{k:\theta_k = \underline{\theta}\}} (1 + \theta_k) I'_k(x) \\
&= \sum_{\{k:\theta_k > \underline{\theta}\}} (1 + \theta_k) I'_k(x) + (1 + \underline{\theta}) \sum_{\{k:\theta_k = \underline{\theta}\}} I'_k(x)
\end{aligned}$$

$$\geq (1+\underline{\theta})\sum_{k=1}^{n} I_k'(x) = (1+\underline{\theta})I'(x).$$

This implies that optimal indemnity functions should satisfy

$$\sum_{\{k:\theta_k=\underline{\theta}\}} I_k^{*\prime}(x) = I'(x) \text{ and } I_i^{*\prime}(x) = 0 \text{ if } \theta_i > \underline{\theta},$$

which naturally yields the results of this proposition. □

Proposition 6.1 helps us to reduce the case with n reinsurers to the case with only 1 reinsurer. In that case, the objective function in Problem 2 can be written as follows:

$$\begin{aligned} L(I) &= \rho(X - I(X) + \pi(I) = \rho(X - I(X)) + \pi(X) = \rho(X) - \rho(I(X)) + (1+\underline{\theta})\mathbf{E}[I(X)] \\ &= \rho(X) - \int_0^M ((1+\underline{\theta})S_X(x) - g(S_X(x)))I'(x)dx. \end{aligned}$$

Then, the optimal indemnity function I is readily obtained, and summarized in the next proposition.

Proposition 6.2. *If $\gamma_i = 0$ for all $i \in \{1,2,\ldots,n\}$, then the optimal indemnity I that solves Problem 2 with $\pi(I) = (1+\underline{\theta})\mathbf{E}[I(X)]$ is given by $I^*(x) = \int_0^x I^{*\prime}(t)dt$, where*

$$I^{*\prime}(x) = \mathbb{1}_{\{x:(1+\underline{\theta})S_X(x)<g(S_X(x))\}}(x) + \xi(x)\mathbb{1}_{\{x:(1+\underline{\theta})S_X(x)=g(S_X(x))\}}(x), \tag{6.1}$$

where $\xi(x) \in [0,1]$.

In summary, this section shows that if all reinsurers use expected value premium principles, then we can derive the optimal reinsurance contracts explicitly. While this may be a very restrictive special case, we wish to point out that this is the main case under which we derive full closed-form expressions of I. In Sections 4 and 5, we obtained only closed-form solution given the aggregate indemnity $I(X)$, which was subsequently determined numerically. The theoretical results derived in this section agree well with those of Boonen and Ghossoub (2021) in a multiple-reinsurer setting.

7 Conclusion

This paper studies the optimal reinsurance problem between one insurer and multiple reinsurers by providing the concept of a representative reinsurer in the context of moment-based premium principles. The representative reinsurer's premium principle, which admits an analytical form and possesses monotonicity and convexity properties, simplifies the problem and enables the use of convex and quadratic programming approaches to find solutions. The results demonstrate that when reinsurers apply the same safety loading factor for the first moment of the ceded loss, the complexity of the original problem is significantly reduced. Furthermore, the idea of the representative reinsurer can be extended to cases where reinsurers apply premium principles based on higher-order moments. This research provides an effective method for solving the optimal reinsurance problem and contributes to the literature on computational (re)insurance contracting.

This paper paves the way for several future research opportunities. For example, the potential applications of the representative reinsurer concept can be further explored in case of variance or standard deviation premium principles. This paper only focuses on the reinsurance demand from the insurer's perspective, the optimal pricing strategies (e.g., the determination of the safety loading factors for the two moments of indemnities) from the reinsurers' perspectives can also be investigated in a Stackelberg game, where the Nash equilibrium among the reinsurers can be studied in a competitive market.

Acknowledgements

The authors are very grateful for the insightful and constructive comments and suggestions from two anonymous reviewers, which have greatly improved the presentation of the paper. The authors are ranked in alphabetical order. Wenjun Jiang acknowledges the financial support received from the Natural Sciences and Engineering Research Council of Canada (grant RGPIN-2020-04204), Alberta Innovates and the University of Calgary. Yiying Zhang acknowledges the National Natural Science Foundation of China (No. 12101336, 72311530687), Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515011806), and Shenzhen Science and Technology Program (No. JCYJ20230807093312026).

Disclosure statements

No potential competing interests were reported by the authors.

References

B. Acciaio. *Two problems related to utility theory under unusual assumptions.* PhD thesis, University of Perugia, 2006.

K. J. Arrow. Optimal insurance and generalized deductibles. *Scandinavian Actuarial Journal*, 1974 (1):1–42, 1974.

P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

A. V. Asimit, V. Bignozzi, K. C. Cheung, J. Hu, and E.-S. Kim. Robust and Pareto optimality of insurance contracts. *European Journal of Operational Research*, 262(2):720–732, 2017.

A. V. Asimit, T. Gao, J. Hu, and E.-S. Kim. Optimal risk transfer: a numerical optimization approach. *North American Actuarial Journal*, 22(3):341–364, 2018.

C. Bernard, X. He, J.-A. Yan, and X. Y. Zhou. Optimal insurance design under rank-dependent expected utility. *Mathematical Finance*, 25(1):154–186, 2015.

C. Bernard, F. Liu, and S. Vanduffel. Optimal insurance in the presence of multiple policyholders. *Journal of Economic Behavior & Organization*, 180:638–656, 2020.

C. Birghila, T. J. Boonen, and M. Ghossoub. Optimal insurance under maxmin expected utility. *Finance and Stochastics*, 27(2):467–501, 2023.

T. J. Boonen and M. Ghossoub. On the existence of a representative reinsurer under heterogeneous beliefs. *Insurance: Mathematics and Economics*, 88:209–225, 2019.

T. J. Boonen and M. Ghossoub. Optimal reinsurance with multiple reinsurers: distortion risk measures, distortion premium principles, and heterogeneous beliefs. *Insurance: Mathematics and Economics*, 101:23–37, 2021.

T. J. Boonen and M. Ghossoub. Bowley vs. Pareto optima in reinsurance contracting. *European Journal of Operational Research*, 307(1):382–391, 2023.

T. J. Boonen and W. Jiang. Mean–variance insurance design with counterparty risk and incentive compatibility. *ASTIN Bulletin: The Journal of the IAA*, 52(2):645–667, 2022.

T. J. Boonen and Y. Zhang. Bowley reinsurance with asymmetric information: a first-best solution. *Scandinavian Actuarial Journal*, 2022(6):532–551, 2022.

T. J. Boonen, K. S. Tan, and S. C. Zhuang. The role of a representative reinsurer in optimal reinsurance. *Insurance: Mathematics and Economics*, 70:196–204, 2016.

K. Borch. An attempt to determine the optimum amount of stop loss reinsurance. *Transactions of the XVI International Congress of Actuaries*, 1:597–610, 1960.

K. Borch. Equilibrium in a reinsurance market. *Econometrica*, 30(3):424–444, 1962.

J. Cai and Y. Chi. Optimal reinsurance designs based on risk measures: A review. *Statistical Theory and Related Fields*, 4(1):1–13, 2020.

J. Cai, K. S. Tan, C. Weng, and Y. Zhang. Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics*, 43(1):185–196, 2008.

J. Cao, D. Li, V. R. Young, and B. Zou. Reinsurance games with two reinsurers: Tree versus chain. *European Journal of Operational Research*, 310(2):928–941, 2023a.

J. Cao, D. Li, V. R. Young, and B. Zou. Reinsurance games with n variance-premium reinsurers: from tree to chain. *ASTIN Bulletin: The Journal of the IAA*, page 1–23, 2023b. doi: 10.1017/asb.2023.24.

J. Cao, D. Li, V. R. Young, and B. Zou. Optimal insurance to maximize exponential utility when premium is computed by a convex functional. *SIAM Journal on Financial Mathematics*, 15(1):SC15–SC27, 2024.

K. C. Cheung. Optimal reinsurance revisited–a geometric approach. *ASTIN Bulletin: The Journal of the IAA*, 40(1):221–239, 2010.

Y. Chi. Reinsurance arrangements minimizing the risk-adjusted value of an insurer's liability. *ASTIN Bulletin: The Journal of the IAA*, 42(2):529–557, 2012.

Y. Chi and H. Meng. Optimal reinsurance arrangements in the presence of two reinsurers. *Scandinavian Actuarial Journal*, 2014(5):424–438, 2014.

Y. Chi and K. S. Tan. Optimal incentive-compatible insurance with background risk. *ASTIN Bulletin: The Journal of the IAA*, 51(2):661–688, 2021.

Y. Chi and W. Wei. Optimal insurance with background risk: An analysis of general dependence structures. *Finance and Stochastics*, 24(4):903–937, 2020.

Y. Chi and S. C. Zhuang. Optimal insurance with belief heterogeneity and incentive compatibility. *Insurance: Mathematics and Economics*, 92:104–114, 2020.

D. Curtiss. Recent extentions of Descartes' rule of signs. *Annals of Mathematics*, 19(4):251–278, 1918.

M. Denuit, J. Dhaene, M. Goovaerts, and R. Kaas. *Actuarial theory for dependent risks: measures, orders and models*. John Wiley & Sons, 2006.

H. U. Gerber and G. Pafumi. Utility functions: from risk theory to finance. *North American Actuarial Journal*, 2(3):74–91, 1998.

M. Ghossoub, W. Jiang, and J. Ren. Optimal insurance for a prudent decision maker under heterogeneous beliefs. *European Actuarial Journal*, pages 1–28, 2022.

M. Ghossoub, B. Li, and B. Shi. Bowley-Optimal Convex Premium Principles under Exponential Utility. *Available at SSRN 4438479*, 2023.

A. Golubin. Pareto-optimal insurance policies in the models with a premium based on the actuarial value. *Journal of Risk and Insurance*, 73(3):469–487, 2006.

G. Huberman, D. Mayers, and C. W. Smith Jr. Optimal insurance policy indemnity schedules. *The Bell Journal of Economics*, pages 415–426, 1983.

L. Lin, F. Liu, J. Liu, and L. Yu. The optimal reinsurance strategy with price-competition between two reinsurers. *arXiv preprint arXiv:2305.00509*, 2023.

S. Malamud, H. Rui, and A. Whinston. Optimal reinsurance with multiple tranches. *Journal of Mathematical Economics*, 65:71–82, 2016.

M. Shaked and J. G. Shanthikumar. *Stochastic Orders*. Springer, 2007.

K. S. Tan, C. Weng, and Y. Zhang. VaR and CTE criteria for optimal quota-share and stop-loss reinsurance. *North American Actuarial Journal*, 13(4):459–482, 2009.

Q. Wang, R. Wang, and Y. Wei. Distortion riskmetrics on general spaces. *ASTIN Bulletin: The Journal of the IAA*, 50(3):827–851, 2020.

S. S. Wang. Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, 17(1):43–54, 1995.

S. S. Wang. A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance*, 67(1):15–36, 2000.

G. Werner and C. Modlin. Basic ratemaking. In *Casualty Actuarial Society*, volume 4, pages 1–320, 2010.

Z. Q. Xu, X. Y. Zhou, and S. C. Zhuang. Optimal insurance under rank-dependent utility and incentive compatibility. *Mathematical Finance*, 29(2):659–692, 2019.

M. E. Yaari. The dual theory of choice under risk. *Econometrica: Journal of the Econometric Society*, 55(1):95–115, 1987.

V. R. Young. Optimal insurance under Wang's premium principle. *Insurance: Mathematics and Economics*, 25(2):109–122, 1999.

V. R. Young. Premium principles. *Wiley StatsRef: Statistics Reference Online*, 2014.

M. B. Zhu, M. Ghossoub, and T. J. Boonen. Equilibria and efficiency in a reinsurance market. *Insurance: Mathematics and Economics*, 113:24–49, 2023.

S. C. Zhuang, C. Weng, K. S. Tan, and H. Assa. Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics*, 67:65–76, 2016.

A Higher-order moment-based premium principles

In this appendix, we extend the results of Section 3 to the case where each reinsurer charges the insurer according to the higher-order moment-based premium principle, i.e.

$$\pi_i(I_i) = \sum_{s=1}^{N} \frac{\theta_{is}}{s} \mathbf{E}[I_i(X)^s], \tag{A.1}$$

where $N \geq 2$ is the maximum order of the moments included in the premium principles, and $\theta_{is} \geq 0$ is the safety loading factor applied to the s-th moment of I_i. Similar as Section 3, we first solve Problem (4.1) in this context. The following theorem characterizes the solution to Problem (4.1).

Theorem A.1. *If at least one of* $\{\theta_{is}\}_{s=2,\ldots,N}$ *is positive for all* $i \in \{1, 2, \ldots, n\}$*, the optimal indemnity functions* $(I_1^*, I_2^*, \ldots, I_n^*)$ *that solve Problem (4.1) are given by*

$$I_i^*(x) = \begin{cases} r(x; \theta_{i1}, \ldots, \theta_{iN}, \lambda(x)), & \text{if } \theta_{i1} \leq \lambda(x), \\ 0, & \text{if } \theta_{i1} > \lambda(x), \end{cases} \tag{A.2}$$

where $r(x; \theta_{i1}, \ldots, \theta_{iN}, \lambda(x))$ *is the unique non-negative root of the function*

$$L_i(y; x) := \sum_{s=1}^{N} \theta_{is} y^{s-1} - \lambda(x), \tag{A.3}$$

where $\lambda(x)$ *satisfies*

$$\sum_{\{k: \theta_{i1} \leq \lambda(x)\}} r(x; \theta_{i1}, \ldots, \theta_{iN}, \lambda(x)) = I(x). \tag{A.4}$$

Proof. The proof follows that for Theorem 4.1. Instead of solving Problem (4.1) directly, we solve the following problem first:

$$\begin{aligned} \min_{I_1, \ldots, I_n \in \mathcal{I}_0(I)} \quad & \int_0^M \sum_{k=1}^{n} \left(\sum_{s=1}^{N} \frac{\theta_{ks}}{s} I_k(x)^s \right) dF_X(x) \\ \text{s.t.} \quad & \sum_{k=1}^{n} I_k(x) \geq I(x) \text{ for all } x \in [0, M]. \end{aligned} \tag{A.5}$$

Since the objective function of (A.5) is increasing in $I_1, \ldots, I_n$, we directly conclude that the optimal indemnities $I_1^*, \ldots, I_n^*$ for (A.5) will always make the constraint binding. To solve (A.5), we adopt the element-wise minimization, which turns the problem (A.5) into

$$\min_{I_1, \ldots, I_n \in \mathcal{I}_0(I)} \quad L(I_1, \ldots, I_n) := \sum_{k=1}^{n} \left(\sum_{s=1}^{N} \frac{\theta_{ks}}{s} I_k(x)^s \right) - \lambda(x) \left(\sum_{k=1}^{n} I_k(x) - I(x) \right) \tag{A.6}$$

with x being any given point within $[0, M]$, where $\lambda(x) \geq 0$ is the Lagrangian multiplier. The first-order condition $\frac{\partial L}{\partial I_i} = 0$ yields

$$\sum_{s=1}^{N} \theta_{is} I_i(x)^{s-1} - \lambda(x) = 0.$$

Let $L_i(y; x) = \sum_{s=1}^{N} \theta_{is} y^{s-1} - \lambda(x)$, which is a polynomial function of y. Note that the coefficients $\{\theta_{is}\}_{s=1,\ldots,N}$ are all non-negative, thus the coefficients of $L_i(y; x)$ have either no sign change if $\theta_{i1} \geq \lambda(x)$ or one sign change if $\theta_{i1} < \lambda(x)$. By using the Descartes' rule of signs (Curtiss, 1918), $L_i(y; x)$ has one non-negative root if $\lambda(x) \geq \theta_{i1}$, which we denote by $r(x; \theta_{i1}, \ldots, \theta_{iN}, \lambda(x))$, and no such root if $\lambda(x) < \theta_{i1}$. Here, the Lagrangian multiplier $\lambda(x)$ needs to satisfy

$$\sum_{\{k:\ \theta_{k1} \leq \lambda(x)\}} r(x; \theta_{k1}, \ldots, \theta_{kN}, \lambda(x)) = I(x), \tag{A.7}$$

from which one also concludes that $r(x; \theta_{i1}, \ldots, \theta_{iN}, \lambda(x)) \leq I(x)$ for any $i \in \{1, 2, \ldots, n\}$.

Since $L_i(y;x)$ is increasing in y when $y \geq 0$, we get that the monotonicity of $r(x;\theta_{i1},\ldots,\theta_{iN},\lambda(x))$ aligns with that of $\lambda(x)$. Since $I(x)$ is increasing, we conclude from (A.7) that $\lambda(x)$ is an increasing function, which leads $r(x;\theta_{i1},\ldots,\theta_{iN})$ to be an increasing function. Taking derivative of both sides of (A.7) yields

$$\sum_{\{k:\ \theta_{k1}\leq\lambda(x)\}} r'(x;\theta_{k1},\ldots,\theta_{kN},\lambda(x)) = I'(x) \in [0,1].$$

Thus, all $r(x;\theta_{i1},\ldots,\theta_{iN},\lambda(x))$ are in the class $\mathcal{I}$. Hence, the indemnity functions characterized via

$$I_i^*(x) = r(x;\theta_{i1},\ldots,\theta_{iN},\lambda(x)) \text{ if } \theta_{i1} \leq \lambda(x) \quad \text{and} \quad I_i^*(x) = 0 \text{ if } \theta_{i1} > \lambda(x)$$

also solve the problem (4.1) under the higher-order moment-based premium principles. This concludes the proof. □

For any $x \in [0, M]$, we define $H_i^*(x) = \sum_{s=1}^N \theta_{is} I_i^*(x)^{s-1}$ for $i \in \{1,2,\ldots,n\}$. As per the proof for Theorem A.1, for $j \in \{k : \theta_{k1} \leq \lambda(x)\}$ we have

$$H_j^*(x) = \sum_{s=1}^N \theta_{js} r(x;\theta_{j1},\ldots,\theta_{jN},\lambda(x))^{s-1} = \lambda(x) < \min_{\{k:\theta_{k1}>\lambda(x)\}}\{\theta_{k1}\} = \min_{\{k:\theta_{k1}>\lambda(x)\}}\{H_k^*(x)\}.$$

It is thus straightforward that

$$\mathcal{K}^*(x) = \operatorname*{arg\,min}_{1\leq k\leq n}\{H_k^*(x)\} = \{k : \theta_{k1} \leq \lambda(x)\}.$$

Similar to the implications of Theorem 4.1, Theorem A.1 implies that the participation of a reinsurer into the risk sharing depends solely on the loading factor for the first-order moment, while other moments contribute to the determination of the proportion of loss that the reinsurer is to bear.

The following theorem concludes the optimal value for the objective function of (4.1) in the general case.

Theorem A.2. *Under the condition of Theorem A.1, the optimal value for the objective function in Problem (4.1) is*

$$\pi(I) = \min_{\overrightarrow{I}\in\overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^n \pi_k(I_k(X)) = \int_0^M H^*(x) S_X(x) dI(x), \tag{A.8}$$

where $H^*(x) = \min_{1\leq k\leq n}\{H_k^*(x)\}$.

Proof. We obtain

$$\begin{aligned}\min_{\overrightarrow{I}\in\overrightarrow{\mathcal{I}}(I)} \sum_{k=1}^n \pi_k(I_k(X)) &= \sum_{k=1}^n \pi_k(I_k^*(X)) \\ &= \int_0^M \left(\sum_{k=1}^n H_k^*(x) I_k^{*\prime}(x)\right) S_X(x)dx \\ &= \int_0^M \left(\sum_{k\in\mathcal{K}^*(x)} H_k^*(x) I_k^{*\prime}(x)\right) S_X(x)dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^M \left(H^*(x) \sum_{k \in \mathcal{K}^*(x)} I_k^{*\prime}(x) \right) S_X(x) dx \\
&= \int_0^M H^*(x) I'(x) S_X(x) dx.
\end{aligned}
$$

This concludes the proof. □

The following theorem states that the premium principle in (A.8) is still monotonic and convex.

Theorem A.3. *The premium principle $\pi(I)$ in (A.8) is monotonic and convex.*

Proof. It is easy to verify that the polynomial function $\sum_{s=1}^{N} \frac{\theta_{is}}{s} x^s$ is convex for $x \geq 0$ if all the coefficients $\theta_{is} \geq 0$. Thus, $\pi_i(I_i)$ is convex in I_i. Based on this fact and Theorem A.1, we can show that Theorem 4.3 still holds under the higher-order-moment-based premium principle (A.1). The proof is similar to that of Theorem 4.3 and thus omitted. □

Though Theorems A.1∼A.3 are natural extensions of Theorems 4.1∼4.3, the development of numerical schemes to explicitly solve Problem 1 might face non-trivial challenges. It is noteworthy that under the second-order moment-based premium principle, the optimal indemnities as shown in Theorem 4.1 admit parametric forms and the representative reinsurer's premium principle as shown by Theorem 4.2 also admits closed form. These explicit forms greatly facilitate the use of numerical approach. However, under the higher-order moment-based premium principles, as shown by Theorems A.1 and A.2, the parametric or closed forms are generally no longer derivable, which becomes the main obstacle for applying the similar numerical approach.