Optimal insurance design in the presence of government financial assistance

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Abstract

This paper revisits the study of insurance demand in the context of potential government financial assistance, such as *ex post* disaster relief and *ex ante* premium subsidies. We impose the incentive-compatibility condition on the indemnity, and assume that the premium is determined by the actuarial-value-based premium principle. By applying Ohlin's lemma, we characterize the optimal forms of the indemnity function under independence between the relief event and the insurable loss. The optimal parameters of the indemnity function are derived, and both analytical and numerical comparative studies are conducted to demonstrate the effects of disaster relief and premium subsidies on the demand for insurance. Furthermore, we study two forms of dependence between the relief event and the insurable loss. First, we study one specific yet common loss-dependent relief probability case. Second, we study special cases of conditional insurable loss distributions using the hazard rate ordering. Also, we study the effect of premium subsidies on the insurance demand, and show that premium subsidies increase the demand for insurance under increasing absolute risk aversion. The results provide new insights into the study of natural hazard insurance demand in the presence of government interventions.

Key words: Optimal insurance, disaster relief fund, premium subsidy, deductible function, incentive-compatibility condition.

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1 Introduction

Each year, natural disasters, such as earthquakes, storms, and floods, cause significant personal and economic losses globally. According to the 2023 report by Munich Re,[1] these disasters resulted in global losses of approximately $250 billion. Natural hazard insurance plays a central role in providing individuals and businesses with financial protection against these losses. Such insurance can be purchased from insurance companies to recover from specific natural hazards. However, for some major systemic catastrophes, the government may provide assistance and broader support, such as infrastructure reconstruction and essential services restoration. The benefits of natural hazard insurance and government assistance are therefore related, and can be complementary to each other. Government assistance programs are typically financed through tax revenue or dedicated disaster relief funds, and governments may also subsidize insurance premiums to enhance the affordability of insurance in high-risk areas.

To properly budget for *ex ante* premium subsidies and *ex post* disaster relief fund, it is crucial for the government to understand the insurance demand of individuals or corporations with and without any forms of government assistance. It is noteworthy that the government assistance can be regarded as a special kind of background "risk" of the insurance buyer, and the literature on optimal insurance contracting under background risks is rich. In the presence of the additive background risk, Chi & Wei (2018) examine the optimality of the stop-loss function under the higher-order risk attitudes of the insurance buyer. Chi & Tan (2021) study the effect of the stochastic dependence between the background risk and the insurable risk on the optimal insurance contract under the incentive-compatibility condition, and show that the optimal contract would change substantially once the incentive-compatibility condition gets removed. Another background risk is often referred to as the counterparty's risk or insurer's default risk. Some recent representative works are, for example, Boonen & Jiang (2022) and Boonen & Jiang (2023).

Both disaster relief assistance and premium subsidies are government intervention measures, which aim to address the rapid growth of perils caused by natural hazards in a public-private partnership context. In practice, if the government provides disaster relief funds to the policyholders, it typically distributes a fixed amount or portion of funds to the insureds; see Kelly & Kleffner (2003), Linnerooth-Bayer et al. (2005), and Raschky & Weck-Hannemann (2007), to name a few. There exist various disaster-related programs in the United States that are coordinated by the Federal Emergency Management Agency (FEMA), including the National Flood Insurance Program (NFIP), the Individual Assistance (IA) program, and so on. FEMA offers subsidized rates to reduce flood insurance premiums for policyholders in flood-prone regions. If a flood-related disaster is federally declared, an NFIP policyholder may file a claim with the NFIP to receive compensation for their flood-related losses and may also be eligible to apply for the IA program to receive additional assistance. For example, Akbulut-Yuksel et al. (2023) show that the individuals residing in flooded areas experience a rise in income from the floods that struck the state of Queensland in Australia in 2010 with the government's post-disaster relief funds. The role of government interventions has been extensively investigated in the literature on insurance economics and policy making, particularly in flood and crop insurance markets where substantial empirical analysis has already been conducted. Moreover, Deryugina & Kirwan (2018) show that disaster relief anticipation is qualitatively and quantitatively important for insurance demand by studying the US crop insurance markets. The authors also point out that eliminating disaster payments is

[1] See https://www.munichre.com/content/dam/munichre/mrwebsiteslaunches/2023-annual-report/MunichRe-Group-Annual-Report-2023-en.pdf/_jcr_content/renditions/original./MunichRe-Group-Annual-Report-2023-en.pdf

something the government has yet to commit to.

Despite the social benefit created by the government disaster relief assistance, the insurance demand may be reduced as well. Anticipating public charitable assistance, individuals may intentionally reduce their insurance spending. Such behavior is called "charity hazard", which has been shown by, for example, Browne & Hoyt (2000), Van Asseldonk et al. (2002), Raschky & Weck-Hannemann (2007), Miglietta et al. (2020), and Robinson et al. (2021). While government relief funds can generally be seen as a risk management tool for individuals, as addressed in Raschky et al. (2013), the payment event and size are somewhat *ad hoc*, depending on the political and societal circumstances, and generally do not cover all kinds of losses. Hence, government disaster relief assistance is subject to uncertainty, leaving individuals in an uncertain position to seek other risk-hedging tools.

It is worth mentioning that the provision of government relief usually depends on the severity of natural disasters, which means that there is a possibility for the relief action. Inspired by this fact, Hinck (2024) studies the design of optimal insurance contracts with government disaster relief payments where the government's relief event is modeled by a binary random variable. He derives the shape of optimal insurance contracts within a framework with and without ambiguity on the relief probability. In Hinck (2024), the relief payment function is required to be a twice continuously differentiable function, which excludes some practical cases such as that the relief fund is capped for each individual (or the population in a disaster area). Furthermore, the insurance compensation function is not required to satisfy the incentive compatibility condition, which may result in *ex post* moral hazard issues. Hinck (2024) identifies sufficient conditions, such as a constant relief probability, under which the incentive compatibility condition is satisfied.

To improve the willingness of (catastrophe) insurance purchase, governments may take out a portion (or the whole) of the relief fund as *ex ante* premium subsidies. Another reason is that premium subsidies may perform better than relief payments in reducing the retained loss of individuals or insureds. For example, by analyzing frost insurance demand of German winegrowers, Philippi & Schiller (2024) demonstrate that the implementation of a premium subsidy in an immature market with low levels of participation, presumably caused by strong anticipation of disaster relief, is effective in increasing overall frost insurance demand. Hence, premium subsidies can be used to address low demand for natural hazard insurance when it is partly caused by governmental disaster relief payments. In particular, premium subsidies have been widely used in crop insurance markets. For instance, the U.S. government has been actively shifting from providing *ex post* disaster relief toward providing *ex ante* premium subsidies in crop insurance markets; see, for example, Glauber (2013), Yu et al. (2018), and Tsiboe & Turner (2023).

This paper contributes to the literature in several ways. By incorporating the disaster relief funds, a general bivariate function is applied to model the terminal retained loss of the decision maker (DM) as a consequence of the combined action of the insurer and government. An optimal insurance demand problem is studied in detail when the relief payment is modeled as a binary random variable. Under the setting in which the loss is independent of the relief payment, the deductible policy is proved optimal, with the optimal deductible levels derived for several special cases. Besides that, the effect of the amount of payment and premium subsidies are analytically studied, with surprising examples provided when the DM holds different risk attitudes. Moreover, we study the insurance demand problem under the dependence of the relief payment and loss, where the optimal parametric ceded loss function is derived for the case in which the relief payment probability is increasing with the size of loss, and an improvement technique is proposed for the case when the DM is faced with different loss distributions under different amounts of relief payment. Compared with the two recent representative works on the coupling of insurance and government financial assistance, i.e., Hinck (2024) and Philippi & Schiller (2024), our paper focuses on a class of indemnity functions that satisfy the incentive-compatibility condition, which is now popular in many well-known research articles, and employs the stochastic ordering approach to derive the optimal contract, which is proven more efficient in the absence of some differentiability conditions. The economic implications of the results of Philippi & Schiller (2024) are also extended to a more general class of indemnity functions.

This paper is set out as follows. Section 2 introduces the disaster relief mechanism and formulates the insurance demand problem in the paper. When the relief event is independent of the insurable loss, Section 3 shows that the solution is of the deductible form, and analyzes the effect of the disaster relief fund and premium subsidy on the insurance demand. Section 4 studies the case where the relief probability depends on the loss size via a particular step function. The optimal retained loss function is proven to have multiple layers. Further, in the sense of stochastic ordering, the insurance demand is studied when loss distribution varies with the amount of paid relief fund. Section 5 concludes the paper and suggests potential research directions. A proof of Theorem 2.1 is provided in Appendix A.

2 Problem formulation

We focus on a one-period economy that is built on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a non-negative random variable representing the loss faced by a DM, with continuous support $[0, M]$ for some $M > 0$. The cumulative distribution function (CDF) and survival function of X are denoted by F_X and S_X, respectively. In the case of eligible disasters occurring, the government may provide disaster relief funds to individuals. Let Y denote the payment that the government can pledge to the DM, which is a random variable with the following binary distribution:

$$Y = \begin{cases} 0, & \text{if no relief is provided;} \\ \ell, & \text{if relief is provided.} \end{cases}$$

where $\ell > 0$ is a constant, representing the relief budget level. The decision of providing disaster financial assistance is up to the government, and the relief probability $p := \mathbb{P}(Y = \ell) \in [0, 1]$ can be assumed to be either constant or dependent on X. Here, $p = 0$ results in no government disaster relief payments and it becomes a standard insurance problem. The case of $p = 1$ indicates the deterministic participation of government relief.

Suppose that the DM aims to protect herself via insurance. An insurance contract typically consists of a ceded loss function I (also known as the *indemnity function*) and its associated premium $\pi(I)$. To alleviate potential *ex post* moral hazard, we exogenously impose the so-called incentive compatibility condition on the indemnity function motivated by Huberman et al. (1983), limiting I to the following set:

$$\mathcal{I} := \{ I : [0, M] \mapsto [0, M] \mid I(0) = 0,\ 0 \leq I(x_2) - I(x_1) \leq x_2 - x_1,\ \forall\, 0 \leq x_1 \leq x_2 \leq M \}.$$

The functions within $\mathcal{I}$ satisfy several desirable properties. For example, the ceded loss is always increasing, and its increment does not exceed the increment of the total loss. It is worth noting that the functions within $\mathcal{I}$ are 1-Lipschitz continuous.

We define $G(x, y)$ as the bivariate function of the retained loss and the government's relief fund, representing the DM's final retained loss after receiving the relief fund. In consideration of the government's relief fund, the DM aims to seek the optimal ceded loss function I^* from the insurer by solving the following optimization problem:

$$\max_{I\in\mathcal{I}} \mathbb{E}\left[u\left(W_0 - G(X - I(X), Y) - \pi(I(X))\right)\right]. \tag{1}$$

Throughout this paper, we assume that

(a). The utility function $u : \mathbb{R} \mapsto \mathbb{R}$ is increasing, continuously differentiable, and concave;

(b). $\pi(I(X)) = h(\mathbb{E}[I(X)])$ is an actuarial-value-based premium principle, where h is referred to as the *pricing function* and satisfies $h(0) = 0$ and $h'(x) \geq 1$ for $x \in [0, M]$;

(c). $G(x, y)$ is a non-negative and convex function of x for each fixed y, and satisfies $x - y \leq G(x, y) \leq x$. Moreover, $G(x, y)$ is decreasing in y, for any fixed x.

Note that the premium is charged based on the promised insurance indemnification instead of the actual compensation since the disaster relief fund is paid (if any) after the insurance indemnification. Two special cases of the function h are $h(x) = (1 + \theta)x$ for $\theta \geq 0$, which yields the expected value premium principle: $\pi(I(X)) = (1 + \theta)\mathbb{E}[I(X)]$, and $h(x) = ((1 + \theta)x - s)_+$ for $\theta, s \geq 0$, which yields $\pi(I(X)) = \left((1 + \theta)\mathbb{E}[I(X)] - s\right)_+$. Moreover, the assumptions on G imply that

(i). the retained loss remains unchanged if no disaster relief funds are provided and reduces from $X - I(X)$ to $G(X - I(X), \ell)$ otherwise;

(ii). with the relief fund, the marginal increment of the final retained loss is increasing with respect to the original amount of retained loss;

(iii). the final retained loss is further reduced after receiving a larger amount of relief fund.

Two special cases of the function G are $G(x, y) = (x - y)_+$ and $G(x, y) = x - (\beta x) \wedge y$, $\beta \in (0, 1)$, where we use the notation $z_+ := \max\{z, 0\}$ and $z_1 \wedge z_2 := \min\{z_1, z_2\}$. Then, the terminal retained loss can be written as

$$G(X - I(X), \ell) = (X - I(X) - \ell)_+$$

and

$$G(X - I(X), \ell) = X - I(X) - (\beta(X - I(X))) \wedge \ell,$$

respectively. Here, in the first special case, the government pledges the amount ℓ, but this is capped by the constraint that the government relief does not exceed the original retained risk. Likewise, for the second special case with $\beta = 1$, the retained risk after government relief is at most ℓ, but it is smaller if the original retained risk is already smaller than ℓ.

If $\ell = M$, then the random variable $G(X - I(X), M)$ is the same as the retained risk after government relief in Hinck (2024), with the exception that we do not impose G being twice continuously differentiable (or even differentiable). This makes our setting more general than Hinck (2024).

Let $R(X) = X - I(X)$ be the retained loss of the DM after receiving the indemnity from the insurer. Problem (1) can be written as

$$\max_{R\in\mathcal{I}} \mathbb{E}\left[u\left(W_0 - G(R(X), Y) - \pi(X - R(X))\right)\right]. \tag{2}$$

The following theorem shows the existence of solutions to (2), and provides a sufficient condition for the uniqueness of solution.

Theorem 2.1. *There exists a solution $R^* \in \mathcal{I}$ to Problem (2). If h is convex and at least one of $-u$, G and h is strictly convex, then R^* is unique $\mathbb{P}$-almost surely, that is $\mathbb{P}(R^*(X) = \tilde{R}(X)) = 1$ for all $\tilde{R}$ that also solve Problem (2).*

The proof of Theorem 2.1 is similar to the proof of Theorem 2.1 in Liang et al. (2023). For completeness, we provide a complete proof in Appendix A. In the proof of Theorem 2.1, we use a compactness argument on $\mathcal{I}$, for which we need X to be bounded. If existence of solutions to Problem (2) happens to hold true, we remark that the results in the rest of the paper also hold for $M = \infty$ provided that $\mathbb{E}[X] < \infty$.

3 Independence between the loss and government relief

3.1 General result

This section studies the case where there is independence between the government's relief decision and the size of loss X.

Assumption 1. *The random variables X and Y are independent.*

Under Assumption 1, Problem (2) can be written as:

$$\begin{aligned} \max_{R\in\mathcal{I}} \; \Big\{ & p\mathbb{E}[u(W_0 - G(R(X), \ell) - \pi(X - R(X)))] \\ & + (1-p)\mathbb{E}[u(W_0 - R(X) - \pi(X - R(X)))]\Big\}, \end{aligned} \tag{3}$$

where we recall that p is the government relief probability $\mathbb{P}(Y = \ell)$.

We next provide the parametric solution to the problem (3). The proof of the following theorem relies on an improvement technique based on the convex order proposed in Ohlin (1969). This improvement technique shows that the deductible indemnity is optimal for any concave expected utility objectives among all indemnities with a fixed expectation and premium. This technique is also used by Cheung (2010), Sung et al. (2011), and Chi & Tan (2013), but is different from the approach of Hinck (2024).

Theorem 3.1. *Under Assumption 1, the solution to Problem (3) is of the form $R^*(x) = x \wedge d$ for some $d \geq 0$.*

Proof: We start the proof with a definition. We say that a continuous function f up-crosses (resp. down-crosses) another continuous function g at x_0 if there exist $\epsilon_1, \epsilon_2 > 0$ such that $f(x) \leq g(x)$ (resp. $f(x) \geq g(x)$) for $x \in (x_0 - \epsilon_1, x_0)$ and $f(x) \geq g(x)$ (resp. $(f(x) \leq g(x))$ for $x \in (x_0, x_0 + \epsilon_2)$ with the inequalities being strict for some $x \in (x_0 - \epsilon_1, x_0 + \epsilon_2)$.

Next, we proceed with the proof the theorem. To solve Problem (3), we consider its variant with a fixed budget:

$$\begin{aligned} &\max_{R\in\mathcal{I}} p\mathbb{E}[u(W_0 - G(R(X), \ell) - P)] + (1-p)\mathbb{E}[u(W_0 - R(X) - P)] \\ &\text{s.t. } P = h(\mathbb{E}[X - R(X)]). \end{aligned} \tag{4}$$

Let $v_1(x) = \mathbb{E}[u(W_0 - G(x, \ell) - P)]$ and $v_2(x) = u(W_0 - x - P)$, then it is easy to verify that v_1 and v_2 are both concave functions. This simplifies Problem (4) as follows:

$$\begin{aligned} &\max_{R\in\mathcal{I}} p\mathbb{E}[v_1(R(X))] + (1-p)\mathbb{E}[v_2(R(X))], \\ &\text{s.t. } \mathbb{E}[R(X)] = \mathbb{E}[X] - h^{-1}(P) = \tilde{P} \geq 0. \end{aligned} \tag{5}$$

Given any $R \in \mathcal{I}$ that satisfies $\mathbb{E}[R(X)] = \tilde{P}$, we can always find a $d \geq 0$ such that $\mathbb{E}[X \wedge d] = \mathbb{E}[R(X)] = \tilde{P}$. Let $R^*(x) = x \wedge d$, it is straightforward that $R(x)$ up-crosses $R^*(x)$ at some point $x_0 \in [0, M]$. As such, by applying Lemma 3 of Ohlin (1969), we have $R^*(X) \leq_{cx} R(X)$, where $\leq_{cx}$ denotes the convex order. Since $-v_1$ and $-v_2$ are both convex functions, we have

$$\mathbb{E}[-v_1(R^*(X))] \leq \mathbb{E}[-v_1(R(X))], \quad \mathbb{E}[-v_2(R^*(X))] \leq \mathbb{E}[-v_2(R(X))],$$

which leads to

$$p\mathbb{E}[v_1(R^*(X))] + (1-p)\mathbb{E}[v_2(R^*(X))] \geq p\mathbb{E}[v_1(R(X))] + (1-p)\mathbb{E}[v_2(R(X))].$$

This completes the proof. ■

Theorem 3.1 indicates that the optimal form of the ceded loss function is a classical deductible function $I^*(x) = x - R^*(x) = (x-d)_+$, which simplifies Problem (3) as follows:

$$\max_{d \geq 0} \Big\{ p\mathbb{E}[u(W_0 - G(X \wedge d, \ell) - \pi((X-d)_+))] \\ + (1-p)\mathbb{E}[u(W_0 - X \wedge d - \pi((X-d)_+))] \Big\}. \tag{6}$$

In the next subsection, some analytical forms of the optimal deductible level will be presented for special cases of (6).

Our problem (1) is comparable to the optimal insurance problem under background risk that was considered by Dana & Scarsini (2007) or many other relevant literature:

$$\max_{I \in \tilde{\mathcal{I}}} \mathbb{E}[u(W_0 - X_1 - X_2 + I(X_1) - \pi(I(X_1)))], \tag{7}$$

where X_1 is the insurable risk, X_2 is a background risk that may depend on X_1, and $\tilde{\mathcal{I}}$ is a more general class of indemnity functions without considering the incentive-compatibility condition. Dana & Scarsini (2007) show that if (a) $X_1 + X_2$ is stochastically increasing with respect to X_1 and (b) X_2 is stochastically decreasing with respect to X_1, then a generalized deductible policy is optimal to the problem (7), where $I^*(x) = 0$ for $x \in [0, d]$ and $I^{*\prime}(x) \in [0, 1]$ for $x \in (d, M]$. Note that in our case, $X_1 = X$ and $X_2 = G(X - I(X), \ell) - X$, and it is easy to check that the conditions (a) and (b) hold for our case given $I \in \mathcal{I}$. Hence, the optimal indemnity function for our problem can be partially explained by the rationales from Dana & Scarsini (2007). Remarkably, the key difference between our problem and that of Dana & Scarsini (2007) is that our X_2 includes the indemnity function, which justifies the optimality of the stop-loss function, since $I^{*\prime}(x) = 1$ for $x \in (d, M]$ leads to the smallest X_2 among all such generalized deductible policies.

3.2 Optimal deductible levels for special cases of (6)

Following Philippi & Schiller (2024), who studied the optimal proportion for the quota-share insurance under a two-point loss distribution, we first investigate the optimal deductible level for Problem (6) when $p = 1$, i.e., the case where the government always provides disaster relief fund:

$$\max_{d \geq 0} \mathbb{E}[u(W_0 - G(X \wedge d, \ell) - \pi((X-d)_+))]. \tag{8}$$

For the ease of presentation, we write

$$\psi_1(d) = \mathbb{E}[u(W_0 - G(X \wedge d, \ell) - \pi((X-d)_+))], \quad \psi_2(d) = \mathbb{E}[u(W_0 - X \wedge d - \pi((X-d)_+)].$$

Since $G(\cdot,\ell)$ is a convex function on a compact domain, it is absolutely continuous on the interior of the domain. Thus, it is differentiable almost everywhere. The notation $G_1'(x,y)$ denotes the partial derivative of G with respect to its first component x for fixed y, whenever it exists.

Theorem 3.2. *Let* $d_\ell = \inf\{d \in [0,M] : S_X(d)h'(\mathbb{E}[(X-d)_+]) \le G_1'(d,\ell)\}$ *and*

$$\phi_\ell(d) = \frac{\mathbb{E}[u'(W_0 - G(X\wedge d,\ell) - \pi((X-d)_+)]}{u'(W_0 - G(d,\ell) - \pi((X-d)_+))}.$$

The optimal deductible level for Problem (8) is

$$d_\ell^* = \inf\left\{d \in [d_\ell, M] : \phi_\ell(d)h'(\mathbb{E}[(X-d)_+]) \le G_1'(d,\ell)\right\}, \tag{9}$$

where by convention $\inf\emptyset$ *is defined as the right-end point of the domain.*

Proof: Note that

$$\begin{aligned}\psi_\ell(d) = &\int_0^d u(W_0 - G(x,\ell) - \pi((X-d)_+))dF_X(x)\\ &+ \int_d^M u(W_0 - G(d,\ell) - \pi((X-d)_+))dF_X(x).\end{aligned}$$

A simple calculation yields

$$\begin{aligned}\psi_\ell'(d) =& S_X(d)\Big\{\mathbb{E}[u'(W_0 - G(X\wedge d,\ell) - \pi((X-d)_+))]h'(\mathbb{E}[(X-d)_+])\\ &- u'(W_0 - G(d,\ell) - \pi((X-d)_+))G_1'(d,\ell)\Big\}\\ =& S_X(d)u'(W_0 - G(d,\ell) - \pi((X-d)_+))\left\{\phi_\ell(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell)\right\}.\end{aligned}$$

Since h is an increasing convex function and $G(x,y)$ is increasing convex in x, $S_X(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell)$ is decreasing in d. As such, by the definition of d_ℓ, we have

$$S_X(d)h'(\mathbb{E}[(X-d)_+])\begin{cases} > G_1'(d,\ell), & \text{if } d < d_\ell,\\ \le G_1'(d,\ell), & \text{if } d \ge d_\ell,\end{cases}$$

whenever $G_1'(d,\ell)$ exists. We note that

$$\begin{aligned}\phi_\ell(d) =& \frac{\int_0^d u'(W_0 - G(x,\ell) - \pi((X-d)_+))dF_X(x)}{u'(W_0 - G(d,\ell) - \pi((X-d)_+))}\\ &+ \frac{\int_d^M u'(W_0 - G(d,\ell) - \pi((X-d)_+))dF_X(x)}{u'(W_0 - G(d,\ell) - \pi((X-d)_+))} \ge S_X(d).\end{aligned}$$

Therefore, when $d < d_\ell$, we have

$$\phi_\ell(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell) \ge S_X(d)h'(\mathbb{E}[(X-d)_+]) - G'(d,\ell) > 0,$$

whenever $G'(d,\ell)$ exists, which leads to $\psi_\ell'(d) \ge 0$ over $[0, d_\ell)$. Thus, the optimal deductible level is located within $[d_\ell, M]$.

The derivative of $\phi_\ell(d)$ is given by:

$$\begin{aligned}\phi'_\ell(d) =& \frac{\int_0^d u''(W_0 - G(x,\ell) - \pi((X-d)_+))dF_X(x) \cdot h'(\mathbb{E}[(X-d)_+])S_X(d)}{u'(W_0 - G(d,\ell) - \pi((X-d)_+))} \\ &+ \phi_\ell(d) \cdot AR(W_0 - G(d,\ell) - \pi((X-d)_+)) \left[h'(\mathbb{E}[(X-d)_+])S_X(d) - G'_1(d,\ell)\right],\end{aligned}$$

where $AR(x) = -\frac{u''(x)}{u'(x)}$ denotes the Arrow-Pratt absolute risk aversion of the utility function u. It is apparent that $\phi'_\ell(d) \le 0$ for $d \in [d_\ell, M]$. Hence, $\phi_\ell(d)h'(\mathbb{E}[(X-d)_+]) - G'_1(d,\ell)$ is decreasing for $d \in [d_\ell, M]$. As per the definition of d^*_ℓ, we have $\psi'_\ell(d) \ge 0$ on $[0, d^*_\ell)$ and $\psi'_\ell(d) \le 0$ on $[d^*_\ell, M]$. Thus, d^*_ℓ is the optimal deductible level for Problem (8). This completes the proof. ■

Theorem 3.2 complements Theorem 4.2 of Chi (2019), where the optimal deductible level is derived for the Arrow's model but under heterogeneous beliefs between the policyholder and the insurer. Furthermore, the proof of Theorem 3.2 shows that $\psi_1(d)$ is a unimodal quasi-concave function. Similarly, $\psi_2(d)$ is also a unimodal quasi-concave function.[2]

We next presents the optimal deductible level for another special case of (6), specifically, the one where the government provides no disaster relief fund:

$$\max_{d\ge 0} \mathbb{E}[u(W_0 - X \wedge d - \pi((X-d)_+))]. \tag{10}$$

This case corresponds to the scenario: $\ell = 0$ or $G(X \wedge d, \ell) = X \wedge d$. The following result is a corollary of Theorem 3.2.

Corollary 3.1. *Let* $d_0 = \inf\{d \in [0, M] : S_X(d)h'(\mathbb{E}[(X-d)_+]) \le 1\}$ *and*

$$\phi_0(d) = \frac{\mathbb{E}[u'(W_0 - X \wedge d - \pi((X-d)_+)]}{u'(W_0 - d - \pi((X-d)_+))}.$$

The optimal deductible level for Problem (10) *is*

$$d^*_0 = \inf\left\{d \in [d_0, M] : \phi_0(d)h'(\mathbb{E}[(X-d)_+]) \le 1\right\}. \tag{11}$$

3.3 The effect of disaster relief fund on the demand for insurance

Philippi & Schiller (2024) show that the presence of government's disaster relief fund will lower the demand for insurance in a simple setting (i.e., the setting where only quota-share insurance is allowed, and the loss follows a two-point distribution). We will extend the result of Philippi & Schiller (2024) to a more general setting. For that purpose, we adopt the following mild assumption in this section.

Assumption 2. *For* $0 \le y_2 \le y_1 \le M$, $G'_1(x, y_1) \le G'_1(x, y_2)$, *whenever the derivatives exist.*

Assumption 2 implies that: if the government increases the amount of relief fund, the marginal increment of the retained loss gets lowered. This holds for a large class of G. For example, if $G(x,y) = (x-y)_+$, we have

$$G'_1(x,y) = \begin{cases} 1, & \text{if } y < x, \\ 0, & \text{if } y > x. \end{cases}$$

[2]A function f is called quasi-concave when $f(\lambda x + (1-\lambda)y) \ge \min\{f(x), f(y)\}$ for all x, y on the domain and all $\lambda \in [0,1]$.

If $G(x,y) = x - (\beta x) \wedge y$ for some $\beta \in [0,1]$, we have

$$G_1'(x,y) = \begin{cases} 1, & \text{if } y < \beta x, \\ 1-\beta, & \text{if } y > \beta x. \end{cases}$$

An extreme case is when $y = 0$, which yields $G(x,0) = x$. Since $G(x,y)$ is increasing and convex in x and satisfies $G(x,y) \le x$, it naturally follows that $G_1'(x,y) \le 1 = G_1'(x,0)$. We next present Theorem 3.3 to illustrate the relationship between the optimal deductible level and the disaster relief fund level.

Theorem 3.3. *If the absolute risk aversion coefficient* $AR(x) := -\frac{u''(x)}{u'(x)}$ *is decreasing, then the optimal deductible level to Problem (8) is increasing with respect to* ℓ.

Proof: Denote by d_1^* and d_2^* the optimal deductible levels to Problem (8) when $\ell = \ell_1$ and ℓ_2, respectively, where $\ell_1 \ge \ell_2$. Under Assumption 2, we have

$$S_X(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell_1) \ge S_X(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell_2),$$

whenever $G_1'(d,\ell_1)$ and $G_1'(d,\ell_2)$ exist, which results in

$$\begin{aligned} d_1 &= \inf\{d \in [0,M] : S_X(d)h'(\mathbb{E}[(X-d)_+]) \le G_1'(d,\ell_1)\} \\ &\ge \inf\{d \in [0,M] : S_X(d)h'(\mathbb{E}[(X-d)_+]) \le G_1'(d,\ell_2)\} = d_2. \end{aligned}$$

To prove $d_1^* \ge d_2^*$, it suffices to show that

$$\phi_{\ell_1}(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell_1) \ge \phi_{\ell_2}(d)h'(\mathbb{E}[(X-d)_+]) - G_1'(d,\ell_2)$$

for $d \in [d_1, M]$ whenever both $G_1'(d,\ell_1)$ and $G_1'(d,\ell_2)$ exist. In the following, we show that $\phi_{\ell_1}(d) \ge \phi_{\ell_2}(d)$ for $d \in [d_1, M]$ by employing the calculus of variation technique.

To shorten the notations, we let $G_1 := G(X \wedge d, \ell_1)$, $G_2 := G(X \wedge d, \ell_2)$, $\tilde{G}_1 := G(d,\ell_1)$, $\tilde{G}_2 := G(d,\ell_2)$ and $\pi := \pi((X-d)_+)$. We consider the following function

$$\varphi(\epsilon) = \frac{\mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]}{u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi)},$$

for $\epsilon \in [0,1]$, and it is apparent that $\varphi'(\epsilon) \le 0$ on $[0,1]$ implies

$$\phi_{\ell_1}(d) \ge \phi_{\ell_2}(d).$$

A simple calculation yields $\varphi'(\epsilon) \le 0$ if and only if

$$\begin{aligned} &-\mathbb{E}[u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)(G_2 - G_1)]u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\ &+ \mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1))(\tilde{G}_2 - \tilde{G}_1) \le 0. \end{aligned}$$

Since $G_1'(x,\ell_2) - G_1'(x,\ell_1) \ge 0$, we have $G(d,\ell_2) - G(d,\ell_1) \ge G(X \wedge d, \ell_2) - G(X \wedge d, \ell_1)$. This leads to

$$\begin{aligned} &-\mathbb{E}[u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)(G_2 - G_1)]u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\ &+ \mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1))(\tilde{G}_2 - \tilde{G}_1) \\ \le\ &\left(\tilde{G}_2 - \tilde{G}_1\right)\Big\{-\mathbb{E}[u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\ &+ \mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1))\Big\}. \end{aligned} \tag{12}$$

Furthermore, since $G(x, y)$ is increasing in x, we have

$$W_0 - G_1 - \epsilon(G_2 - G_1) - \pi \geq W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi.$$

Due to the decreasingness of $AR(x)$, we have

$$AR(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi) \leq AR(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi),$$

which implies

$$\begin{aligned}
& - u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\
& \leq -u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) - \pi)u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi).
\end{aligned}$$

Hence,

$$\begin{aligned}
& - \mathbb{E}[u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\
& \leq -u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi)\mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)].
\end{aligned}$$

Hence, (12) satisfies

$$\begin{aligned}
& \left(\tilde{G}_2 - \tilde{G}_1\right) \Big\{ - \mathbb{E}[u''(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u'(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1) - \pi) \\
& + \mathbb{E}[u'(W_0 - G_1 - \epsilon(G_2 - G_1) - \pi)]u''(W_0 - \tilde{G}_1 - \epsilon(\tilde{G}_2 - \tilde{G}_1)) \Big\} \leq 0.
\end{aligned}$$

Therefore, we reach the conclusion that $\varphi'(\epsilon) \leq 0$ for $\epsilon \in [0, 1]$, which yields $d_1^* \geq d_2^*$. This completes the proof. ■

The interpretation for Theorem 3.3 is straightforward. The DM becomes wealthier with the relief fund and thus less risk averse under the condition of Theorem 3.3, which leads to less demand for insurance.

Theorem 3.3 also implies that $d_\ell^* \geq d_0^*$. Since $\psi_\ell(d)$ and $\psi_0(d)$ are both unimodal quasi-concave functions, it follows that

$$p\psi_\ell'(d_0^{*-}) + (1-p)\psi_0'(d_0^{*-}) \geq 0 \quad \text{and} \quad p\psi_\ell'(d_\ell^{*+}) + (1-p)\psi_0'(d_\ell^{*+}) \leq 0.$$

Hence, the following corollary is a direct consequence of Theorem 3.3.

Corollary 3.2. *If the absolute risk aversion coefficient $AR(x) := -\frac{u''(x)}{u'(x)}$ is decreasing, then the optimal deductible level d^* of Problem (6) satisfies $d_0^* \leq d^* \leq d_\ell^*$.*

The general analytical result for d^* is difficult to obtain as the sum of two unimodal quasi-concave functions is not necessarily a unimodal quasi-concave function, making the proof of Theorem 3.3 not applicable to the general case.

We remark that decreasing absolute risk aversion (DARA) is quite common in practice, as it shows the DM's greater acceptance of risky situations when becoming wealthier. If the utility function has an increasing absolute risk aversion (IARA) coefficient, then it is unclear whether the DM would demand more insurance or not when provided the disaster relief fund. On the one hand, providing the disaster relief fund incentivizes the so-called "charity hazard", where the DM enjoys the "free lunch" and lowers her demand for insurance. On the other hand, the DM would probably increase her demand for insurance, since providing the disaster relief fund makes the DM wealthier, leading to less willingness to accept the riskier situations. We close this section by presenting the following example.

Example 3.1. *The quadratic utility function is endowed with IARA. In this example, the following settings are adopted.*

(a). The utility function is given by $u(x) = x - \frac{\beta}{2}x^2$*, where* $\beta = 0.001$.[3]

(b). The loss variable X_1 *has a truncated exponential distribution with parameter* $\mu = 1/\lambda = 500$*, truncated at* $M = 5000$*. Its CDF is given by* $F_{X_1}(x) = (1 - \exp(-x/\mu))/(1 - \exp(-M/\mu))$ *for* $x \in [0, M]$*. Moreover, the loss variable* X_2 *has a truncated gamma distribution with parameters* $a = 20$ *and* $b = 25$*, also truncated at* $M = 5000$*, with CDF given by* $F_{X_2}(x) = \gamma(a, bx)/\gamma(a, bM)$ *for* $x \in [0, M]$*, where* γ *is the lower incomplete gamma function.*

(c). The initial wealth is given by $W_0 = 500$*.*

(d). The DM's retained loss after receiving the disaster relief fund is $G(R(X), \ell) = (R(X) - \ell)_+$*.*

(e). The expected-value premium principle is used i.e. $\pi(I(X)) = (1+\theta)\mathbb{E}[I(X)]$*, where* $\theta = 0.05$*.*

If the DM is faced with X_1*, then it is shown in the left panel of Figure 1 that the optimal deductible level* d^*_ℓ *is increasing with respect to* ℓ*, which is in line with the conclusion of Theorem 3.3. However, if the DM is faced with* X_2*, then it is shown in the right panel of Figure 1 that the optimal deductible level* d^*_ℓ *is decreasing with respect to* ℓ*. Thus, when the absolute risk aversion coefficient is increasing, providing disaster relief may still enhance the DM's demand for insurance, which does not concur with the conclusion of Philippi & Schiller (2024) that "an increase in anticipation of disaster relief payments leads to a reduction in the coverage level". We note that our model setting and that of Philippi & Schiller (2024) are different in several assumptions, one major difference is the amount of relief fund. In this example, the relief fund is capped at* ℓ*, while in Philippi & Schiller (2024) the relief fund is proportional to the DM's retained loss without upper limit.*

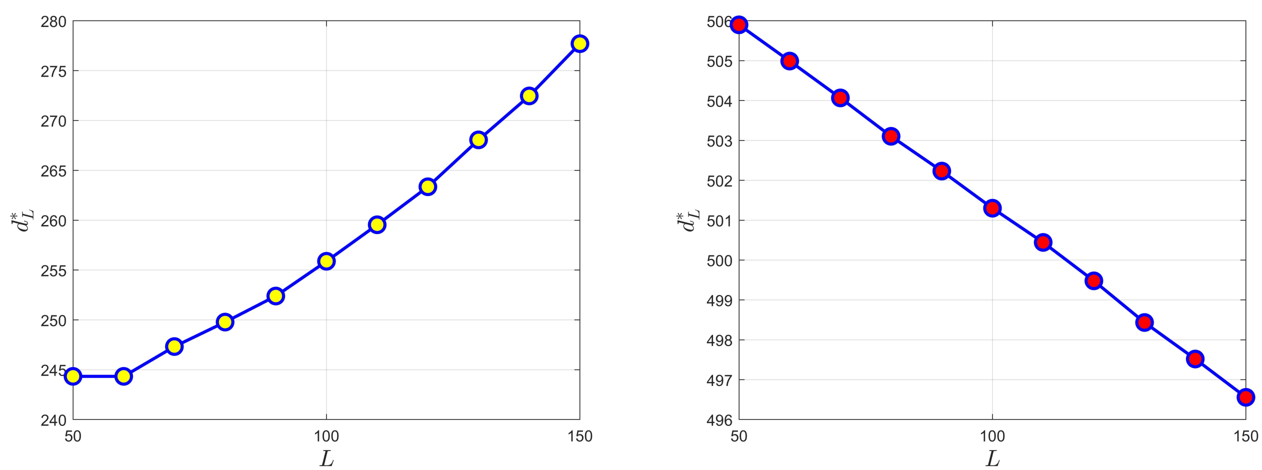

Figure 1: The optimal deductible levels for different ℓ under: (Left) exponential distribution; (Right) gamma distribution.

[3]Here, β is chosen small enough to guarantee a sufficiently large saturation point such that $u(x)$ is increasing for relevant wealth levels.

3.4 The effect of premium subsidy on the demand for insurance

In addition to the government's relief fund, the DM can also be supported via the premium subsidies. In the literature, insurance, such as micro-insurance or health insurance, can be subsidized in different ways, among which lowering the safety loading factor for the expected-value premium principle is quite often used; see Selden (1999), Kovacevic & Pflug (2011), and Philippi & Schiller (2024), where the amount of subsidy is proportional to the expected insurance indemnity. Fixed and varying amounts of subsidies can also be found in, for example, Janzen et al. (2021) and Flores-Contró et al. (2024) when investigating the role of micro-insurance in social protection and poverty reduction. Within the context of our paper, subsidizing the catastrophe insurance seems to be a more efficient way to stimulate the growth of the catastrophe insurance market, and in the meantime reduce the so-called "charity hazard". For the Bernoulli-type loss variable, Philippi & Schiller (2024) show that the DM who has increasing absolute risk aversion coefficient will increase the demand for insurance when receiving the premium subsidy. This is also true under our generalized setting with more general distributions of the loss variable. For that purpose, we slightly abuse the notation and write ϕ_ℓ in Theorem 3.2 as ϕ_h due to the variation in h. If h_1 and h_2 denote the pricing functions before and after receiving the premium subsidy respectively, the following assumption is adopted in this section.

Assumption 3. *$h_1'(x) \geq h_2'(x)$ for $x \in [0, M]$.*

Assumption 3 implies that the premium subsidy increases with respect to the expected indemnity, which motivates the DM to seek more insurance coverage.

We provide two examples of premium subsidy functional forms satisfying Assumption 3. First, the premium changes from $(1+\theta_1)\mathbb{E}[I(X)]$ to $(1+\theta_2)\mathbb{E}[I(X)]$ for some $\theta_2 \in (0, \theta_1)$. Then, $h_1(x) = (1+\theta_1)x$ and $h_2(x) = (1+\theta_2)x$, and the government subsidizes the amount $(\theta_1 - \theta_2)\mathbb{E}[I(X)]$. Second, the premium changes from $(1+\theta)\mathbb{E}[I(X)]$ to $((1+\theta)\mathbb{E}[I(X)] - s)_+$ for some $s \in [0, \infty)$. Then, $h_1(x) = (1+\theta)x$ and $h_2(x) = ((1+\theta)x - s)_+$, and the government subsidizes the amount $(1+\theta)\mathbb{E}[I(X)] \wedge s$.

Theorem 3.4. *Let h_1, h_2 be the pricing functions that satisfy Assumption 3. If the absolute risk aversion coefficient is increasing, then for Problem (8) the optimal deductible level under h_1 is greater than that under h_2.*

Proof: Denote by d_1^* and d_2^* the optimal deductible levels under the pricing functions h_1 and h_2 respectively. Since $h_1'(x) \geq h_2'(x)$ for $x \in [0, M]$, and this leads to

$$\begin{aligned} d_1 =& \inf\{d \in [0, M] : S_X(d) h_1'(\mathbb{E}[(X-d)_+]) \leq G_1'(d, \ell)\} \\ \geq& \inf\{d \in [0, M] : S_X(d) h_2'(\mathbb{E}[(X-d)_+]) \leq G_1'(d, \ell)\} = d_2. \end{aligned}$$

To show $d_1^* \geq d_2^*$, we need to show that

$$\phi_{h_1}(d) h_1'(\mathbb{E}[(X-d)_+]) \geq \phi_{h_2}(d) h_2'(\mathbb{E}[(X-d)_+]) \tag{13}$$

for $d \in [d_1, M]$. Define the function:

$$\eta(\epsilon) = \varrho(\epsilon)\left(h_1'(\mathbb{E}[(X-d)_+]) + \epsilon(h_2'(\mathbb{E}[(X-d)_+]) - h_1'(\mathbb{E}[(X-d)_+]))\right),$$

where $\varrho(\epsilon) = \phi_{h_1 + \epsilon(h_2 - h_1)}(d)$, then we have $\eta'(\epsilon) \leq 0$ for $\epsilon \in [0, 1]$, which implies (13).

In the sequel, let $\pi_1 = h_1(\mathbb{E}[(X-d)_+])$ and $\pi_2 = h_2(\mathbb{E}[(X-d)_+])$. It is straightforward that $\pi_1 \geq \pi_2$ due to Assumption 3. We first show that $\varrho'(\epsilon) \leq 0$ for $\epsilon \in [0,1]$. Note that

$$\varrho'(\epsilon) = \frac{(\pi_2 - \pi_1)\kappa(\epsilon)}{[u'(W_0 - G(d,\ell) - \pi_1 - \epsilon(\pi_2 - \pi_1))]^2},$$

where

$$\begin{aligned}\kappa(\epsilon) = &- \mathbb{E}[u''(W_0 - G(X\wedge d,\ell) - \pi_1 - \epsilon(\pi_2-\pi_1))]u'(W_0 - G(d,\ell) - \pi_1 - \epsilon(\pi_2-\pi_1)) \\ &+ \mathbb{E}[u'(W_0 - G(X\wedge d,\ell) - \pi_1 - \epsilon(\pi_2-\pi_1))]u''(W_0 - G(d,\ell) - \pi_1 - \epsilon(\pi_2-\pi_1)).\end{aligned}$$

If the DM is endowed with an IARA utility function, then by following the proof of Theorem 3.3, we can show that $\kappa(\epsilon) \geq 0$ for $\epsilon \in [0,1]$. As such, $\varrho'(\epsilon) \leq 0$ for $\epsilon \in [0,1]$.

Therefore, for any $\epsilon \in [0,1]$

$$\begin{aligned}\eta'(\epsilon) = &\varrho'(\epsilon)\left[h_1'(\mathbb{E}[(X-d)_+]) + \epsilon(h_2'(\mathbb{E}[(X-d)_+]) - h_1'(\mathbb{E}[(X-d)_+]))\right] \\ &+ \varrho(\epsilon)\left[h_2'(\mathbb{E}[(X-d)_+]) - h_1'(\mathbb{E}[(X-d)_+])\right] \leq 0.\end{aligned}$$

The proof is finished. ■

The result of Theorem 3.4 is not surprising. The premium subsidy makes the DM wealthier, who is thus more risk averse under the condition of Theorem 3.4, which leads to more demand for insurance.

If the DM is endowed with a DARA utility function, then providing premium subsidy makes the DM wealthier, which makes the DM less averse towards risk and thus lowers her demand for insurance. On the other hand, Boonen & Jiang (2024) show that under mild conditions, premium subsidies can still enhance the DM's demand for insurance if her utility function is within a subclass of the so-called Hyperbolic Absolute Risk Aversion (HARA) utility functions. In Boonen & Jiang (2024), premium subsidies are provided by reducing the safety loading factor in the expected-value premium principle. We present an interesting example below to show that the conclusion of Boonen & Jiang (2024) does not hold true in the case the premium subsidies are provided via other functional forms.

Example 3.2. *In this example we investigate the effect of premium subsidies on the DM's insurance demand for DARA utility functions. The settings in the example are given as follows:*

(a). The utility function is given by $u(x) = \log(x)$.

(b). The loss variable X_1 *has a truncated exponential distribution with parameter* $\mu = 500$. *The loss variable* X_2 *has a truncated gamma distribution with* $a = 20$ *and* $b = 25$. *Here, distributions are truncated at* $M = 3000$.

(c). The initial wealth is given by $W_0 = 5000$.

(d). The type I premium subsidy is provided via reducing the safety loading factor for the expected-value premium principle, such that the premium changes from $(1+\theta_1)\mathbb{E}[I(X)]$ *to* $(1+\theta_2)\mathbb{E}[I(X)]$ *for some* $\theta_2 \in (0,\theta_1)$.

(e). The type II premium subsidy is provided via direct premium reduction up to a certain amount, such that the premium changes from $(1+\theta)\mathbb{E}[I(X)]$ *to* $((1+\theta)\mathbb{E}[I(X)] - S)_+$ *for some* $S \in [0,\infty)$.

(f). There is no disaster relief fund, i.e. $\ell = 0$.

As shown by Figure 2, when provided the type I premium subsidy the optimal deductible level is always increasing with θ*, indicating that the demand for insurance increases with more premium subsidy, which agrees with the implication of Theorem 3.4. However, when provided with the type II premium subsidy, the left panel of Figure 3 shows that the optimal deductible level increases with respect to* S *if the DM faces* X_1*, indicating that the demand for insurance decreases with the premium subsidy. This specific counter-intuitive example complements the current results.*

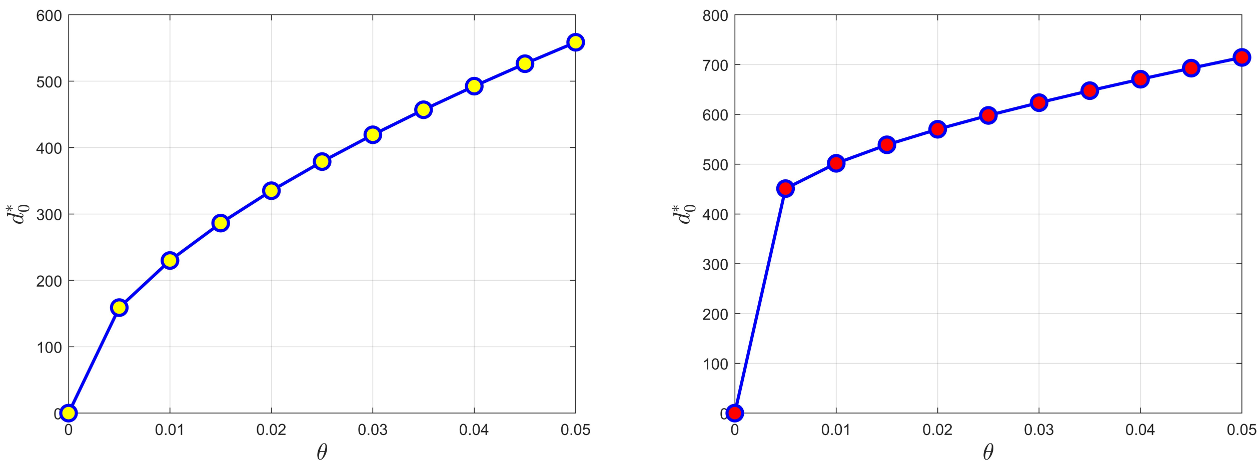

Figure 2: The optimal deductible levels for varying θ under the: (Left) exponential distribution; (Right) gamma distribution.

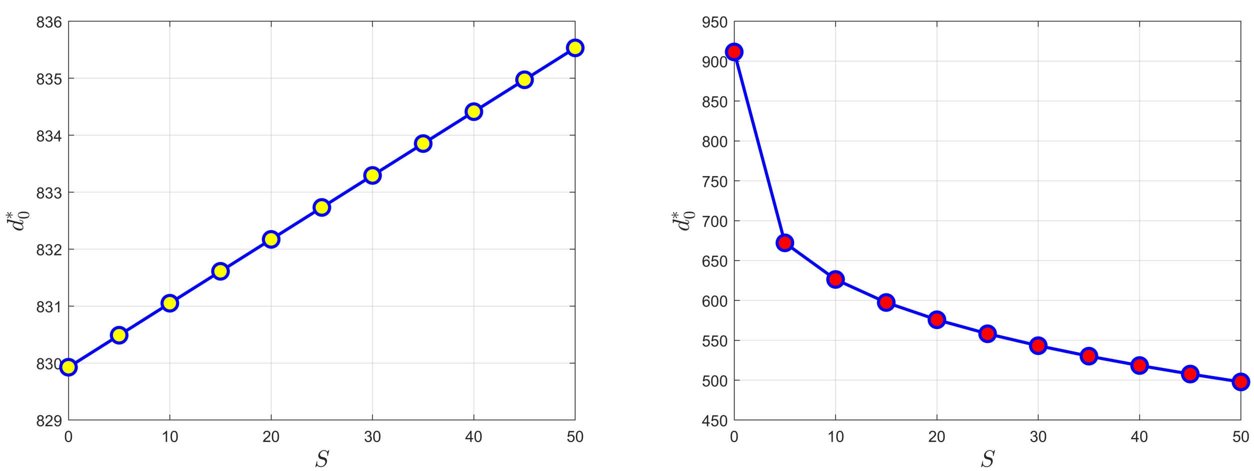

Figure 3: The optimal deductible levels for different S under: (Left) exponential distribution; (Right) gamma distribution.

Our result confirms the finding of Philippi & Schiller (2024), where the change of insurance demand with respect to the premium subsidy is uncertain under general settings due to the possibility of insurance being a Giffen good.[4] The conditions for insurance to be a Giffen good are studied in Hoy & Robson (1981). Philippi & Schiller (2024) also presented several examples showing that insurance is an ordinary good in most real-life situations, under which a reduction in premium would increase the demand for insurance, mitigating the potential "charity hazard" that arises from receiving the disaster relief fund.

We close this section by looking at one specific example for the joint effect of relief fund and premium subsidy on the demand of insurance. Note that the exponential utility function, i.e., $u(x) = e^{-\alpha x}$ for some $\alpha > 0$, has the constant absolute risk aversion coefficient, which is the only member belonging to both the DARA and IARA classes. If we further assume that the premium subsidy is provided by reducing the premium by a fixed proportion $s \in (0, 1)$, such that the actual *ex ante* payment of the DM is $(1 - s)\pi(I(X))$, the optimal deductible level for the problem (8), as per Theorem 3.2, is given by

$$d^* = \inf\{d \in [d_\ell, M] : \Phi(d; \ell, s) \leq 0\},$$

where

$$\Phi(d; \ell, s) = (1 - s)\mathbb{E}[e^{\alpha(G(d,\ell)-G(X\wedge d,\ell))}]h'(\mathbb{E}[(X - d)_+]) - G_1'(d, \ell).$$

By Theorems 3.3 and 3.4, $\Phi(d; \ell, s)$ is increasing with respect to ℓ but decreasing with respect to s. By comparing with the case without the disaster relief fund and premium subsidy, i.e. $\ell = s = 0$, it is straightforward that the deductible level is larger if $\Phi(d; \ell, s) \geq \Phi(d; 0, 0)$, and is smaller if $\Phi(d; \ell, s) \leq \Phi(d; 0, 0)$.

It is notable that our model only concerns the decision of the DM, which makes it irrelevant to know where the premium subsidy comes from. This, however, does matter to the counter-parties who are providing the support to the DM, as their decisions would greatly affect the decisions of the DM. If a cheaper premium is offered by the insurer, then the insurer's decision could be driven by the fact that the increased insurance demand may lead to more profits. If the premium subsidy is provided by the government, then the intention of the government is to reduce the *ex post* disaster relief fund when insurance is more involved. These incentives give rise to different decision-making problems from the counter-parties' perspectives. For example, we assume that the government provides both the disaster relief fund and premium subsidy. To maintain the utility level of the DM under her chosen deductible level d, the indifference curve of (ℓ, s), where s denotes the subsidized portion of the premium, is depicted by

$$\mathbb{E}[u(W_0 - G(X \wedge d, \ell) - (1 - s)\pi((X - d)_+))] = \text{constant},$$

from which one can easily calculate the marginal substitution rate between ℓ and s:

$$\frac{d\ell}{ds} = \frac{\mathbb{E}[u'(W_0 - G(X \wedge d, \ell) - (1 - s)\pi((X - d)_+))\pi((X - d)_+)]}{\mathbb{E}[u'(W_0 - G(X \wedge d, \ell) - (1 - s)\pi((X - d)_+))G_2'(X \wedge d, \ell)]} \leq 0,$$

where $G_2'(\cdot, \cdot)$ denotes the partial derivative of G with respect to y, and $G_2'(X \wedge d, \ell) \leq 0$ is due the assumption (c) after Eq. (1). Thus, the government can reserve a smaller relief fund if more premium subsidies are provided. The optimal budgeting for the *ex ante* premium subsidies and the *ex post* relief fund under the changing demand for insurance is a complicated but interesting and meaningful problem for future research.

[4] A Giffen good is a product that people purchase or consume more when its price rises and vice versa, which violates the law of demand. An ordinary good is a product that people purchase or consume less when its price rises and vice versa, which follows the law of demand.

4 Dependence between the loss and government relief

4.1 Loss-increasing relief probability

It is realistic to assume that the size of loss X and the government's relief decision Y are positively dependent. In this section, we propose a specific dependence structure between X and Y. If the underlying loss variable is large, it may be more likely that a disaster is taking place, and this thus makes the likelihood of receiving disaster relief higher. For example, in the context of homeowner's insurance, disaster relief is most likely to be granted following extreme events such as major floods or earthquakes. To be specific, we assume that the probability of receiving disaster relief payments is linked to the actual magnitude of the loss, as a larger personal loss signifies a greater likelihood of a major event occurring and, consequently, increases the probability of receiving relief payments. This yields a specific structure for the distribution of $Y|X$, and we focus on the following step function for modeling relief probability:

Assumption 4. *The government's relief decision Y is such that $\mathbb{P}(Y = \ell|X) = p(X)$ and $\mathbb{P}(Y = 0|X) = 1 - p(X)$, where*

$$p(X) = \sum_{i=1}^{n} p_i \mathbb{1}_{(a_{i-1},a_i]}(X), \tag{14}$$

with $0 \le p_1 \le \cdots \le p_n \le 1$, and $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = M$.

It can be calculated that $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \ell\mathbb{E}[p(X)]$ and $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \ell\mathbb{E}[Xp(X)]$. Hence

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \ell\left(\mathbb{E}[Xp(X)] - \mathbb{E}[X]\mathbb{E}[p(X)]\right) = \ell\mathrm{Cov}(X, p(X)) \ge 0,$$

where the inequality follows from the Assumption 4 as $p(x)$ is an increasing function. Therefore, X and Y are (weakly) positively dependent.

Theorem 4.1. *Under Assumption 4, the solution to Problem (3) is of the form*

$$R^*(x) = \sum_{i=1}^{n-1} R_i^*(x)\mathbb{1}_{(a_{i-1},a_i]}(x) + R_n^*(x)\mathbb{1}_{(a_{n-1},a_n]}(x), \tag{15}$$

where

$$\begin{aligned} R_i^*(x) &= R^*(a_{i-1}) + (x - a_{i-1}) \wedge d_i^1 + (x - a_{i-1} - d_i^2)_+, \\ R_n^*(x) &= R^*(a_{n-1}) + (x - a_{n-1}) \wedge d_n, \end{aligned}$$

with

$$d_i^1 \in [0, R^*(a_i) - R^*(a_{i-1})],\ d_i^2 - d_i^1 = (a_i - a_{i-1}) - (R^*(a_i) - R^*(a_{i-1})),\ \text{and } d_n \ge 0.$$

Proof: Under Assumption 4, Problem (3) can be written as

$$\begin{aligned} \max_{R\in\mathcal{I}}\ \mathbb{E}\Big[&p(X)u(W_0 - G(R(X),\ell) - \pi(X - R(X))) \\ &+ (1 - p(X))u(W_0 - R(X) - \pi(X - R(X)))\Big]. \end{aligned} \tag{16}$$

Note that

$$
\begin{aligned}
&\mathbb{E}[p(X)u(W_0 - G(R(X),\ell) - \pi(X - R(X)))] \\
=&\sum_{i=1}^{n} p_i \mathbb{E}[u(W_0 - G(R(X),\ell) - \pi(X - R(X)))\mathbb{1}_{(a_{i-1},a_i]}(X)] \\
=&\sum_{i=1}^{n} p_i (F(a_i) - F(a_{i-1}))\mathbb{E}[u(W_0 - G(R(X),\ell) - \pi(X - R(X))\big|X \in (a_{i-1}, a_i]].
\end{aligned}
$$

This simplifies Problem (16) as follows:

$$
\begin{aligned}
\max_{R\in\mathcal{I}} \sum_{i=1}^{n}(F(a_i) - F(a_{i-1}))\mathbb{E}\Big[&p_i u(W_0 - G(R(X),\ell) - \pi(X - R(X))) \\
&+ (1-p_i)u(W_0 - R(X) - \pi(X - R(X)))\Big|X \in (a_{i-1}, a_i]\Big].
\end{aligned} \tag{17}
$$

Let $P \geq 0$. Under the actuarial-value-based premium principle, we can write $P = h(\mathbb{E}[X - R(X)])$ as

$$
\mathbb{E}[R(X)] = \mathbb{E}[X] - h^{-1}(P) = \tilde{P},
$$

and so

$$
\sum_{i=1}^{n} \mathbb{E}[R(X)\mathbb{1}_{(a_{i-1},a_i]}(X)] = \tilde{P},
$$

or

$$
\sum_{i=1}^{n}(F(a_i) - F(a_{i-1}))\mathbb{E}[R(X)|X \in (a_{i-1}, a_i]] = \tilde{P}.
$$

Hence, for the fixed budget P, Problem (17) can be written as

$$
\begin{aligned}
&\max_{R\in\mathcal{I}} \sum_{i=1}^{n}(F(a_i) - F(a_{i-1}))\mathbb{E}\Big[p_i u(W_0 - G(R(X),\ell) - P) + (1-p_i)u(W_0 - R(X) - P)\Big|X \in (a_{i-1}, a_i]\Big], \\
&\text{s.t. } \sum_{i=1}^{n}(F(a_i) - F(a_{i-1}))\mathbb{E}[R(X)|X \in (a_{i-1}, a_i]] = \tilde{P}.
\end{aligned} \tag{18}
$$

To use the arguments of Lemma 3 of Ohlin (1969), we consider an arbitrary $\tilde{R} \in \mathcal{I}$ that satisfies $\mathbb{E}[\tilde{R}(X)] = \tilde{P}$. Then, there exists another $R \in \mathcal{I}$ such that the following conditions hold simultaneously over the interval $(a_{i-1}, a_i]$ for $i = 1, 2, \ldots, n-1$:

$$
\begin{cases}
R(a_{i-1}) = \tilde{R}(a_{i-1}),\ R(a_i) = \tilde{R}(a_i), \\
R(x) = R(a_{i-1}) + (x - a_{i-1}) \wedge d_i^1 + (x - a_{i-1} - d_i^2)_+, \\
d_i^2 - d_i^1 = (a_i - a_{i-1}) - (R(a_i) - R(a_{i-1})), \\
d_i^1 \in [0, R(a_i) - R(a_{i-1})], \\
\mathbb{E}\left[R(X)|X \in (a_{i-1}, a_i]\right] = \mathbb{E}[\tilde{R}(X)|X \in (a_{i-1}, a_i]].
\end{cases} \tag{19}
$$

and the following conditions hold over the interval $(a_{n-1}, a_n]$:

$$
\begin{cases}
R(x) = R(a_{n-1}) + (x - a_{n-1}) \wedge d_n \\
\mathbb{E}\left[R(X)|X \in (a_{n-1}, a_n]\right] = \mathbb{E}[\tilde{R}(X)|X \in (a_{n-1}, a_n]].
\end{cases} \tag{20}
$$

It is easy to find that $\tilde{R}$ up-crosses R over each sub-interval, and as such, by using Lemma 3 of Ohlin (1969), we have

$$\begin{aligned}&\mathbb{E}\Big[p_i u(W_0 - G(\tilde{R}(X),\ell) - P) + (1-p_i)u(W_0 - \tilde{R}(X) - P)\Big|X \in (a_{i-1}, a_i]\Big]\\ \leq&\mathbb{E}\Big[p_i u(W_0 - G(R(X),\ell) - P) + (1-p_i)u(W_0 - R(X) - P)\Big|X \in (a_{i-1}, a_i]\Big]\end{aligned}$$

for $i = 1, 2, \ldots, n$. This leads to the optimal form of $R(x)$ as given by Eqs. (19) and (20). An illustrative comparison between R^* and R is given by Figure 4. ■

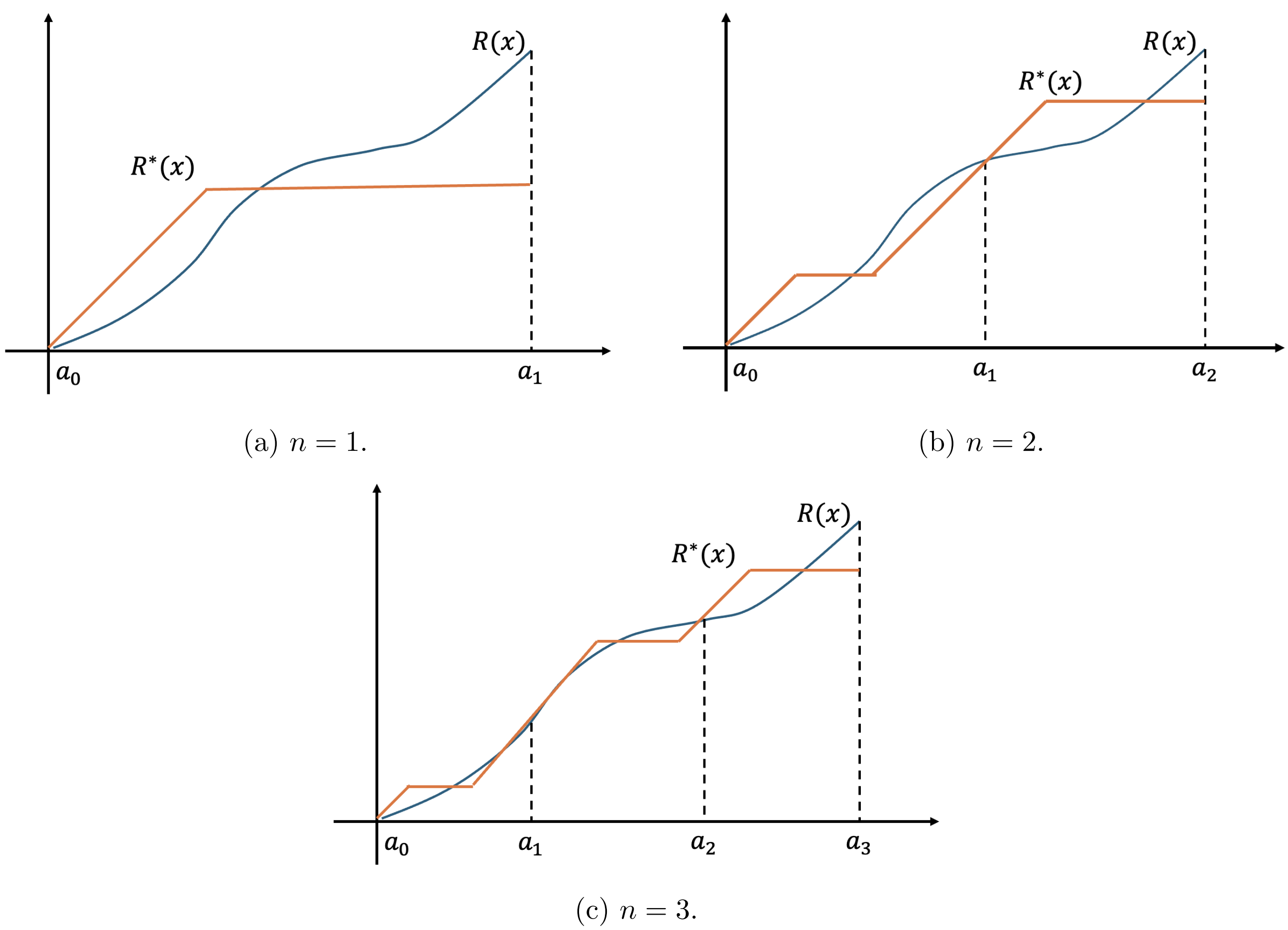

Figure 4: Comparison between R^* and R.

Under a much more general dependence structure between the insurable loss and the background loss, Chi & Wei (2020) show that the optimal indemnity function to the problem (7) is of the multi-layered form, which is consistent with our finding in this section. The multi-layered form is attributed to Assumption 4, under which the DM would like to cede out part of the small- or medium-size losses due to the smaller probabilities of getting the relief fund.

As shown by Theorem 4.1, there are two parameters, i.e. d_i^1 and $R^*(a_i)$, to be optimized over each of the intervals $(a_{i-1}, a_i]$ for $i = 1, 2, \ldots, n-1$, and one parameter d_n to be optimized over the last interval. Thus, through Theorem 4.1 the original infinite-dimensional optimization problem is reduced to a $(2n-1)$-dimensional optimization problem.

Example 4.1. *We present several examples to study the sensitivity of R^* with respect to the model parameters in (4), focusing on a simple case where $n = 2$. We define the common settings for all the examples as follows:*

(a). The utility function is given by $u(x) = -\exp(-x/1000)$.

(b). The loss variable X has the same distribution as X_1 in Example 3.2.

(c). The initial wealth of the DM is irrelevant for exponential utility functions and is normalized such that $W_0 = 0$.

(d). The premium principle is as same as that for Example 3.1.

Figure 5 shows the optimal retained loss functions for different values of a_1, where $p_1 = 0.3$ and $p_2 = 0.6$. Note that the constant relief probability case with $p = 0.3$ can be treated as a special case of (14) with $a_1 = M$. When a_1 becomes smaller, it is more likely for the DM to receive the relief fund, which therefore reduces her demand for insurance. Interestingly, when $a_1 \in (0, M)$, the DM would retain one layer of losses that covers a_1.

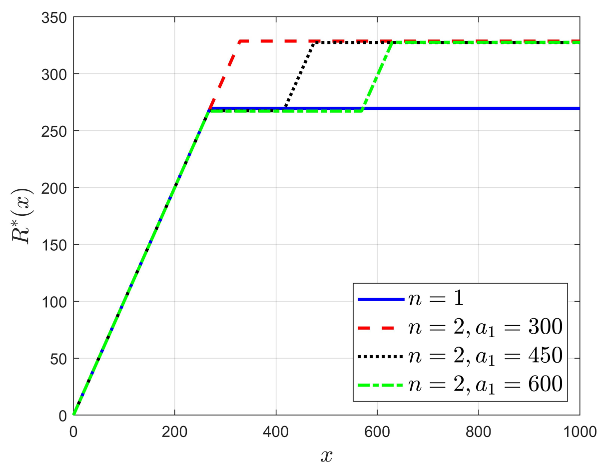

Figure 5: The optimal retained loss function for different values of a_1.

Figure 6 shows the optimal retained loss functions under different p_2 and p_1. With $p_1 = 0.3$ and $a_1 = 450$, the left panel of 6 shows that if p_2 increases, the DM would retain more losses around a_1 without changing her demand for insurance for other layers of losses. With $p_2 = 0.3$ and $a_1 = 450$, the right panel of 6 shows that if p_1 increases, the DM would cede out more small losses while retaining more losses around a_1.

We close this section by comparing the values of the objective function (2) under Assumptions 1 and 4, where in the former case the relief probability is taken as $\mathbb{E}[p(X)]$ where $p(X)$ is as shown in (14). In such a case, the expected relief probabilities are the same under the two different assumptions. The following proposition shows that the DM will benefit from Assumption 4 regarding the dependence between Y and X.

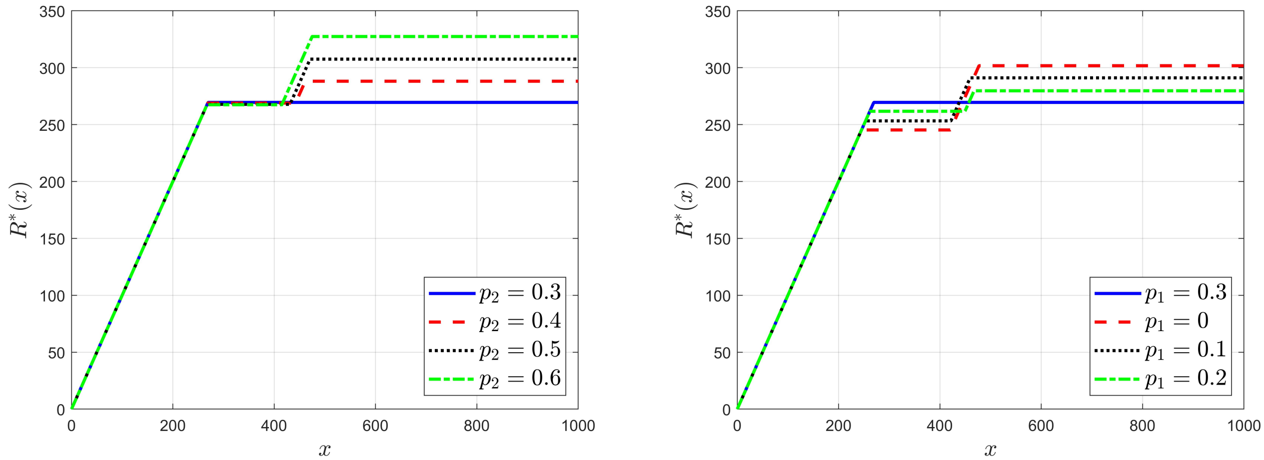

Figure 6: The optimal retained loss function under: (Left) varying p_2; and (Right) varying p_1.

Proposition 4.1. *Under Assumption 4, the following inequality holds:*

$$\begin{aligned}
&\mathbb{E}\Big[p(X)u(W_0 - G(R(X), \ell) - \pi(X - R(X))) \\
&\qquad + (1 - p(X))u(W_0 - R(X) - \pi(X - R(X)))\Big] \\
\geq\ &\mathbb{E}\Big[\mathbb{E}[p(X)]u(W_0 - G(R(X), \ell) - \pi(X - R(X)))\Big] \\
&\qquad + \mathbb{E}\Big[(1 - \mathbb{E}[p(X)])u(W_0 - R(X) - \pi(X - R(X)))\Big].
\end{aligned}$$

Proof: Let $w(x) = W_0 - G(R(x), \ell) - \pi(X - R(X))$ and $\tilde{w}(x) = W_0 - R(x) - \pi(X - R(X))$. Both w and $\tilde{w}$ are decreasing functions. Due to the concavity of u, $G(x, y) \leq x$, and $G'_1(x, y) \leq 1$, if $x_1 < x_2$ and $G(R(x_2), \ell) - G(R(x_1), \ell) > 0$, we have

$$\frac{u(w(x_1)) - u(w(x_2))}{w(x_1) - w(x_2)} \leq \frac{u(\tilde{w}(x_1)) - u(\tilde{w}(x_2))}{\tilde{w}(x_1) - \tilde{w}(x_2)},$$

which implies

$$\frac{u(w(x_1)) - u(w(x_2))}{G(R(x_2), \ell) - G(R(x_1), \ell)} \leq \frac{u(\tilde{w}(x_1)) - u(\tilde{w}(x_2))}{R(x_2) - R(x_1)} \leq \frac{u(\tilde{w}(x_1)) - u(\tilde{w}(x_2))}{G(R(x_2), \ell) - G(R(x_1), \ell)}.$$

From this, it follows that $u(w(x_1)) - u(w(x_2)) \leq u(\tilde{w}(x_1)) - u(\tilde{w}(x_2))$, and thus $u(w(x_1)) - u(\tilde{w}(x_1)) \leq u(w(x_2)) - u(\tilde{w}(x_2))$.

If $x_1 < x_2$ and $G(R(x_2), \ell) - G(R(x_1), \ell) = 0$, then it is straightforward that $u(w(x_1)) - u(\tilde{w}(x_1)) \leq u(w(x_2)) - u(\tilde{w}(x_2))$. Hence, $u(w(x)) - u(\tilde{w}(x))$ is an increasing function.

Since $p(x)$ is also an increasing function, $p(X)$ and $u(w(X)) - u(\tilde{w}(X))$ are comonotonic. The positive correlation between $p(X)$ and $u(w(X)) - u(\tilde{w}(X))$ leads to

$$\mathbb{E}[p(X)\,(u(w(X)) - u(\tilde{w}(X)))] \geq \mathbb{E}[p(X)]\mathbb{E}[u(w(X)) - u(\tilde{w}(X))].$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[p(X)u(W_0 - G(R(X),\ell) - \pi(X - R(X))) \\
&\qquad + (1-p(X))u(W_0 - R(X) - \pi(X - R(X)))\Big] \\
= \ & \mathbb{E}[u(\tilde{w}(X))] + \mathbb{E}[p(X)\left(u(w(X)) - u(\tilde{w}(X))\right)] \\
\geq \ & \mathbb{E}[u(\tilde{w}(X))] + \mathbb{E}[p(X)]\mathbb{E}\left(u(w(X)) - u(\tilde{w}(X))\right) \\
= \ & \mathbb{E}\Big[\mathbb{E}[p(X)]u(W_0 - G(R(X),\ell) - \pi(X - R(X)))\Big] \\
&\qquad + \mathbb{E}\Big[(1-\mathbb{E}[p(X)])u(W_0 - R(X) - \pi(X - R(X)))\Big].
\end{aligned}$$

The proof is complete. ■

The above proposition shows that the DM prefers to obtain more chance of relief in case the insurable loss X is high. From this, we conclude that Assumption 4 is not only more practical but also more advantageous to the DM.

4.2 Different loss distributions for different relief funds

Though the decision of paying relief fund is up to the government, the decision highly depends on the number and sizes of the losses. It is often the case that the government provides financial assistance or recovery programs if most individuals are suffering severe losses, and may provide less assistance or no assistance if losses are not severe. In other words, the bivariate distribution of (X, Y) exhibits positive dependence. By standard conditional probility arguments, we can write the bivariate distribution of (X, Y) as $(X|Y, Y)$, and defining a particular distribution for Y and $X|Y$ separately. Note that such decomposition is without loss of generality, and does not assume any causality of Y on X. The following assumption will be adopted throughout this section.

Assumption 5. *The conditional distribution function of X given Y is given by $\mathbb{P}(X \leq x|Y = \ell) = F_{X_1}(x)$ and $\mathbb{P}(X \leq x|Y = 0) = F_{X_2}(x)$, with the probability density functions f_{X_1} and f_{X_2} respectively. Furthermore, $X_2 \leq_{hr} X_1$ where "$\leq_{hr}$" denotes the hazard rate order.*[5]

In Assumption 5, we use X_1 and X_2 to denote the loss variables that are faced by the DM when the paid relief funds are ℓ and 0, respectively. Due to the well-known relationships between the following stochastic orders, see Shaked & Shanthikumar (2007):[6]

$$Z_1 \leq_{lr} Z_2 \implies Z_1 \leq_{hr} Z_2 \implies Z_1 \leq_{st} Z_2,$$

the hazard rate order is more general than the commonly assumed likelihood ratio order (see also Chi (2019) for related discussions), and implies the usual stochastic order, meaning that X_1 is riskier than X_2. Under the model setup of this paper, $X_2 \leq_{hr} X_1$ is equivalent to saying that $S_{X_1}(x)/S_{X_2}(x)$ is increasing over $[0, M]$.

[5]For two random variables Z_1, Z_2 whose hazard functions are well defined, $Z_1 \leq_{hr} Z_2$ if $\frac{f_{Z_1}(z)}{S_{Z_1}(z)} \geq \frac{f_{Z_2}(z)}{S_{Z_2}(z)}$ for all z within the support of Z_1 and Z_2.

[6]For two random variables Z_1, Z_2, $Z_1 \leq_{lr} Z_2$ if $f_{Z_2}(z)/f_{Z_1}(z)$ is increasing over the union of the supports of Z_1 and Z_2, and $Z_1 \leq_{st} Z_2$ if $S_{Z_1}(z) \leq S_{Z_2}(z)$ for all z within the support of Z_1 and Z_2.

Under Assumption 5, Problem (2) can be written as

$$\max_{R\in\mathcal{I}} \Big\{p\mathbb{E}[u(W_0 - G(R(X_1),\ell) - \pi(X_3 - R(X_3)))] \\ + (1-p)\mathbb{E}[u(W_0 - R(X_2) - \pi(X_3 - R(X_3)))]\Big\}, \tag{21}$$

where $\pi(X_3 - R(X_3))$ is the premium charged by the insurer as per its knowledge regarding the loss.[7] The following theorem presents the solution to Problem (21) for a special case of the distribution of X_3. The proof is similar to that for Theorem 4.1 of Chi (2019), and we provide a proof here for completeness.

Theorem 4.2. *Let Assumption 5 hold. If $X_3 \le_{hr} X_2$, then the solution to Problem (21) is of the form $R_d(x) = x \wedge d$ for some $d \ge 0$.*

Proof: Since $X_2 \le_{hr} X_1$, we have $X_3 \le_{hr} X_i$ for $i = 1, 2$. This implies that $S_{X_i}(x)/S_{X_3}(x)$ is increasing over $[0, M]$.

For any given $R \in \mathcal{I}$, there must exist a $d \ge 0$ such that $\mathbb{E}[X_3 \wedge d] = \mathbb{E}[R(X_3)]$. As such, applying $R_d(x) := x \wedge d$ leads to the same premium as R. Furthermore, for $i = 1, 2$,

$$\begin{aligned}
\mathbb{E}[R(X_i)] - \mathbb{E}[R_d(X_i)] &= \int_0^M S_{X_i}(x)dR(x) - \int_0^M S_{X_i} dR_d(x) \\
&= \int_0^M S_{X_i}(x)\left[R_d'(x) - \mathbb{1}_{\{x\le d\}}(x)\right]dx \\
&= \int_0^M \frac{S_{X_i}(x)}{S_{X_3}(x)} S_{X_3}(x)\left[R'(x) - \mathbb{1}_{\{x\le d\}}(x)\right]dx \\
&\ge \frac{S_{X_i}(d)}{S_{X_3}(d)} \int_0^M S_{X_3}(x)\left[R'(x) - \mathbb{1}_{\{x\le d\}}(x)\right]dx \\
&= \frac{S_{X_i}(d)}{S_{X_3}(d)} \left[\mathbb{E}[R(X_3)] - \mathbb{E}[X_3 \wedge d]\right] = 0.
\end{aligned}$$

It is apparent that R up-crosses R_d only once, which leads to that $F_{R_d(X_i)}$ up-crosses $F_{R(X_i)}$ only once, for $i = 1, 2$. Define $x_0 = \inf\{x \in [0, d] : x \ge R(x)\}$. By the definition of up-crossing, we have $x_0 < d$. Thus, $F_{R_d(X_i)}(t) < F_{R(X_i)}(t)$ for $t \in (x_0, d)$. Applying Theorem 2.3 of Cheung et al. (2015), we get that $R_d(X_i) \le_{icx} R(X_i)$, where "$\le_{icx}$" denotes the increasing convex (or stop-loss) order.[8]

Let $P = \pi(X_3 - R(X_3)) = \pi(X_3 - R_d(X_3))$. Since both functions

$$v_1(x) := -u(W_0 - G(x,\ell) - P) \text{ and } v_2(x) := -u(W_0 - x - P)$$

are increasing and convex, we have

$$\mathbb{E}[v_1(R_d(X_1))] \le \mathbb{E}[v_1(R(X_1)] \text{ and } \mathbb{E}[v_2(R_d(X_2))] \le \mathbb{E}[v_2(R(X_2))].$$

[7]Regarding the financial subsidy of the government on the insurance premium, it can be in general assumed that $X_3 \le_{st} X_1$, meaning that the premium is charged based on a smaller loss compared with the loss when the government would like to provide relief payment. In this case, we have $X_3 - R(X_3) \le_{st} X_1 - R(X_1)$ as the function $x - R(x)$ is increasing and the usual stochastic order is preserved under the increasing transform. Hence, it follows that $\pi(X_3 - R(X_3)) \le \pi(X_1 - R(X_1))$, which implies that the premium charged based on X_3 can be thought as provided with some amount of subsidy from the government.

[8]For two random variables Z_1, Z_2, $Z_1 \le_{icx} Z_2$ if $\mathbb{E}[\phi(Z_1)] \le \mathbb{E}[\phi(Z_2)]$ for all increasing convex function $\phi : \mathbb{R} \mapsto \mathbb{R}$. Or equivalently, $\mathbb{E}[(Z_1 - d)_+] \le \mathbb{E}[(Z_2 - d)_+]$ for all $d \in \mathbb{R}$.

This leads to

$$
\begin{aligned}
&p\mathbb{E}[u(W_0 - G(R_d(X_1), \ell) - \pi(X_3 - R_d(X_3)))] + (1-p)\mathbb{E}[u(W_0 - R_d(X_2) - \pi(X_3 - R_d(X_3)))] \\
\geq\ & p\mathbb{E}[u(W_0 - G(R(X_1), \ell) - \pi(X_3 - R(X_3)))] + (1-p)\mathbb{E}[u(W_0 - R(X_2) - \pi(X_3 - R(X_3)))].
\end{aligned}
$$

The proof is complete. ■

In practice, the insurer would also consider a mixture distribution of F_{X_1} and F_{X_2}, with possibly different weights assigned to these two distributions, that is

$$
F_{X_3} = \tilde{p}F_{X_1} + (1-\tilde{p})F_{X_2}, \tag{22}
$$

where $\tilde{p}$ is not necessarily equal to p. In such a setting, the parameter p can be understood as the DM's subjective perception of the probability of receiving the relief fund. It is easy to find that $\frac{S_{X_3}(x)}{S_{X_1}(x)}$ is decreasing while $\frac{S_{X_3}(x)}{S_{X_2}(x)}$ is increasing, indicating that $X_2 \leq_{hr} X_3 \leq_{hr} X_1$.

If X_3 is modeled via (22), then we have $\mathbb{E}[R(X_3)] = \tilde{p}\mathbb{E}[R(X_1)] + (1-\tilde{p})\mathbb{E}[R(X_2)]$. If one fixes the premium $P > 0$, solving the problem (21) is equivalent to solving

$$
\begin{aligned}
&\max_{R\in\mathcal{I}}\ p\mathbb{E}[u(W_0 - G(R(X_1), \ell) - P] + (1-p)\mathbb{E}[u(W_0 - R(X_2) - P)] \\
&\text{ s.t. } \tilde{p}\mathbb{E}[R(X_1)] + (1-\tilde{p})\mathbb{E}[R(X_2)] = \tilde{P} := \mathbb{E}[X_3] - h^{-1}(P).
\end{aligned} \tag{23}
$$

However, it is difficult to obtain the closed-form solution to Problem (23). For example, if $p = 0$, $\tilde{p} = 1$, and $\pi(\cdot)$ is the expected-value premium principle, then the main problem studied in Ghossoub et al. (2023) with an increasing likelihood ratio is a special case of (23), for which the optimal ceded loss function takes on either a multi-layered or a general coinsurance form.

Though the general solution is difficult to obtain, we propose the following "improvement technique" to seek a solution that outperforms the given one under certain conditions. To do this, we define the following layered retained loss function

$$
R_{d_1,d_2,d_3}(x) = x \wedge d_1 + (x \wedge d_3 - d_2)_+, \tag{24}
$$

where $0 \leq d_1 \leq d_2 \leq d_3 \leq M$.

Proposition 4.2. *For any given $R \in \mathcal{I}$ that satisfies the constraint of (23), if one of the following conditions hold, then applying R_{d_1,d_2,d_3} yields a higher expected utility for Problem (23) than R:*

(i). the function R up-crosses R_{d_1,d_2,d_3}, and the equation system

$$
\mathbb{E}[R(X_1)] = \mathbb{E}[R_{d_1,d_2,d_3}(X_1)], \quad \mathbb{E}[R(X_2)] = \mathbb{E}[R_{d_1,d_2,d_3}(X_2)] \tag{25}
$$

has one solution for (d_1, d_2, d_3) that satisfies $0 \leq d_1 \leq d_2 \leq d_3 \leq M$;

(ii). the function R crosses R_{d_1,d_2,d_3} three times, and the equation system

$$
\mathbb{E}[R(X_1)] = \mathbb{E}[R_{d_1,d_2,d_3}(X_1)], \quad \mathbb{E}[R(X_2)] = \mathbb{E}[R_{d_1,d_2,d_3}(X_2)] \tag{26}
$$

has one solution for (d_1, d_2, d_3) that satisfies $0 \leq d_1 \leq d_2 \leq d_3 \leq M$ and

$$
\mathbb{E}[(R(X_2) - R(\xi_2))_+] \geq \mathbb{E}[(R_{d_1,d_2,d_3}(X_2) - R(\xi_2))_+], \tag{27}
$$

where ξ_2 is the down-crossing point.

Proof: By applying Ohlin's lemma, (i) can be proved directly.

To prove (ii), we note that if R crosses R_{d_1,d_2,d_3} three times, it can only up-cross R_{d_1,d_2,d_3} at some $x \in [d_1, d_2]$, then down-cross R_{d_1,d_2,d_3} at some $x \in [d_2, d_3]$, and up-cross R_{d_1,d_2,d_3} at some $x \in [d_3, M]$. Denote by ξ_1, ξ_2, ξ_3 the three distinctive crossing points, where $\xi_1 \in [d_1, d_2], \xi_2 \in [d_2, d_3]$ and $\xi_3 \in [d_3, M]$ (see Figure 7 for an illustrative plot). let

$$F_1(x) = \mathbb{P}(R(X_1) \leq x), \; F_2(x) = \mathbb{P}(R_{d_1,d_2,d_3}(X_1) \leq x),$$
$$\tilde{F}_1(x) = \mathbb{P}(R(X_2) \leq x), \; \tilde{F}_2(x) = \mathbb{P}(R_{d_1,d_2,d_3}(X_2) \leq x),$$

we directly have that F_1 and F_2 (as well as $\tilde{F}_1$ and $\tilde{F}_2$) also cross each other three times. Since R up-crosses R_{d_1,d_2,d_3} at ξ_1, we define

$$x_1 = \inf\{x \in [0, \xi_1] : R_{d_1,d_2,d_3}(x) > R(x)\}.$$

Then, as per the definition of up-crossing, we have $x_1 < \xi_1$. It is straightforward that $F_2(x) < F_1(x)$ and $\tilde{F}_2(x) < \tilde{F}_1(x)$ for $x \in (x_1, \xi_1)$.

Since $\mathbb{E}[(R(X_2) - R(\xi_2))_+] \geq \mathbb{E}[(R_{d_1,d_2,d_3}(X_2) - R(\xi_2))_+]$, we have

$$\begin{aligned}
&\mathbb{E}[(R(X_1) - R(\xi_2))_+] - \mathbb{E}[(R_{d_1,d_2,d_3}(X_1) - R(\xi_2))_+] \\
=& \int_0^M S_{X_1}(x) d(R(x) - R(\xi_2))_+ - \int_0^M S_{X_1}(x) d(R_{d_1,d_2,d_3}(x) - R(\xi_2))_+ \\
=& \int_{\xi_2}^M S_{X_1}(x) R'(x) dx - \int_{\xi_2}^M S_{X_1}(x) \mathbb{1}_{\{x \leq d_3\}}(x) dx \\
=& \int_{\xi_2}^M S_{X_1}(x) \left[R'(x) - \mathbb{1}_{\{x \leq d_3\}}(x)\right] dx \\
=& \int_{\xi_2}^M \frac{S_{X_1}(x)}{S_{X_2}(x)} S_{X_2}(x) \left[R'(x) - \mathbb{1}_{\{x \leq d_3\}}(x)\right] dx \\
\geq& \frac{S_{X_1}(d_3)}{S_{X_2}(d_3)} \int_{\xi_2}^M S_{X_2}(x) \left[R'(x) - \mathbb{1}_{\{x \leq d_3\}}(x)\right] dx \\
=& \frac{S_{X_1}(d_3)}{S_{X_2}(d_3)} \left\{\mathbb{E}[(R(X_2) - R(\xi_2))_+] - \mathbb{E}[(R_{d_1,d_2,d_3}(X_2) - R(\xi_2))_+]\right\} \geq 0.
\end{aligned}$$

Therefore, $\mathbb{E}[(R_{d_1,d_2,d_3}(X_1) - R(\xi_2))_+] \leq \mathbb{E}[(R(X_1) - R(\xi_2))_+]$.

By applying the "Karlin-Novikoff-Stoyan-Taylor crossing conditions" (see Theorem 2.4 of Cheung et al. (2015)), we have $R_{d_1,d_2,d_3}(X_i) \leq_{icx} R(X_i)$ for $i = 1, 2$. The rest of the proof follows the same way as that for Theorem 4.2. The proof is complete. ■

Similar to most existent literature, the above "improvement technique" aims to reduce the original infinite-dimensional optimization problem to a finite-dimensional one. It is worth mentioning that the conditions presented in Proposition 4.2 are sufficient conditions, which may not cover all the scenarios. In case such a R_{d_1,d_2,d_3} is not available, one can examine if there exists a retained loss function that has more layers, i.e.

$$R_{d_1,d_2,d_3,d_4,d_5}(x) = x \wedge d_1 + (x \wedge d_3 - d_2)_+ + (x \wedge d_5 - d_4)_+$$

with $0 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5 \leq M$, satisfies the conditions of Theorem 2.4 Case 2 of Cheung et al. (2015). This serves as an alternative approach if Proposition 4.2 cannot further improve the given retained loss function.

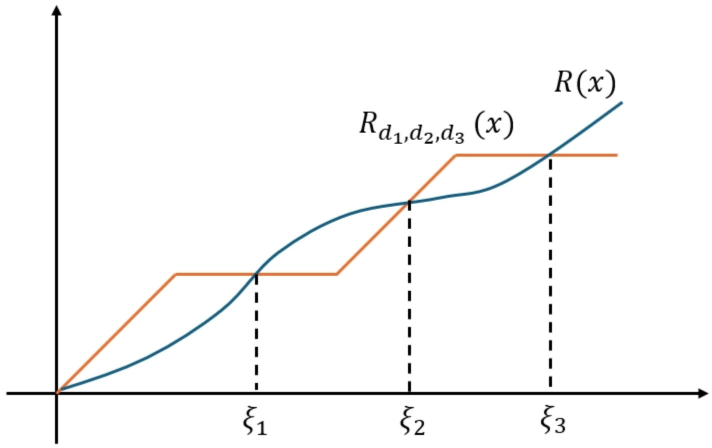

Figure 7: An illustrative plot for case (ii) of Proposition 4.2.

Example 4.2. *In this example, we apply the improvement technique from Proposition 4.2 to the following setting:*

(a). *The utility function is given by* $u(x) = -\exp(-x/1000)$ *and initial wealth is* $W_0 = 0$.

(b). *The loss variables* X_1 *and* X_2 *have truncated exponential distributions with parameters* $\mu_1 = 400$ *and* $\mu_2 = 350$*, respectively. The truncation point is* $M = 3000$.

(c). *The expected-value premium principle is applied, i.e.* $\pi(I(X)) = (1+\theta)\mathbb{E}[I(X)]$*, where* $\theta = 0.1$.

(d). *The relief fund payment probabilities are* $p = 0.7$ *and* $\tilde{p} = 0.8$ *from the perspectives of DM and insurer, respectively.*

(e). *The DM's retained loss after receiving the disaster relief fund is* $G(R(X), \ell) = (R(X) - \ell)_+$*, where* $\ell = 100$.

As proportional insurance is a widely used example in actuarial science, we assume that the DM is initially suggested to purchase $I(x) = (1-\alpha)x$ *from the insurer, where* α *is the proportion of the retained loss. This leads to the following optimization problem*

$$\max_{\alpha\in[0,1]} \Big\{ p\mathbb{E}[u(W_0 - G(\alpha X_1, \ell) - (1+\theta)(1-\alpha)\mathbb{E}[X_3])] + (1-p)\mathbb{E}[u(W_0 - \alpha X_2 - (1+\theta)(1-\alpha)\mathbb{E}[X_3])] \Big\},$$

for which the optimal proportion is given by $\alpha^* \approx 0.448$.

In the next step, we apply Proposition 4.2 (ii) to identify R_{d_1,d_2,d_3} *that outperforms* $R(x) := \alpha^* x$*. For that purpose, we solve the equation system*

$$\begin{cases} \alpha^*\mathbb{E}[X_1] = \mathbb{E}[R_{d_1,d_2,d_3}(X_1)], \\ \alpha^*\mathbb{E}[X_2] = \mathbb{E}[R_{d_1,d_2,d_3}(X_2)], \\ \alpha^*\mathbb{E}[(X_2 - \xi_2)_+] = \mathbb{E}[(R_{d_1,d_2,d_3}(X_2) - \alpha^*\xi_2)_+], \end{cases} \tag{28}$$

where we set the inequality of (27) to be equality to figure out the three parameters d_1, d_2, d_3*. However, the above system of equations is non-linear, and this makes the numerical search for a* *solution complicated. We therefore consider the following optimization problem instead of solving the equation system*

$$\min_{(d_1,d_2,d_3)\in\mathcal{D}} \ (\alpha^*\mathbb{E}[X_1]-\mathbb{E}[R_{d_1,d_2,d_3}(X_1)])^2+(\alpha^*\mathbb{E}[X_2]-\mathbb{E}[R_{d_1,d_2,d_3}(X_2)])^2 + (\alpha^*\mathbb{E}[(X_2-\xi_2)_+]-\mathbb{E}[(R_{d_1,d_2,d_3}(X_2)-\alpha^*\xi_2)_+])^2, \tag{29}$$

where

$$\mathcal{D} := \left\{(d_1,d_2,d_3)\in[0,M]^3 : \ A[d_1,d_2,d_3]^T \le [0,0,M,0,0]^T\right\},$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -\alpha^* & 0 \\ -1 & 1 & \alpha^*-1 \end{bmatrix}.$$

Here, restricting (d_1, d_2, d_3) *to* $\mathcal{D}$ *is to guarantee that* R *crosses* R_{d_1,d_2,d_3} *three times.*

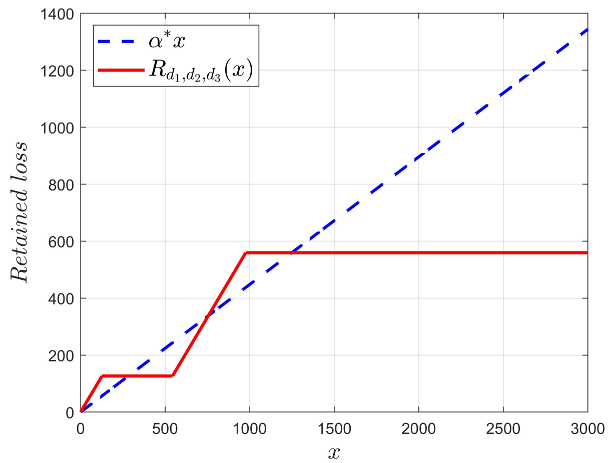

Figure 8: The comparison between $R(x) := \alpha^* x$ and $R_{d_1,d_2,d_3}(x)$.

Using the "fmincon" function in MATLAB, we solve Problem (29), which yields

$$d_1^* = 126.71, \quad d_2^* = 543.80, \quad d_3^* = 976.57,$$

and the corresponding retained loss function is illustrated in Figure 8. It can be easily calculated that

$$p\mathbb{E}[\alpha^* X_1] + (1-p)\mathbb{E}[\alpha^* X_2] \approx p\mathbb{E}[R_{d_1^*,d_2^*,d_3^*}(X_1)] + (1-p)\mathbb{E}[R_{d_1^*,d_2^*,d_3^*}(X_2)]$$

and

$$\alpha^*\mathbb{E}[(X_2-\xi_2)_+] \approx \mathbb{E}[(R_{d_1^*,d_2^*,d_3^*}(X_2)-\alpha^*\xi_2)_+].$$

Finally, we verify that $R_{d_1^*,d_2^*,d_3^*}$ *indeed improves the expected utility:*

$$p\mathbb{E}[u(W_0 - G(\alpha^* X_1,\ell) - (1+\theta)(1-\alpha^*)\mathbb{E}[X_3])] + (1-p)\mathbb{E}[u(W_0 - \alpha^* X_2 - (1+\theta)(1-\alpha^*)\mathbb{E}[X_3])] \approx -1.071,$$

$$p\mathbb{E}[u(W_0 - G(R_{d_1^*,d_2^*,d_3^*}(X_1),\ell) - (1+\theta)\mathbb{E}[X_3 - R_{d_1^*,d_2^*,d_3^*}(X_3)])] + (1-p)\mathbb{E}[u(W_0 - R_{d_1^*,d_2^*,d_3^*}(X_2) - (1+\theta)\mathbb{E}[X_3 - R_{d_1^*,d_2^*,d_3^*}(X_3)])] \approx -1.060.$$

5 Conclusion

This paper studies the insurance demand in the presence of government financial assistance, including disaster relief payment and premium subsidies. The optimal retained loss function is proven to be of the deductible form when the relief event is independent of the insurable loss. The effects of disaster relief assistance and premium subsidies on the demand for insurance have been investigated, and the impact can be significantly different under different risk aversion attitudes. We have also extended the results to cases in which the relief event and insurable loss are dependent, where two general dependent settings are considered. The optimal ceded loss functions are proven to be of the multiple-layered deductible treaties, which are rarely seen in the current literature on catastrophe insurance design.

Note that the present study does not take into account the profit of the insurer. For future works, it would be interesting to introduce the insurer's profit when maximizing the DM's terminal expected utility. For example, the insurer might want to seek the optimal pricing strategy by considering the Bowley solution; see, for example, Chan & Gerber (1985), Cheung et al. (2019). Another potential future research direction is the problem of allocating a fixed amount of budget relief fund from the perspective of the government. More specifically, let K be the capital reserved by the government for upholding the subsistence of its citizens. Consider a city or country consisting of m geographical regions suffering from some natural disaster such as flooding or storms. The government aims at distributing the budget K to these m regions as relief payments. The government is interested in seeking the best allocation policy by using, for example, the expected-shortfall or mean-variance models; see, for example, Dhaene et al. (2012) and Xu & Mao (2013).

It is worth noting that few studies consider the joint performance of insurance premium subsidies and relief funds. For example, Van Asseldonk et al. (2013) explore the trade-offs between providing catastrophic assistance and subsidizing insurance premiums. They highlight policy options that successfully stabilize income while limiting distortions of public intervention. By analyzing EU crop insurance based on public-private partnership, Liesivaara & Myyrä (2017) show that farmers' willingness to pay for crop insurance is conditional on the prospect of government disaster relief, and the possibility for disaster relief payments leads to extensive use of taxpayers' money if crop insurance premiums are subsidized. Other relevant studies are Bulut (2017) and Möllmann et al. (2019). The current coexistence of disaster relief funds and insurance premium subsidies motivates us to consider a DM who wants to make an agreement on the disaster insurance contract with an insurer by taking into account *ex ante* premium subsidies as well as the possibility of the government's *ex post* disaster relief payment. This issue is briefly addressed in the final part of Section 3.4. However, when considered under more general conditions, such as budget constraints for financial assistance or the use of general indemnity functions, the problem becomes more complex, yet intriguing and significant. This topic remains open for future research.

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Declarations of interest

No potential competing or conflict interests were reported by the authors.

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A Proof of Theorem 2.1

Let the set $\mathcal{I}$ be equipped with the L_∞ norm, then the objective function of Problem (2) is continuous w.r.t $R \in \mathcal{I}$. Similar to the proof of Theorem 2.1 of Liang et al. (2023), we note that: (i) the set $\mathcal{I}$ is uniformly equicontinuous due to the 1-Lipschitz continuity; (ii) the set $\mathcal{I}$ is uniformly bounded (by M); (iii) the 1-Lipschitz continuity can be preserved under the uniform convergence. Hence, Arzelà-Ascoli theorem is applicable here, from which we can derive that the set $\mathcal{I}$ is sequentially compact, and thus compact. Therefore, the maximum of the objective function of (2) is attainable over $\mathcal{I}$. This completes the proof of the existence.

For the uniqueness, let $R_1, R_2 \in \mathcal{I}$ denote the solutions to (2), i.e.,

$$\begin{aligned}
&\max_{R\in\mathcal{I}} \mathbb{E}[u(W_0 - G(R(X), Y) - h(\mathbb{E}[X - R(X)]))] \\
&= \mathbb{E}[u(W_0 - G(R_1(X), Y) - h(\mathbb{E}[X - R_1(X)]))] \\
&= \mathbb{E}[u(W_0 - G(R_2(X), Y) - h(\mathbb{E}[X - R_2(X)]))],
\end{aligned}$$

where $\mathbb{P}(R_1(X) = R_2(X)) < 1$. Let $R_3 = \epsilon R_1 + (1-\epsilon)R_2$ for some $\epsilon \in (0,1)$. If h is convex, the following holds:

$$\begin{aligned}
&\mathbb{E}[u(W_0 - G(R_3(X), Y) - h(\mathbb{E}[X - R_3(X)]))] \\
&= \mathbb{E}[u(W_0 - G(\epsilon R_1(X) + (1-\epsilon)R_2(X), Y) - h(\mathbb{E}[X - \epsilon R_1(X) - (1-\epsilon)R_2(X)]))] \\
&\geq \mathbb{E}\Big[u(W_0 - \epsilon G(R_1(X), Y) - (1-\epsilon)G(R_2(X), Y) \\
&\qquad - \epsilon h(\mathbb{E}[X - R_1(X)]) - (1-\epsilon)h(\mathbb{E}[X - R_2(X)]))\Big] \\
&\geq \epsilon\mathbb{E}[u(W_0 - G(R_1(X), Y) - h(\mathbb{E}[X - R_1(X)]))] \\
&\qquad + (1-\epsilon)\mathbb{E}[u(W_0 - G(R_2(X), Y) - h(\mathbb{E}[X - R_2(X)]))] \\
&= \max_{R\in\mathcal{I}} \mathbb{E}[u(W_0 - G(R(X), Y) - h(\mathbb{E}[X - R(X)]))],
\end{aligned}$$

where the first inequality is due to the convexity of h, and the the second inequality is due to the convexity of $-u, G$, and h. At least one of these two inequalities is strict due to the strict convexity of $-u, G$, or h. This yields a direct contradiction to the fact that R_1, R_2 are the solutions to Problem (2). As such, we have $\mathbb{P}(R_1(X) = R_2(X)) = 1$. This completes the proof.