BOWLEY INSURANCE WITH EXPECTED UTILITY MAXIMIZATION OF THE POLICYHOLDERS

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Abstract

This paper studies the Bowley solution for a sequential game within the expected utility framework. We assume that the policyholders are expected utility maximizers, and there exists a representative policyholder who faces a fixed loss with given probability and no loss otherwise. This policyholder selects the optimal indemnity function in response to the pricing kernel set by the insurer. Knowing the policyholder's choice of indemnity function, the insurer adjusts the pricing kernel to maximize its expected net profit. This pricing kernel is of our central interest in this paper, and in our setting the pricing kernel can be evaluated via the safety loading factor in an expected value premium principle. For a wide class of utility functions, we show that the optimal safety loading factor increases with respect to both the policyholder's risk aversion level and the probability of zero loss. We also show that the insurance contract corresponding to the Bowley solution is Pareto dominated in the sense that both parties' interests can be further improved, which shows the inefficiency of Bowley solution. Some numerical examples are presented to illustrate the main results, and it is shown that both the policyholder and insurer can strictly benefit from the Bowley solution.

Keywords: Bowley insurance, expected net profit, safety loading factor, Pareto optimality.

JEL code: C71, G22

1 Introduction

This paper studies an important notion of equilibria in economics – the Bowley solution, which consists of an indemnity function and a pricing kernel, where the latter is used to calculate the premium of the indemnity function. The Bowley solution is obtained in a sequential game in which the insurer selects the pricing kernel and the policyholders select the indemnity functions by maximizing their expected utilities under the given pricing kernel. The insurer maximizes its expected profit based on the policyholders' optimal responses. Chan and Gerber (1985) were the first to seek the optimal pair of indemnity function and pricing kernel in the expected utility framework via a power expansion. They present closed-form approximations for specific utility functions.

In this paper, we present a model in which the representative policyholder is facing a fixed positive loss with given probability, and no loss otherwise. This allows us to write the pricing function as a function of the safety loading factor in an expected value premium principle. Also, in this setting, any insurance indemnity can be viewed as of the deductible form. To obtain Bowley solutions, we take the perspective of a monopolistic insurer, and this insurer selects the safety loading factor to maximize its expected profit against a representative policyholder under symmetric information. Under a mild regularity condition on the utility function of the policyholder, we show that the deductible increases with respect to the safety loading factor. From the perspective of the insurer, there is thus a trade-off: a larger safety loading leads to a larger profit for a given indemnity, but on the other hand the indemnity itself is decreasing with respect to the safety loading factor. These two effects offset each other, which makes the selection of the optimal safety loading factor a non-trivial optimization problem. In this paper, we derive the Bowley solution analytically when the representative policyholder adopts, for example, exponential utility functions, quadratic utility functions, or logarithmic utility functions.

Recently, Cheung et al. (2019) study the Bowley solution within a modern risk management framework, in which the insurer aims to minimize a distortion risk measure and the reinsurer aims to maximize its expected profit. Other developments focusing on the Bowley solution along these two tracks, i.e. utility maximization and risk minimization, can be found in, for example, Chi et al. (2020), Boonen and Ghossoub (2022), Gavagan et al. (2022), Yuan et al. (2022) and some references therein.

We show that the Bowley solution is Pareto dominated in our setting. This means that it does not result in a Pareto optimal insurance contract, and there exists another insurance contract that is weakly better for both agents and strictly better for at least one agent. Such Pareto optimal contract can be obtained by maximizing the utility of the policyholder under a participation constraint of the insurer (Raviv, 1979). Because we do not consider any costs for the insurer, Pareto optimal insurance contracts are not expected to contain deductible points (Raviv, 1979), and we show in this paper that the Bowley solution does not coincide with any of them.

There are two main implications for the results in this paper. First, the Bowley solution is inefficient since its resultant insurance contract is Pareto dominated by another insurance contract.

A social planner can thus intervene and design an insurance contract that is better for both the policyholder and insurer and strictly better for at least one of them. There is thus scope for a government to intervene and make the market efficient. Second, in contrast to the Bowley solutions for convex and comonotonic-additive preferences (Boonen and Ghossoub, 2022), we show that the representative policyholder can also strictly benefit from the optimal insurance transaction. It must be the strictly concave expected utility preference of the representative policyholder that leads to the situation in which the insurer cannot fully exploit the representative policyholder by making her indifferent from the status quo.

The study of optimal insurance contracting is pioneered by the seminal works of Borch (1960) and Arrow (1974). A classic result of Arrow (1974) states that if the policyholder maximizes the expectation of a strictly concave utility function and purchases an indemnity via an expected value premium principle, then it is optimal for the policyholder to purchase full insurance above a deductible. This result has been generalized later by Schlesinger (1981), where he links the deductible to the degree of risk aversion. The deductible in the contract vanishes if and only if the expected value premium principle has no safety loading factor. In the context of proportional insurance, this was found earlier by Mossin (1968). These optimal insurance results have been extended in many directions. To name but a few, some extensions consider multiple policyholders in the insurance market (Cheung et al., 2014; Boonen and Liu, 2022), or alternative preferences of the policyholder (Doherty and Eeckhoudt, 1995; Braun and Muermann, 2004; Bernard et al., 2015; Chi et al., 2022), or asymmetric information regarding the policyholder's loss distribution (Chiappori and Salanie, 2000; Finkelstein and Poterba, 2004) or risk preferences (Landsberger and Meilijson, 1994; Boonen et al., 2021). This article mainly contributes to the literature of optimal insurance by studying the impact of a particular mechanism (i.e., the sequential game) on the optimal indemnities when there is a monopolistic insurer in the market.

The paper is set out as follows. The model is defined in Section 2. Section 3 provides the Bowley solution in the sequential game. Section 4 shows the inefficiency of such solution. Section 5 provides a numerical illustration. Section 6 concludes. All the proofs are delegated to Appendix A.

2 Problem setup

We focus on the market with finitely many policyholders who are faced with non-negative random losses $X_1, X_2, \dots, X_n$, which are realized at the end of a given future reference period. We assume that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.), and we write $X_i \overset{d}{=} X$ for all $i = 1, \dots, n$. Here, X is the risk faced by an agent representative of all the policyholders, and we assume that X has only two possible realizations: $X = 0$ with probability $p \in (0, 1)$ and $X = x_0 > 0$ with probability $1-p$ (a *Bernoulli*-type distribution). This agent considers to purchase an insurance contract that is summarized by a pair (I, P), where $I(X)$ is an indemnity function of the underlying loss X such that $I(0) = 0$ and $I(x_0) \in [0, x_0]$, and the premium $P \geq 0$. The premium P is calculated via a linear function π:

$$\pi(I) = \mathbb{E}[\zeta I(X)] = \zeta_1(1-p)I(x_0),$$

where ζ is a non-negative pricing kernel (random variable) which takes the value ζ_0 when no loss is incurred and ζ_1 otherwise. For $\theta = \zeta_1 - 1$, it follows that $\pi(I)$ is equivalent to the expected value premium principle:

$$\pi(I) = (1+\theta)(1-p)I(x_0) = (1+\theta)\mathbb{E}[I(X)].$$

To obtain a premium that exceeds the actuarial premium, we impose $\theta \geq 0$ (or $\zeta_1 \geq 1$), and θ is called the safety loading (or risk loading) factor. In the sequel, we refer to θ as the premium strategy of the insurer, and we refer to the pair (I, θ) as an insurance strategy.

Suppose that the representative policyholder is endowed with initial wealth w and is an expected utility maximizer. We assume that the policyholder's adopted utility function u is increasing, strictly concave and twice continuously differentiable. Then, given the safety loading factor $\theta \geq 0$, the representative policyholder would like to select the indemnity function which maximizes its expected utility, which is formulated as follows.

Problem 1 (Representative policyholder's problem)**.**

$$\begin{cases} \max\limits_{I \in \mathcal{I}} \mathbb{E}[u(w - X + I(X) - \pi(I))], \\ \;s.t.\; \pi(I) = (1+\theta)\mathbb{E}[I(X)], \end{cases}$$

where $\mathcal{I}$ is the set of admissible indemnity functions:

$$\mathcal{I} := \left\{ I : \{0, x_0\} \mapsto [0, x_0] \;\middle|\; I(0) = 0, I(x_0) \in [0, x_0] \right\}. \tag{2.1}$$

The optimal indemnity function to Problem 1 depends on θ, and we will show later that it is unique for each value of θ. Therefore, we are allowed to define I^*_θ as the optimal indemnity function to Problem 1. Knowing the policyholder's choice of I, the insurer selects the safety loading factor θ that maximizes its expected net profit (ENP):

$$\text{ENP} = n\left((1+\theta)\mathbb{E}[I^*_\theta(X)] - \mathbb{E}[I^*_\theta(X)]\right) = n\theta\mathbb{E}[I^*_\theta(X)].$$

The number of policyholders n is thus irrelevant for the value of the optimal safety loading θ, and thus we set $n = 1$ in the rest of this paper, without loss of generality. The insurer's problem is then formulated as follows.

Problem 2 (Insurer's problem)**.**

$$\max_{\theta \in [0,\infty)} \theta\mathbb{E}[I^*_\theta(X)].$$

We refer to strategies (I^*_θ, θ) with θ solving Problem 2 as *Bowley* solutions. Note that purchasing no insurance is always feasible for the policyholder, and thus we do not need to include the participation constraint $\mathbb{E}[u(w - X + I^*_\theta(X) - (1+\theta)\mathbb{E}[I^*_\theta(X)])] \geq \mathbb{E}[u(w - X)]$ to guarantee individual rationality of the policyholder. Also, $\theta \geq 0$ guarantees individual rationality of the insurer, i.e. make the insurer's ENP non-negative.

3 Bowley solution

3.1 The solution to the policyholder's problem

We write $y = I(x_0)$. Then, the policyholder's objective function can be written as

$$\begin{aligned} G(y, \theta) :&= \mathbb{E}[u(w - X + I(X) - \pi(I))] \\ &= u(w - (1+\theta)(1-p)y)p + u(w - x_0 + (p + \theta p - \theta)y)(1-p). \end{aligned}$$

For convenience, we denote by $G'_i(y, \theta)$ the partial derivative of $G(y, \theta)$ with respect to its i-th argument, and by $G''_{ij}(y, \theta)$ its double partial derivative with respect to its i-th and j-th argument, where $i, j \in \{1, 2\}$. We can readily verify that $G''_{11}(y, \theta) < 0$, and

$$\begin{aligned} G'_1(x_0, \theta) &= \left\{-(1+\theta)(1-p)p + (p + \theta p - \theta)(1-p)\right\} u'(w - (1+\theta)(1-p)x_0) \\ &= \theta(p-1)u'(w - (1+\theta)(1-p)x_0) \leq 0. \end{aligned}$$

Hence, the maximum of $G(y, \theta)$ is reached when $y \leq x_0$. Furthermore, we derive

$$G'_1(0, \theta) = -(1+\theta)(1-p)u'(w)p + (p + \theta p - \theta)u'(w - x_0)(1-p).$$

If $G'_1(0, \theta) \leq 0$, or equivalently

$$\theta \geq \frac{p}{\frac{u'(w-x_0)}{u'(w-x_0)-u'(w)} - p}, \tag{3.1}$$

then $\arg\max_{y \in [0, x_0]} \; G(y, \theta) = \{0\}$. We refer to solutions with $y = 0$ as trivial, as there is no insurance. To avoid this trivial solution, we make the following assumption.

Assumption 1.

$$0 \leq \theta < \frac{p}{\frac{u'(w-x_0)}{u'(w-x_0)-u'(w)} - p}.$$

Since $\frac{u'(w-x_0)}{u'(w-x_0)-u'(w)} \geq 1$, Assumption 1 implies $\theta < \frac{p}{1-p}$. Under Assumption 1, the maximum of $G(y; \theta)$ is reached within $(0, x_0]$, and the optimal coverage level y^* can be easily identified by using the first-order condition, i.e. $G'_1(y^*, \theta) = 0$. Furthermore, since $G''_{11}(y, \theta) < 0$, such y^* is unique in $[0, x_0]$. This allows us to define

$$y^*(\theta) := \underset{y \in [0, x_0]}{\arg\max} \; G(y, \theta),$$

for θ satisfying Assumption 1. By the implicit function theorem, $y^*(\cdot)$ is continuously differentiable (e.g., Theorem 11.1 of Loomis and Sternberg, 1990). The following lemma gives a sufficient condition for $y^*(\cdot)$ to be decreasing.

Lemma 3.1. *Let Assumption 1 hold. If $\theta \mapsto G'_1(y, \theta)$ is decreasing for all $y \in [0, x_0]$, then $y^*(\cdot)$ is decreasing.*

With Lemma 3.1, it can be proved that for a large class of utility functions, namely the utility functions with hyperbolic absolute risk aversion (HARA) (Gollier, 2001), with some restrictions on parameters, $y^*(\cdot)$ is decreasing. The HARA utility function takes the form

$$u(x) = \xi \left(\eta + \frac{x}{\gamma} \right)^{1-\gamma} \quad \text{for} \quad \eta + \frac{x}{\gamma} > 0. \tag{3.2}$$

where the parameters ξ and γ are such that $\frac{\xi(1-\gamma)}{\gamma} > 0$ to guarantee that $u(\cdot)$ is increasing and strictly concave (Gollier, 2001). The inverse of its Arrow-Pratt measure of risk aversion is given by

$$-\frac{u'(z)}{u''(z)} = \eta + \frac{z}{\gamma},$$

which is a linear function of z. This class includes many well known utility functions, such as exponential ($\gamma \to \infty$), quadratic ($\gamma = -1$), logarithmic ($\eta = 0$, $\gamma = 1$), and power utility ($\eta = 0$, $\gamma \neq 1$) functions. We present an important result in the following proposition.

Proposition 3.1. *If the utility function is a HARA utility function with $\gamma \leq 1$ or an exponential utility function, then $y^*(\theta)$ is decreasing.*

For a HARA utility function with $\gamma > 1$, we next provide a counter-example of the statement in Proposition 3.1. Let $u(x) = -x^{-1}$ for $x > 0$ (which is a power utility function with $\gamma = 2$), and $w = 710, x_0 = 700$, and $p = 0.6$. The upper bound of θ in Assumption 1 is given by $\frac{p}{\frac{u'(w-x_0)}{u'(w-x_0)-u'(w)} - p} \approx 1.499$. The function $y^*(\theta)$ is displayed in Figure 1, and this function is clearly not decreasing.

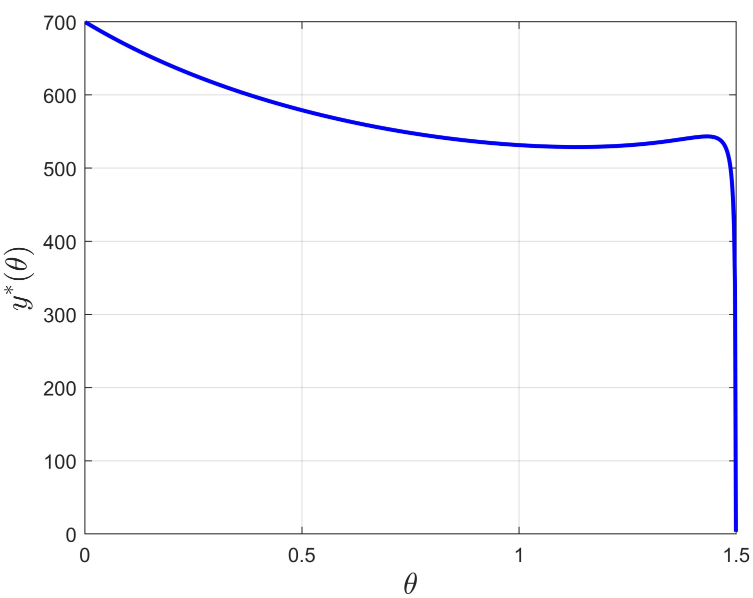

Figure 1: An example where $y^*(\theta)$ is not decreasing.

Lemma 3.1 and Proposition 3.1 actually present a very general sufficient conditions to make the insurance a non-Giffen good. That is, a higher price leads to a lower demand for the insurance. Some related discussions on non-Giffen goods can be found in Hoy and Robson (1981).

We conclude this subsection by presenting the following lemma, which shows the strict monotonicity of $G(y^*(\theta), \theta)$.

Lemma 3.2. *Let Assumption 1 hold. Then, the mapping $\theta \mapsto G(y^*(\theta), \theta)$ is strictly decreasing.*

3.2 The solution to the insurer's problem

For the insurer, the objective function is given by

$$H(\theta) := \theta \mathbb{E}[I^*_\theta(X)] = \theta y^*(\theta)(1-p).$$

Note that if θ satisfies (3.1), then $y^*(\theta) = 0$, which results in $H(\theta) = 0$. Hence, the insurer's problem can be written as

$$\max_{\theta \in [0, \bar{\theta}]} H(\theta), \tag{3.3}$$

where

$$\bar{\theta} = \frac{p}{\frac{u'(w-x_0)}{u'(w-x_0)-u'(w)} - p}.$$

Note that if $\theta \geq \bar{\theta}$, then $y^*(\theta) = 0$ which results in $H(\theta) = 0$. Moreover, for $\theta \in (0, \bar{\theta})$, it holds that $H(\theta) > 0$, and thus the ENP of the insurer is strictly positive.

In the following three subsections, we will derive the implicit or explicit solutions to Problem (3.3) for some popular special cases of the HARA utility function.

3.2.1 Exponential utility function

Suppose that the utility function of the policyholder is given by

$$u(x) = -e^{-\phi x}, \quad x \in \mathbb{R}, \tag{3.4}$$

where $\phi > 0$ is called the absolute risk aversion parameter. We next show that the first-order condition is necessary and sufficient for optimality of the safety loading.

Theorem 3.1. *Under the exponential utility function (3.4), the safety loading factor θ^* that solves (3.3) is given by the unique root of*

$$\phi x_0 - \log \frac{(1+\theta)p}{p+\theta p-\theta} + \frac{1}{1+\theta} - \frac{p}{p+\theta p - \theta} = 0. \tag{3.5}$$

Since the left hand side of Eq. (3.5) is increasing with respect to ϕ and decreasing with respect to θ, a larger ϕ leads to a larger θ^*. Therefore, the insurer would apply a larger safety loading factor when the policyholder is more risk averse.

Moreover, it holds that

$$\phi x_0 - \log \frac{(1+\theta)p}{p+\theta p-\theta} + \frac{1}{1+\theta} - \frac{p}{p+\theta p - \theta} = \phi x_0 - \log \frac{1+\theta}{1+\theta-\frac{\theta}{p}} + \frac{1}{1+\theta} - \frac{1}{1+\theta-\frac{\theta}{p}},$$

which is increasing with respect to p. Therefore, a larger zero-loss probability p leads to a larger safety loading factor θ.

3.2.2 Quadratic utility function

Suppose that the policyholder's utility function is given by

$$u(x) = -\frac{\beta}{2}x^2 + x, \quad x \in (-\infty, \frac{1}{\beta}), \tag{3.6}$$

where $\frac{1}{\beta}$ is called the saturation point. We assume that $w < \frac{1}{\beta}$, so that terminal wealth cannot exceed the saturation point.

Theorem 3.2. *Under the quadratic utility function (3.6), the safety loading factor θ^* that solves (3.3) is given by*

$$\theta^* = \frac{C + \sqrt{C^2 + \beta^2 p(1-p)x_0^2}}{\beta(1-p)x_0},$$

where $C = \beta p x_0 + \beta w - \beta x_0 - 1$.

The Arrow-Pratt measure of risk aversion for the quadratic utility function is $-\frac{u''(x)}{u'(x)} = \frac{\beta}{1-\beta x}$, which is increasing in β. Note that

$$\begin{aligned}\theta^* =& \frac{C + \sqrt{C^2 + \beta^2 p(1-p)x_0^2}}{\beta(1-p)x_0}\\ =& \frac{(\sqrt{C^2 + \beta^2 p(1-p)x_0^2} + C)(\sqrt{C^2 + \beta^2 p(1-p)x_0^2} - C)}{\beta(1-p)x_0(\sqrt{C^2 + \beta^2 p(1-p)x_0^2} - C)}\\ =& \frac{p x_0}{\sqrt{(C/\beta)^2 + p(1-p)x_0^2} - \frac{C}{\beta}},\end{aligned}$$

which is increasing in β. Therefore, similar to the exponential utility case, if the policyholder is more risk averse, then the insurer will charge a larger safety loading factor.

By substituting $C = \beta p x_0 + \beta w - \beta x_0 - 1$ into the above expression of θ^*, we get that

$$\theta^* = \frac{\beta p x_0}{\sqrt{(1-p)(2\beta x_0(1-\beta w) + \beta^2 x_0^2) + (1-\beta w)^2} - \beta p x_0 - \beta w + \beta x_0 + 1},$$

which is increasing in p. Similar to the exponential utility case, we thus find that a larger zero-loss probability leads to a larger safety loading factor.

3.2.3 Logarithmic utility function

Suppose that the policyholder's utility function is given by

$$u(x) = \log(x), \quad x > 0. \tag{3.7}$$

Logarithmic utility is special case of a utility function that is endowed with constant relative risk aversion. We assume in this section that $w - x_0 > 0$ such that terminal wealth of the policyholder is always positive.

Theorem 3.3. *Under the logarithmic utility function (3.7), the safety loading factor θ^* that solves (3.3) is given by*

$$\theta^* = \frac{x_0 p}{w - x_0 p + \sqrt{w^2 - w x_0}}.$$

Since the above θ^* is increasing in p, a larger zero-loss probability p leads to a larger safety loading factor θ^*, which is in line with the results for the exponential and quadratic utilities. Also, since $w > x_0$, it holds that θ^* in Theorem 3.3 is decreasing in w. That is, the wealthier the policyholder is, the smaller the safety loading must become in order to maximize profit for the insurer. A wealthier individual has lower Arrow-Pratt measure of risk aversion, and thus demands less insurance. The insurer then needs to reduce the safety loading factor to attract such individuals.

3.2.4 Common findings with the three utility functions

Considering all the three cases of the utility function of the policyholder, we find that θ^* increases in the zero-loss probability p. Moreover, although the policyholder's risk-aversion level is measured in different ways, we find for the exponential and quadratic utility functions that θ^* increases in the policyholder's risk-aversion level.

Also, we find that the coverage y^* increases in both the zero-loss probability p and the policyholder's risk-aversion level (for the exponential and quadratic utility functions). This shows that the policyholder is more averse to the low-probability-high-consequence event. So, the insurer has then an incentive to increase the safety loading factor in order to maximize its ENP.

4 Inefficiency of the Bowley solution

In this section we will show the inefficiency of the Bowley solution. This implies that the policyholder and insurer can benefit from cooperating, and bargain for an insurance contract jointly. Jointly selecting an insurance contract can be done in a Pareto-optimal way, and this means that one party's interest cannot be further improved without harming the other party's interest. In line with Boonen and Ghossoub (2022), we define Pareto optimality on insurance contracts rather than on the strategies (I, θ) that were used to define Bowley solutions. Recall that an insurance contract is given by a pair (I, P), with premium $P \geq 0$. Assuming that the policyholder and insurer adopt the same objective functions as stated in Section 2, a formal definition of the Pareto-optimal insurance contract is given below.

Definition 4.1. *An insurance contract (I, P) is said to be Pareto optimal if there does not exist another so-called Pareto-dominating contract $(\tilde{I}, \tilde{P})$ such that*

$$\mathbb{E}[u(w - X + I(X) - P)] \leq \mathbb{E}[u(w - X + \tilde{I}(X) - \tilde{P})],$$
$$P - \mathbb{E}[I(X)] \leq \tilde{P} - \mathbb{E}[\tilde{I}(X)],$$

with at least one of the two inequalities being strict.

We call a contract *Pareto dominated* if it is not Pareto optimal. To find the Pareto-optimal insurance policy, we solve the following problem.

Problem 3 (Pareto-optima)**.**

$$\max_{(I,P)\in\mathcal{I}\times[0,\infty)} \; P-\mathbb{E}[I(X)]$$
$$s.t.\ \mathbb{E}[u(w-X+I(X)-P)]\geq L,$$

where L refers to some reservation utility level of the policyholder. Solving such a constrained optimization problem is a common way to obtain Pareto-optimal insurance contracts, and we refer the interested readers to Miettinen (1999) for a comprehensive review of the technique. The following proposition is provided to show the connection between the solution to Problem 3 and the Pareto-optimal contract as defined by Definition 4.1.

Proposition 4.1. *If there exists a non-trivial solution to Problem 3, then it is Pareto optimal.*

We are now able to compare the Bowley solution (i.e., Problems 1 and 2) with the Pareto optima in Problem 3. Notably, we find that there exists a Pareto-dominating contract to the Bowley solution under a mild condition, which shows the inefficiency of the sequential game.

Theorem 4.1. *If $y^*(\cdot)$ is decreasing, then the Bowley solution (I^*,θ^*) leads to a Pareto dominated insurance contract. That is, $(I^*,\pi(I^*))$ with $\pi(I^*)=(1+\theta^*)\mathbb{E}[I^*(X)]$ is Pareto dominated.*

Recall from Lemma 3.1 and Proposition 3.1 that the condition that $y^*(\cdot)$ is decreasing holds for a variety of HARA utility functions.

5 Numerical examples

In this section, we will investigate through numerical examples the impact of the policyholder's utility function and the zero-loss probability on the policyholder's EU and the insurer's ENP. We will also demonstrate numerically that the Bowley solution is Pareto dominated. To interpret the size of expected utility increases, we convert the policyholder's EU to its certainty equivalent wealth (CEW), which is given by

$$\text{CEW}=u^{-1}\left(\mathbb{E}[u(w-X+I(X)-\pi(I))]\right).$$

To eliminate the effect of initial CEW, it is more meaningful to compare the ENP versus the change of CEW, which is given by

$$\Delta\text{CEW}=u^{-1}\left(\mathbb{E}[u(w-X+I(X)-\pi(I))]\right)-u^{-1}\left(\mathbb{E}[u(w-X)]\right).$$

Clearly, ΔCEW is positive if and only the policyholder strictly benefits from purchasing the insurance contract. In the following, we assume that the policyholder is an exponential utility maximizer with absolute risk aversion parameter ϕ (see Eq. (3.4)). The loss amount is assumed to be given by $x_0=1000$, and the initial wealth of the policyholder is irrelevant due to exponential utilities, and therefore we set it equal to zero: $w=0$.

5.1 The Bowley solution

In the first example, we fix the zero-loss probability, which is given by $p = 0.6$, and vary the value of the absolute risk aversion parameter ϕ from 0.0018 to 0.0022. The corresponding policyholder's ΔCEW and insurer's ENP of the Bowley solution are shown in Figure 2. In this example, a larger ϕ leads to an increase of both the policyholder's ΔCEW and the insurer's ENP. This tells us that a more risk-averse policyholder is more likely to buy insurance, and also the insurer benefits from facing a more risk-averse policyholder.

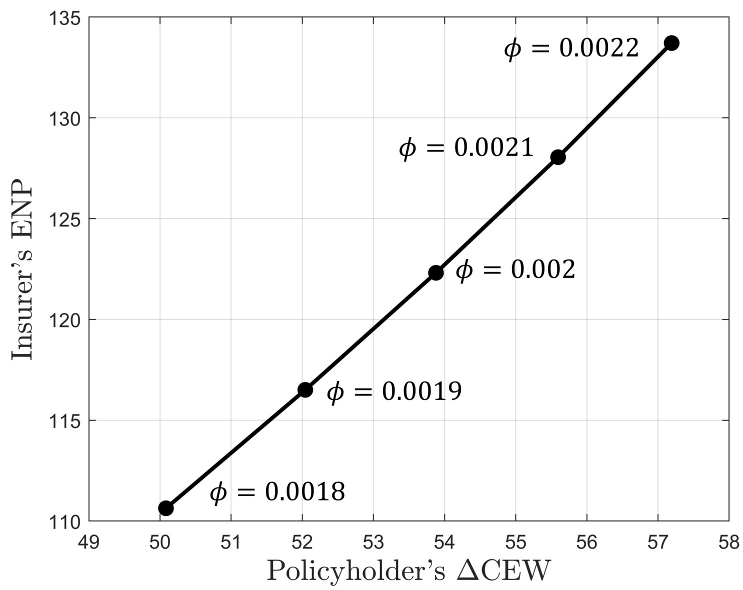

Figure 2: The impact of the absolute risk aversion parameter ϕ on ΔCEW and ENP.

In the second example, we fix the policyholder's expected loss $(1 - p)x_0 = 400$ and change the value of p from 0.3 to 0.8, i.e., while varying p we simultaneously adjust $x_0 = 400/(1 - p)$. Figure 3 displays the effect of p, together with x_0, on both the ΔCEW and the ENP of the Bowley solution. We find that both the policyholder's ΔCEW and the insurer's ENP increase with respect to p. Note that x_0 increases with respect to p in this example. Hence, because the policyholder is more averse to the low-probability-high-consequence events, the policyholder demands more insurance when faced with such events and the increase in insurance demand is beneficial for both parties.

We next fix the loss amount $x_0 = 1000$, and only investigate the impact of the no-loss probability p on the ΔCEW and the ENP of the Bowley solution. As shown in Figure 4, the effect of p is not as straightforward as in the second example. Both the ΔCEW and the ENP first increase and then decrease with respect to p. When p is small, an increase in p means that there is more uncertainty in X, which yields more benefits from insurance. On the other hand, if p is large, an increase in p means that the loss variable X has a smaller expected value, which in return leads to a decrease in the benefits from insurance. Interestingly, both effects on the benefits for the policyholder and the insurer usually move in the same direction.

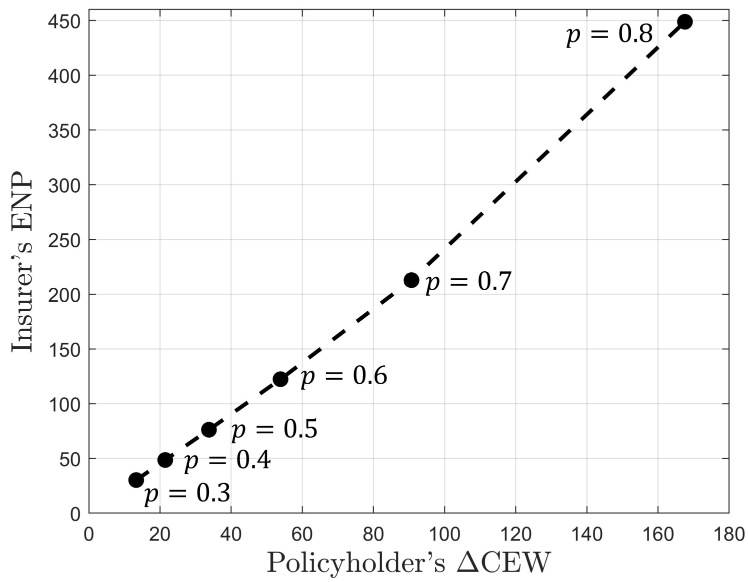

Figure 3: The impact of the no-loss probability p on ΔCEW and ENP when $\mathbb{E}[X]$ is fixed.

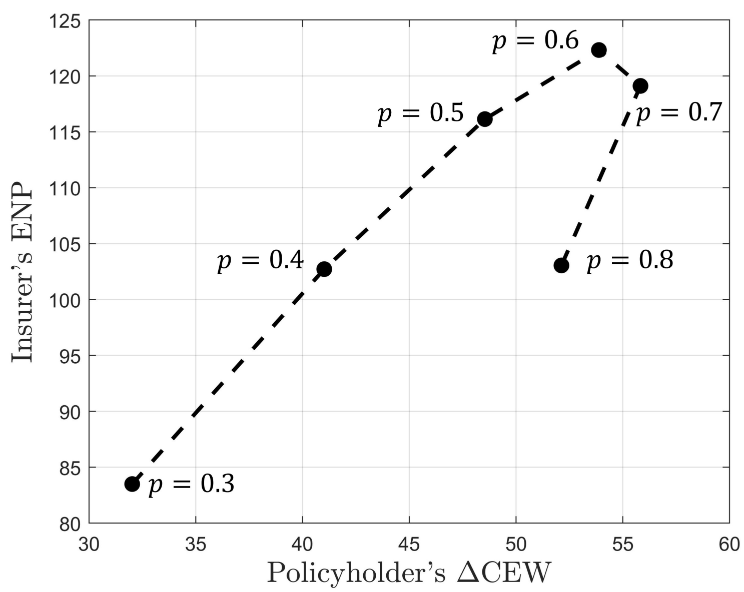

Figure 4: The impact of the no-loss probability p on the ΔCEW and the ENP when x_0 is fixed.

5.2 The comparison between the Bowley solution and the Pareto frontier

We next fix both the zero-loss probability $p = 0.6$ and the policyholder's aversion parameter $\phi = 0.002$. Figure 5 shows the comparison between the Bowley solution and the frontier of Pareto-optimal insurance contracts. As already shown analytically in Theorem 4.1, Figure 5 confirms that the insurance contract corresponding to the Bowley solution is not on the Pareto frontier, and hence there exists a Pareto-dominating contract. For instance, if we fix the policyholder's ΔCEW, which is achieved from the Bowley solution, the insurer can increase her ENP by approximately 60, and this increase in ENP can be obtained via a social planner that solves Problem 3 with $L = u\left(u^{-1}(\mathbb{E}[u(w-X)]) + \Delta\text{CEW}\right)$.

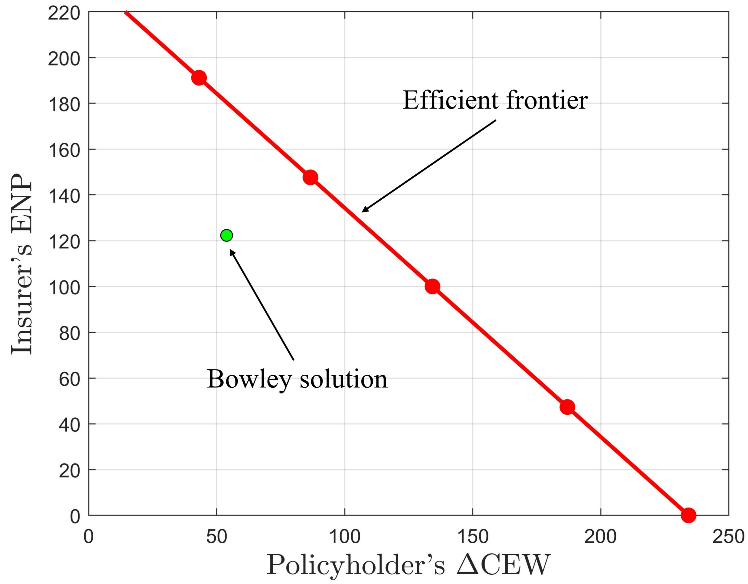

Figure 5: Comparison between insurance contract corresponding to the Bowley solution and the Pareto frontier. The Pareto frontier is approximated via solving Problem 3 for various choices of L.

6 Concluding remarks and future research

This paper studies the optimal insurance problem to find the Bowley solution in a sequential game. The representative policyholder is assumed to be a price-taker and adjusts her demand for insurance according to the pricing kernel set by the insurer. Anticipating the optimal response of the policyholder, the insurer then selects the pricing kernel to maximize its expected net profit. In the setting with a Bernoulli-type loss, finding the pricing kernel is equivalent to finding the safety loading factor. For three different classes of the utility function (exponential, quadratic and logarithmic utility functions), we show that the safety loading factor in the Bowley solution is increasing with respect to both the policyholder's risk aversion level and the probability of zero loss. To show the inefficiency of such sequential game, we compare the Bowley solution with that of a cooperative optimization problem. We show in this paper that the insurance contract corresponding to the Bowley solution is Pareto dominated by the solution to a cooperative optimization problem. In other words, if the policyholder and insurer choose to collaborate instead of acting as the follower and leader, it is possible for both of them to further improve their utilities. Moreover, when compared to the situation where no transaction happens between the policyholder and insurer, we show that the Bowley solution may lead to strict utility improvements for both agents.

Our study differs from Boonen and Ghossoub (2022) in that we assume that there are only two future states of the world with one no-loss state. Therefore, we study fewer states of the future loss X. Furthermore, a fundamental difference between our model setup and that of Boonen and Ghossoub (2022) is the assumption on the preferences of the leader and follower in the sequential game. If the policyholder's preference is captured by a convex and comonotonic-additive risk measure[1] and the insurer's preference is captured by a translation-invariant risk measure[2], Boonen and Ghossoub (2022) show the equivalence between the Bowley solution and the Pareto optimal insurance contract in which the policyholder (the follower in the sequential game) is indifferent with the status quo. However, if the policyholder aims to maximize her expected utility under a strictly concave utility function and the insurer is risk neutral, we show in this paper that the Bowley solution is always Pareto dominated. Moreover, as we have shown in Section 5, in our setting it is possible for both the policyholder and insurer to strictly benefit from the Bowley solution, which is in sharp contrast to Boonen and Ghossoub (2022). In summary, Boonen and Ghossoub (2022) show the efficiency of Bowley solutions and the indifference of the policyholder therein, but these results are not necessarily true if one were to study such problem within the expected utility framework.

This paper can be extended along several directions. One interesting direction is considering a more general distribution of the loss. This brings a more complex pricing kernel to the problem and yields non-trivial challenges to solving the sequential game. Another interesting direction is modifying the insurer's objective by including cost of capital. Both extensions are left for future research.

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[1] A risk measure ρ is called:

- convex if for all random variables (Y, Z) and all $\lambda \in [0,1]$, $\rho(\lambda Y + (1-\lambda)Z) \leq \lambda\rho(Y) + (1-\lambda)\rho(Z)$;
- comonotonic-additive if $\rho(Y+Z) = \rho(Y) + \rho(Z)$, for all (Y, Z) that are comonotonic, where a comonotonic pair (Y, Z) is such that $\left[Y(\omega) - Y(\omega')\right]\left[Z(\omega) - Z(\omega')\right] \geq 0$ for any two states of the world ω and ω'.

[2] A risk measure ρ is called translation-invariant if $\rho(Y+c) = \rho(Y) + c$, for all random variables Y and all $c \in \mathbb{R}$.

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A Proofs

Proof of Lemma 3.1. Let $\theta_1 < \theta_2$. Under Assumption 1, assume that y_1, y_2 satisfy the following two first-order conditions:

$$G_1'(y_1, \theta_1) = 0, \quad G_1'(y_2, \theta_2) = 0,$$

then $G_1'(y_2, \theta_2) = G_1'(y_1, \theta_1) \geq G_1'(y_1, \theta_2)$. Since $G_1'(\cdot, \theta)$ is strictly decreasing, we get that $y_1 \geq y_2$. This ends the proof. □

Proof of Proposition 3.1. Note that

$$\begin{aligned} G_{12}''(y,\theta) = (1-p)\big\{&-u'(w-(1+\theta)(1-p)y)p-(1+\theta)u''(w-(1+\theta)(1-p)y)p(p-1)y \\ &+(p-1)u'(w-x_0+(p+\theta p-\theta)y)+(p+\theta p-\theta)u''(w-x_0+(p+\theta p-\theta)y)(p-1)y\big\}, \end{aligned} \tag{A.1}$$

where the first three terms in the curly brackets are all negative. We list the following three sufficient conditions for $y^*(\theta)$ to be decreasing.

(i). $u'''(\cdot) \leq 0$

If summing up the second and last terms in the curly brackets of Eq. (A.1), we get

$$\begin{aligned}&(1-p)y\left\{(p+\theta p)u''(w-(1+\theta)(1-p)y)-(p+\theta p-\theta)u''(w-x_0+(p+\theta p-\theta)y)\right\}\\ \leq\ &(p+\theta p)(1-p)y\left\{u''(w-y+(p+\theta p-\theta)y)-u''(w-x_0+(p+\theta p-\theta)y)\right\}.\end{aligned}$$

Therefore, if $u'''(\cdot) \leq 0$, then

$$u''(w-y+(p+\theta p-\theta)y) \leq u''(w-x_0+(p+\theta p-\theta)y),$$

from which we get $G''_{12}(y,\theta) \leq 0$. Then, as per Lemma 3.1, $y^*(\theta)$ is decreasing.

(ii). $A_u(w-x_0+x) \leq \frac{1}{x}$ for $x \geq 0$

If summing up the third and last terms in the curly brackets of Eq. (A.1), we get that

$$(p-1)\left\{u'(w-x_0+(p+\theta p-\theta)y)+u''(w-x_0+(p+\theta p-\theta)y)(p+\theta p-\theta)y\right\}.$$

If $A_u(w-x_0+x) \leq \frac{1}{x}$ for $x \geq 0$, then we have

$$A_u(w-x_0+(p+\theta p-\theta)y) \leq \frac{1}{(p+\theta p-\theta)y},$$

and thus

$$u'(w-x_0+(p+\theta p-\theta)y)+u''(w-x_0+(p+\theta p-\theta)y)(p+\theta p-\theta)y \geq 0.$$

Then, by substituting the inequalities in Eq. (A.1), we get $G''_{12}(y,\theta) \leq 0$. Then $y^*(\theta)$ is decreasing by Lemma 3.1.

(iii). $\frac{u'(w-x_0+(p+\theta p-\theta)y)}{u'(w-y+(p+\theta p-\theta)y)}$ decreases in θ

The first-order condition $G'_1(y,\theta)=0$ is equivalent to

$$\frac{u'(w-x_0+(p+\theta p-\theta)y)}{u'(w-y+(p+\theta p-\theta)y)} = \frac{(1+\theta)p}{p+\theta p-\theta}, \tag{A.2}$$

we note that the right hand side of Eq. (A.2) is increasing with respect to θ under Assumption 1. If $G'_1(y_1,\theta_1) = G'_1(y_2,\theta_2) = 0$ with $\theta_1 \leq \theta_2$, and the left hand side of Eq. (A.2) is decreasing with respect to θ, which holds for the exponential utility function, then

$$\begin{aligned}\frac{u'(w-x_0+(p+\theta_1 p-\theta_1)y_1)}{u'(w-y_1+(p+\theta_1 p-\theta_1)y_1)} &= \frac{(1+\theta_1)p}{p+\theta_1 p-\theta_1}\\ &\leq \frac{(1+\theta_2)p}{p+\theta_2 p-\theta_2}\\ &= \frac{u'(w-x_0+(p+\theta_2 p-\theta_2)y_2)}{u'(w-y_2+(p+\theta_2 p-\theta)y_2)}\\ &\leq \frac{u'(w-x_0+(p+\theta_1 p-\theta_1)y_2)}{u'(w-y_2+(p+\theta_1 p-\theta_1)y_2)}.\end{aligned}$$

Since $\frac{u'(w-x_0+(p+\theta p-\theta)y)}{u'(w-y+(p+\theta p-\theta)y)} = \frac{u'(w-x_0+(p+\theta p-\theta)y)}{u'(w-(1-p)(1+\theta)y)}$ is decreasing in y, we have $y_1 \geq y_2$. Thus, $y^*(\theta)$ is decreasing.

Recall that the general form of the utility function with harmonic absolute risk aversion (HARA) is

$$u(x) = \xi \left(\eta + \frac{x}{\gamma}\right)^{1-\gamma} \quad \text{for} \quad \eta + \frac{x}{\gamma} > 0,$$

whose first three derivatives of u are given by (see also Gollier, 2001):

$$\begin{aligned}
u'(x) &= \xi \frac{1-\gamma}{\gamma} \left(\eta + \frac{x}{\gamma}\right)^{-\gamma}, \\
u''(x) &= -\xi \frac{1-\gamma}{\gamma} \left(\eta + \frac{x}{\gamma}\right)^{-\gamma-1}, \\
u'''(x) &= \xi \frac{(1-\gamma)(1+\gamma)}{\gamma^2} \left(\eta + \frac{x}{\gamma}\right)^{-\gamma-2}.
\end{aligned}$$

If $\gamma \in [0,1]$, then

$$A_u(w - X_0 + x) = -\frac{u''(w - x_0 + x)}{u'(w - x_0 + x)} = \frac{\gamma}{w - x_0 + x + \eta\gamma} \leq \frac{1}{x},$$

which satisfies Condition (ii).

If $\gamma \in [-1, 0)$, then since $\frac{\xi(1-\gamma)}{\gamma} > 0$, we have

$$u'''(x) = \xi \frac{(1-\gamma)(1+\gamma)}{\gamma^2} \left(\eta + \frac{x}{\gamma}\right)^{-\gamma-2} = \xi \frac{1-\gamma}{\gamma} \cdot \frac{1+\gamma}{\gamma} \left(\eta + \frac{x}{\gamma}\right)^{-\gamma-2} \leq 0,$$

which satisfies Condition (i).

If $\gamma \in (-\infty, -1)$, then it follows that

$$\begin{aligned}
\frac{u'(w - x_0 + (p + \theta p - \theta)y)}{u'(w - y + (p + \theta p - \theta)y)} &= \left(\frac{\eta\gamma + w - x_0 + (p + \theta p - \theta)y}{\eta\gamma + w - y + (p + \theta p - \theta)y}\right)^{-\gamma} \\
&= \left(1 + \frac{y - x_0}{\eta\gamma + w - y + (p + \theta p - \theta)y}\right)^{-\gamma},
\end{aligned}$$

which is decreasing with respect to θ, which fulfills Condition (iii).

Recall that the case of exponential utility function is already shown within Condition (iii). This concludes the proof. □

Proof of Lemma 3.2. Note that $y^*(\theta)$ satisfies (A.2), that is

$$\frac{u'(w - x_0 + (p + \theta p - \theta)y^*(\theta))}{u'(w - y^*(\theta) + (p + \theta p - \theta)y^*(\theta))} = \frac{(1+\theta)p}{p + \theta p - \theta}.$$

Then,

$$\begin{aligned}
\frac{dG(y^*(\theta), \theta)}{d\theta} =& u'(w - (1+\theta)(1-p)y^*(\theta))p \left[-(1-p)y^*(\theta) - (1+\theta)(1-p)y^{*\prime}(\theta)\right] \\
&+ u'(w - x_0 + (p + \theta p - \theta)y^*(\theta))(1-p) \left[(p-1)y^*(\theta) + (p + \theta p - \theta)y^{*\prime}(\theta)\right] \\
=& u'(w - x_0 + (p + \theta p - \theta)y^*(\theta)) \frac{p + \theta p - \theta}{1+\theta} \left[(p-1)y^*(\theta) - (1+\theta)(1-p)y^{*\prime}(\theta)\right] \\
&+ u'(w - x_0 + (p + \theta p - \theta)y^*(\theta))(1-p) \left[(p-1)y^*(\theta) + (p + \theta p - \theta)y^{*\prime}(\theta)\right] \\
=& u'(w - x_0 + (p + \theta p - \theta)y^*(\theta))(1-p)(p + \theta p - \theta) \left[-\frac{1-p}{p + \theta p - \theta} - \frac{1}{1+\theta}\right] y^*(\theta) < 0.
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 3.1. The first-order condition $G_1'(y^*, \theta) = 0$ yields

$$-(1+\theta)(1-p)u'(w-(1+\theta)(1-p)y^*)p + (p+\theta p-\theta)u'(w-x_0+(p+\theta p-\theta)y^*)(1-p) = 0,$$

or, equivalently,

$$\frac{u'(w-x_0+(p+\theta p-\theta)y^*)}{u'(w-(1+\theta)(1-p)y^*)} = \frac{(1+\theta)p}{p+\theta p-\theta}.$$

Since u is an exponential utility function, we get

$$y^* = x_0 - \frac{1}{\phi}\log\frac{(1+\theta)p}{p+\theta p-\theta}.$$

Then, Problem (3.3) becomes:

$$\max_{\theta\in[0,\bar{\theta}]} \tilde{H}(\theta) := \theta\left(x_0 - \frac{1}{\phi}\log\frac{(1+\theta)p}{p+\theta p-\theta}\right). \tag{A.3}$$

Below we derive the first- and second-order derivatives of $\tilde{H}$:

$$\begin{aligned}
\tilde{H}'(\theta) &= x_0 - \frac{1}{\phi}\log\frac{(1+\theta)p}{p+\theta p-\theta} + \theta\left[-\frac{1}{\phi}\cdot\frac{1}{1+\theta} + \frac{1}{\phi}\cdot\frac{p-1}{p+\theta p-\theta}\right] \\
&= x_0 - \frac{1}{\phi}\log\frac{(1+\theta)p}{p+\theta p-\theta} - \frac{1}{\phi}\cdot\frac{\theta}{1+\theta} + \frac{1}{\phi}\cdot\frac{\theta p-\theta}{p+\theta p-\theta} \\
&= x_0 - \frac{1}{\phi}\log\frac{(1+\theta)p}{p+\theta p-\theta} + \frac{1}{\phi}\cdot\frac{1}{1+\theta} - \frac{1}{\phi}\cdot\frac{p}{p+\theta p-\theta},
\end{aligned}$$

and

$$\tilde{H}''(\theta) = -\frac{1}{\phi}\cdot\frac{1}{1+\theta} + \frac{1}{\phi}\cdot\frac{p-1}{p+\theta p-\theta} - \frac{1}{\phi}\cdot\frac{1}{(1+\theta)^2} + \frac{1}{\phi}\cdot\frac{p(p-1)}{(p+\theta p-\theta)^2} < 0.$$

Therefore, $\tilde{H}$ is strictly concave.

From the first-order condition, we derive that

$$\bar{\theta} = \frac{p(e^{\phi x_0}-1)}{e^{\phi x_0} - p(e^{\phi x_0}-1)}.$$

Let $L(\theta) = \phi x_0 - \log\frac{(1+\theta)p}{p+\theta p-\theta} + \frac{1}{1+\theta} - \frac{p}{p+\theta p-\theta}$, which is clearly continuous. A direct computation yields $L(0) = \phi x_0 > 0$ and

$$L(\bar{\theta}) = (1-p)(1-e^{\phi x_0}) - p(1-e^{-\phi x_0}) < 0.$$

Therefore, there must exist a root of $L(\theta)$ on $(0, \bar{\theta})$. Since $\tilde{H}''(\theta) < 0$, this root is unique. This ends the proof. □

Proof of Theorem 3.2. The first-order condition $G_1'(y^*, \theta) = 0$ yields

$$\frac{-\beta(w-x_0+(p+\theta p-\theta)y^*)+1}{-\beta(w-y+(p+\theta p-\theta)y^*)+1} = \frac{(1+\theta)p}{p+\theta p-\theta},$$

and thus

$$y^* = \frac{\beta p x_0(1+\theta) + \beta\theta w - \beta\theta x_0 - \theta}{\beta(p-p\theta^2+\theta^2)}.$$

Then, Problem (3.3) can be written as

$$\max_{\theta\in[0,\bar{\theta}]} \tilde{H}(\theta) = \frac{\beta p x_0(\theta^2+\theta)+\beta\theta^2 w-\beta\theta^2 x_0-\theta^2}{\beta(p-p\theta^2+\theta^2)}.$$

Then, $\tilde{H}$ can be simplified to

$$\tilde{H}(\theta) = \frac{\frac{\beta p x_0}{\theta}+C}{\frac{\beta p}{\theta^2}+\beta(1-p)},$$

where $C=\beta p x_0+\beta w-\beta x_0-1<0$. Then,

$$\tilde{H}'(\theta) = \frac{-\beta^2 p(1-p)x_0\theta^2+2C\beta p\theta+\beta^2p^2x_0}{\theta^4\left(\frac{\beta p}{\theta^2}+\beta(1-p)\right)^2}.$$

Let $L(\theta)=-\beta^2p(1-p)x_0\theta^2+2C\beta p\theta+\beta^2p^2x_0$. Since $L(0)=\beta^2p^2x_0>0$, $L(\theta)$ has only one positive root, and that is given by

$$\theta^* = \frac{C+\sqrt{C^2+\beta^2p(1-p)x_0^2}}{\beta(1-p)x_0}. \tag{A.4}$$

Therefore, $\tilde{H}'(\theta)\geq 0$ on $[0,\theta^*]$ and $\tilde{H}'(\theta)<0$ on (θ^*,∞). Moreover,

$$\theta^* = \frac{C+\sqrt{C^2+\beta^2p(1-p)x_0^2}}{\beta(1-p)x_0} > \frac{C+|C|}{\beta(1-p)x_0}=0. \tag{A.5}$$

Under the quadratic utility function,

$$\bar{\theta} = \frac{p\beta x_0}{1-\beta w+\beta x_0-p\beta x_0} = -\frac{p\beta x_0}{C}.$$

To compare Eq. (A.5) with $\bar{\theta}$, we use via direct calculations the following:

$$\begin{aligned}
&C+\sqrt{C^2+\beta^2p(1-p)x_0^2}>0,\\
&C^2+\beta^2p(1-p)x_0^2 > -C\sqrt{C^2+\beta^2p(1-p)x_0^2},\\
&\beta^2p(1-p)x_0^2 > -C^2-C\sqrt{C^2+\beta^2p(1-p)x_0^2},\ \text{and thus}\\
&-\frac{p\beta x_0}{C} > \frac{C+\sqrt{C^2+\beta^2p(1-p)x_0^2}}{\beta(1-p)x_0}.
\end{aligned}$$

Hence, $\bar{\theta}>\theta^*$. This concludes the proof. □

Proof of Theorem 3.3. The first-order condition $G_1'(y^*,\theta)=0$ yields

$$\frac{w-(1+\theta)(1-p)y^*}{w-x_0+(p+\theta p-\theta)y^*} = \frac{(1+\theta)p}{p+\theta p-\theta},$$

which implies

$$y^* = \frac{x_0(1+\theta)p-w\theta}{(1+\theta)(p+\theta p-\theta)}.$$

Then Problem (3.3) becomes

$$\max_{\theta\in[0,\bar{\theta}]} \tilde{H}(\theta) = \frac{\theta^2(x_0p-w)+\theta x_0 p}{(1+\theta)(p+\theta p-\theta)},$$

where under the logarithmic utility function

$$\bar{\theta} = \frac{x_0 p}{w - x_0 p}.$$

The first-order derivative of $\tilde{H}(\theta)$ is given by

$$\begin{aligned}\tilde{H}'(\theta) =& \frac{\left[2\theta(x_0 p - w) + x_0 p\right](1+\theta)(p+\theta p-\theta) - \left[\theta^2(x_0 p - w) + \theta x_0 p\right]\left[p+\theta p-\theta+(1+\theta)(p-1)\right]}{(1+\theta)^2(p+\theta p-\theta)^2} \\ =& \frac{\theta^2(x_0p^2 - 2wp + w) + \theta(2x_0p^2 - 2wp) + x_0p^2}{(1+\theta)^2(p+\theta p-\theta)^2}.\end{aligned}$$

Let $L(\theta) = \theta^2(x_0p^2 - 2wp + w) + \theta(2x_0p^2 - 2wp) + x_0p^2$. Since $L(\theta)$ is a quadratic function of θ, we have the following three situations.

- If $x_0p^2 - 2wp + w > 0$, then since

  $$\Delta = (2x_0p^2 - 2wp)^2 - 4(x_0p^2 - 2wp + w)x_0p^2 = 4(w^2p^2 - x_0wp^2) > 0,$$

  there are two roots of $L(\theta)$, i.e.

  $$\theta_1 = \frac{wp - x_0p^2 - \sqrt{w^2p^2 - x_0wp^2}}{x_0p^2 - 2wp + w}, \quad \theta_2 = \frac{wp - x_0p^2 + \sqrt{w^2p^2 - x_0wp^2}}{x_0p^2 - 2wp + w}.$$

  Furthermore, since $L(0) = x_0p^2 > 0$, both roots are positive. Hence, $\tilde{H}'(\theta) \geq 0$ for $\theta \in [0, \theta_1]$, $\tilde{H}'(\theta) \leq 0$ for $\theta \in (\theta_1, \theta_2]$, and $\tilde{H}'(\theta) > 0$ for $\theta \in (\theta_2, \infty)$. Moreover, since

  $$x_0(x_0p^2 - 2wp + w) = (w - x_0p + \sqrt{w^2 - wx_0})(w - x_0p - \sqrt{w^2 - wx_0}),$$

  we have

  $$\begin{aligned}\theta_1 &= \frac{x_0p}{w - x_0p + \sqrt{w^2 - wx_0}} < \frac{x_0p}{w - x_0p} = \bar{\theta}, \\ \theta_2 &= \frac{x_0p}{w - x_0p - \sqrt{w^2 - wx_0}} > \frac{x_0p}{w - x_0p} = \bar{\theta}.\end{aligned}$$

  Therefore, $\theta^* = \theta_1$.

- If $x_0p^2 - 2wp + w = 0$, then $L(\theta)$ is decreasing because $2x_0p^2 - 2wp < 0$. Hence, $L(\theta)$ has one positive root $\theta_3 = \frac{x_0p}{2w - 2x_0p}$. Therefore, $\theta_3 = \frac{\bar{\theta}}{2} < \bar{\theta}$. Hence, $\tilde{H}'(\theta) \geq 0$ for $\theta \in [0, \theta_3]$ and $\tilde{H}'(\theta) < 0$ for $\theta \in (\theta_3, \bar{\theta})$. In this situation, $\theta^* = \theta_3$.

- If $x_0p^2 - 2wp + w < 0$, then since $L(0) = x_0p^2 > 0$, $L(\theta)$ only has one positive root, i.e.

  $$\theta_4 = \frac{wp - x_0p^2 - \sqrt{w^2p^2 - x_0wp^2}}{x_0p^2 - 2wp + w}.$$

  Hence, $\tilde{H}'(\theta) \geq 0$ for $\theta \in [0, \theta_4]$ and $\tilde{H}'(\theta) < 0$ for $\theta \in (\theta_4, \infty)$. Similar to the discussion of the first situation, we have $\theta^* = \theta_4$.

Notice that in the second situation, i.e. when $x_0p^2 - 2wp + w = 0$, we get $p = \frac{w-\sqrt{w^2-wx_0}}{x_0}$. Hence, θ_3 can also be written in the form

$$\theta_3 = \frac{x_0p}{2w - 2x_0p} = \frac{x_0p}{w - x_0p + \sqrt{w^2 - wx_0}}.$$

Thus, $\theta_1 = \theta_3 = \theta_4$. This concludes the proof. □

Proof of Proposition 4.1. We first show that a contract (I, P) that solves Problem 3 must make its constraint binding. If not, i.e.

$$\mathbb{E}[u(w - X + I(X) - P)] > L,$$

then since $P \mapsto \mathbb{E}[u(w - X + I(X) - P)]$ is decreasing, there exists a $\tilde{P} > P$ such that

$$\mathbb{E}[u(w - X + I(X) - \tilde{P})] = L.$$

Moreover, we have $\tilde{P} - \mathbb{E}[I(X)] > P - \mathbb{E}[I(X)]$, which contradicts with that (I, P) solves Problem 3.

Next we prove the statement of this proposition. Suppose that a contract (I, P) that solves Problem 3 is Pareto dominated. Then, there exists another contract $(\tilde{I}, \tilde{P})$ such that

$$P - \mathbb{E}[I(X)] \le \tilde{P} - \mathbb{E}[\tilde{I}(X)] \tag{A.6}$$

and

$$\mathbb{E}[u(w - X + I(X) - P)] \le \mathbb{E}[u(w - X + \tilde{I}(X) - \tilde{P})], \tag{A.7}$$

with at least one inequality being strict. If the inequality in (A.6) is strict, then this directly contradicts with that (I, P) solves Problem 3. If the inequality in (A.7) is strict, then similar to the first part of proof, there must exist another $\hat{P} > \tilde{P}$ such that

$$\mathbb{E}[u(w - X + I(X) - P)] = \mathbb{E}[u(w - X + I(X) - \hat{P})]$$

and

$$P - \mathbb{E}[I(X)] \le \tilde{P} - \mathbb{E}[\tilde{I}(X)] < \hat{P} - \mathbb{E}[\tilde{I}(X)],$$

which gives rise to the contradiction again. This ends the proof. □

Proof of Theorem 4.1. We denote by (I^*, θ^*) the Bowley solution. Let $y^* = I^*(x_0)$ and $P^* = (1 + \theta^*)y^*$. Let $L = G(y^*, \theta^*)$, then Problem 3 can be reformulated as

$$\begin{cases} \max\limits_{y \in [0,x_0], \theta \in [0,\infty)} \quad \theta y \\ \qquad\qquad \text{s.t. } G(y, \theta) \ge G(y^*, \theta^*), \end{cases} \tag{A.8}$$

where the constant factor $(1 - p)$ in the objective function is omitted. We show below that there exists a pair $(y^{Par}, \theta^{Par}) \in [0, x_0] \times [0, \infty)$ such that $G(y^{Par}, \theta^{Par}) = G(y^*, \theta^*)$ and $\theta^{Par} y^{Par} > \theta^* y^*$.

First, note that $G(y, \theta^*)$ is strictly concave in y and reaches a maximum at $y^* \in (0, x_0)$. Therefore,

$$G(x_0, \theta^*) < G(y^*, \theta^*), \text{ or, equivalently,}$$
$$u(w - (1 + \theta^*)(1 - p)x_0) < G(y^*, \theta^*).$$

Thus, there exists a $\theta^{Par} < \theta^*$ such that

$$u(w - (1 + \theta^{Par})(1 - p)x_0) = G(x_0, \theta^{Par}) \le G(y^*, \theta^*).$$

Let $\tilde{y}$ be the solution to $G_1'(y, \theta^{Par}) = 0$, then $\tilde{y} \ge y^*$ since the function $y^*(\cdot)$ is assumed to be decreasing. Since $G(\tilde{y}, \theta^{Par}) > G(y^*, \theta^*)$ based on Lemma 3.2, there exists a $y^{Par} \in (\tilde{y}, x_0]$ such that $G(y^{Par}, \theta^{Par}) = G(y^*, \theta^*)$.

Now consider the following two random variables (i.e., the random wealth of the policyholder under the policies (y^{Par}, θ^{Par}) and (y^*, θ^*))

$$Z_1 = \begin{cases} u(w - (1 + \theta^{Par})(1 - p)y^{Par}), & \text{with probability } p, \\ u(w - x_0 + (p + \theta^{Par}p - \theta^{Par})y^{Par}), & \text{with probability } 1 - p, \end{cases}$$

$$Z_2 = \begin{cases} u(w - (1 + \theta^*)(1 - p)y^*), & \text{with probability } p, \\ u(w - x_0 + (p + \theta^*p - \theta^*)y^*), & \text{with probability } 1 - p. \end{cases}$$

Since $G(y^{Par}, \theta^{Par}) = G(y^*, \theta^*)$ (or $\mathbb{E}[Z_1] = \mathbb{E}[Z_2]$), the following inequality must hold:

$$\begin{aligned} &u(w - x_0 + (p + \theta^*p - \theta^*)y^*) < u(w - x_0 + (p + \theta^{Par}p - \theta^{Par})y^{Par}) \\ &\le u(w - (1 + \theta^{Par})(1 - p)y^{Par}) < u(w - (1 + \theta^*)(1 - p)y^*). \end{aligned}$$

Therefore, $F_{Z_1}(z)$ and $F_{Z_2}(z)$ cross only once, i.e. there exists a $z_0 \in \mathbb{R}$ such that $F_{Z_1}(z) \le F_{Z_2}(z)$ for $z \le z_0$ and $F_{Z_1}(z) \ge F_{Z_2}(z)$ for $z \ge z_0$. Since u is strictly concave (which is assumed in Section 2), $u^{-1}(\cdot)$ is a strictly convex function, then analogous to the proof of Lemma 2 of Ohlin (1969), we let $g(z) = u^{-1}(z) - az - b$, where $y = az + b$ is the tangent line of $u^{-1}(z)$ at $z = z_0$. Then, $g(z)$ is also a strictly convex function, which satisfies $g(z) \ge 0$, $g(z_0) = 0$, $g'(z) < 0$ for $z < z_0$ and $g'(z) > 0$ for $z > z_0$. Hence,

$$\begin{aligned} \mathbb{E}[u^{-1}(Z_1)] - \mathbb{E}[u^{-1}(Z_2)] &= \int_{\mathbb{R}} u^{-1}(z) \left\{ dF_{Z_1}(z) - dF_{Z_2}(z) \right\} \\ &= \int_{\mathbb{R}} g(z) \left\{ dF_{Z_1}(z) - dF_{Z_2}(z) \right\} \\ &= -\int_{\mathbb{R}} g'(z)(F_{Z_1}(z) - F_{Z_2}(z))dz, \end{aligned}$$

where the second equation is due to $\mathbb{E}[Z_1] = \mathbb{E}[Z_2]$ and the third equation has been proved in Ohlin (1969). Note that $g'(z)$ and $F_{Z_1}(z) - F_{Z_2}(z)$ are of the same sign, and $g'(z) < 0$ and $F_{Z_1}(z) < F_{Z_2}(z)$ for any $z \in (u(w - x_0 + (p + \theta^*p - \theta^*)y^*), u(w - x_0 + (p + \theta^{Par}p - \theta^{Par})y^{Par}))$. Therefore,

$$-\int_{\mathbb{R}} g'(z)(F_{Z_1}(z) - F_{Z_2}(z))dz < 0,$$

which leads to $\mathbb{E}[u^{-1}(Z_1)] < \mathbb{E}[u^{-1}(Z_2)]$, or equivalently $\theta^{Par}y^{Par} > \theta^* y^*$.[3]

Now, this leads to $P^{Par} - \mathbb{E}[I^{Par}(X)] = \theta^{Par}(1-p)y^{Par} > \theta^*(1-p)y^* = P^* - \mathbb{E}[I^*(X)]$, for $I^{Par}(x_0) := y^{Par}$ and $P^{Par} := (1+\theta^{Par})\mathbb{E}[I^{Par}(X)]$, and moreover, $\mathbb{E}[u(w-X+I^{Par}(X)-P^{Par})] = G(y^{Par}, \theta^{Par}) = G(y^*, \theta^*) = \mathbb{E}[u(w-X+I^*(X)-P^*)]$. This yields a Pareto domination of (I^*, P^*) by (I^{Par}, P^{Par}), and this ends the proof. $\square$

[3] This result is related to Ohlin's lemma (Ohlin, 1969), which implies that $\mathbb{E}[u^{-1}(Z_1)] \leq \mathbb{E}[u^{-1}(Z_2)]$ because u^{-1} is convex. However, we needed to prove a strict inequality when u is strictly concave (and u^{-1} is strictly convex).