BOWLEY VS. PARETO OPTIMA IN REINSURANCE CONTRACTING

TIM J. BOONEN

AMSTERDAM SCHOOL OF ECONOMICS

UNIVERSITY OF AMSTERDAM

ROETERSSTRAAT 11, 1018 WB

AMSTERDAM

THE NETHERLANDS

T.J.BOONEN@UVA.NL

MARIO GHOSSOUB*

UNIVERSITY OF WATERLOO

DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

200 UNIVERSITY AVE. W.

WATERLOO, ON, N2L 3G1

CANADA

MARIO.GHOSSOUB@UWATERLOO.CA

* Corresponding author.

Key Words and Phrases: Risk Management, Optimal Reinsurance, Pareto Optimality, Bowley Optimality, Convex Risk Measures.

JEL Classification: C02, D86, G22.

2010 Mathematics Subject Classification: 91B30.

We are grateful to the Editor and three anonymous reviewers for comments and suggestion. We thank Frank Riedel, Hans Schumacher, Felix Liebrich, as well as audiences at the 24^{th} International Congress on Insurance: Mathematics and Economics, the 2021 Actuarial Research Conference, the 11^{th} Conference in Actuarial Science & Finance on Samos, the 2022 Workshop on Risk Measures and Uncertainty in Insurance at the House of Insurance of the University of Hannover, the Centre Interdisciplinaire en Modélisation Mathématique de l'Université Laval, KU Leuven, and the University of Amsterdam. Mario Ghossoub acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (NSERC Grant No. 2018-03961).

Abstract. The notion of a Bowley optimum has gained recent popularity as an equilibrium concept in problems of risk sharing and optimal reinsurance. In this paper, we examine the relationship between Bowley optimality and Pareto efficiency in a problem of optimal reinsurance, under fairly general preferences. Specifically, we show that Bowley-optimal contracts are indeed Pareto efficient but they make the insurer indifferent with the status quo (hence providing a partial first welfare theorem). Moreover, we show that only those Pareto-efficient contracts that make the insurer indifferent between suffering the loss and entering into the reinsurance contract are Bowley optimal (hence providing a partial second welfare theorem). We interpret these result as indicative of the limitations of Bowley optimality as an equilibrium concept in this literature. We also discuss relationships with competitive equilibria, and we provide illustrative examples.

1. Introduction

In the context of optimal contract design in reinsurance markets, Bowley solutions follow from a sequential procedure: (i) first, the reinsurer selects a pricing kernel, and in response, the insurer will select the indemnity function that minimizes their risk exposure given that pricing kernel; and (ii) second, knowing the insurer's demand as a function of the pricing kernel, the reinsurer then selects the pricing kernel that minimizes their risk exposure. Bowley solutions were first introduced by Bowley (1928) in the context of a bilateral monopoly, and then first applied to optimal reinsurance design by Chan and Gerber (1985). This paper aims to examine micro-equilibrium properties of Bowley solutions in problems of optimal reinsurance contracting. Specifically, we study the relationship between Bowley optimality and Pareto efficiency for a broad class of risk measures, since the latter is the standard notion of optimality typically used in the related literature.

Chan and Gerber (1985) characterize Bowley solutions when the insurer and reinsurer are risk-averse Expected-Utility (EU) maximizers. These results are extended to more general risk exchanges by Taylor (1992). Bowley solutions were then largely ignored in the literature until the recent work of Cheung et al. (2019), who focused on preferences given by distortion risk measures rather than EU-preferences. The work of Cheung et al. (2019) has reignited the interest in the Bowley solution as an optimality concept in the optimal reinsurance literature. Indeed, Li and Young (2021) study Bowley solutions with preferences and premiums given by a mean-variance form, and Boonen et al. (2021) and Boonen and Zhang (2021) study Bowley solutions with asymmetric information about the preferences of the insurer. Chi et al. (2020) construct a sequential game inspired by Bowley solutions, in which the reinsurer determines the premium budget and the insurer optimizes an EU objective under constraints on the first two moments of the indemnity.

Related to Bowley solutions, Stackelberg equilibria have gained popularity in industrial economics. In a Stackelberg equilibrium, two competitive firms compete in setting quantities in a duopoly. As a key difference with Bowley solutions, the two firms both set their quantities, and the policyholders are jointly modelled via an inverse demand function that leads to the price. On the other hand, in a Bowley solution, the monopolistic leader (reinsurer) and follower (insurer) bargain with each other about a reinsurance contract, which consists of an indemnity and a corresponding premium. Subsequently, the monopolistic leader sets the prices and the follower selects the optimal indemnity. Using this terminology, the approaches of Chen and Shen (2018), Gavagan et al. (2022), and Yuan et al. (2022) are closer to a Bowley solution than to a Stackelberg equilibrium.

The present work is closest in spirit to Cheung et al. (2019), but our main objective as well as our class of preferences are different. Our focus is on preferences that are translation-invariant, and sometimes also assumed convex, comonotonic-additive, and/or continuous; while the focus in Cheung et al. (2019) is on the smaller class of convex distortion risk measures (i.e., concave distortion functions). Convex distortion risk measures are known to be coherent (Artzner et al., 1999; Wang et al., 1997); and the minimization of risk measures plays a central role in the formulation of optimal (re)insurance problems (Asimit et al., 2017; Assa et al., 2021; Balbás et al., 2011; Tan et al., 2020; Torraca and Fanzeres, 2021). Moreover, whereas the objective of Cheung et al. (2019) is to construct Bowley solutions explicitly, our primary concern is deriving some key properties of Bowley solutions and examining how they relate to Pareto optima. Specifically, we show that Bowley solutions lead to Pareto-efficient contracts that make the insurer indifferent with the status quo (Theorem 3.1), but we also show the converse: only those Pareto-optimal solutions that make the insurer indifferent with the status quo are Bowley solutions (Theorem 3.2).[1] In other words, Bowley solutions are precisely the solutions that are Pareto optimal and make the insurer indifferent with the status quo. While Pareto optimality is a reasonable and well-accepted efficiency property, the indifference of the insurer with the status quo can be perceived as undesirable. This paper thus also aims to provide a warning about the applicability of Bowley solutions. In fact, if one wishes to design a market mechanism such that the insurer strictly benefits from purchasing reinsurance, Bowley solutions may not suffice.[2] Instead, complete market and comonotone market competitive equilibria (as in Boonen, 2015; Boonen et al., 2021) lead to Pareto optima in which the insurer may strictly benefit from the reinsurance arrangement. Additionally, one could consider the symmetric or asymmetric Nash bargaining solution (see Kalai, 1977).

The rest of this paper is set out as follows. Section 2 presents the model setting, and defines Bowley solutions and Pareto optima. Section 3 establishes a link between Bowley solutions and Pareto optima. In the context of distortion risk measures, this is then compared with two notions of competitive equilibria and the asymmetric Nash bargaining solution in Section 4. Sections 5 and 6 provide examples with distortion risk measures and the Tail Value-at-Risk (TVaR), respectively. Section 7 concludes.

2. Setup

2.1. **Feasible Indemnity Functions.** An insurer faces a random loss X taken to be a bounded nonnegative random variable on a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$, with supremum $M < +\infty$. Let $\mathcal{F} := \sigma\{X\}$ denote the sigma-algebra generated by X on Ω, and let $B(\mathcal{F})$ denote the vector space of all bounded, $\mathbb{R}$-valued, and $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$, with positive cone $B^+(\mathcal{F})$. When endowed with the supnorm $\|.\|_{sup}$, given by $\|Y\|_{sup} := \sup\{|Y(s)| : s \in S\} < +\infty$, $B(\mathcal{F})$ is a Banach space (Dunford and Schwartz, 1958, IV.5.1). By Doob's Measurability Theorem (Aliprantis and Border, 2006, Theorem 4.41), for any $Y \in B(\mathcal{F})$ there exists a bounded, Borel-measurable map $I : \mathbb{R} \to \mathbb{R}$ such that $Y = I \circ X$. Moreover, $Y \in B^+(\mathcal{F})$ if and only if the function I is nonnegative. Let $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of essentially bounded $\mathbb{R}$-valued and $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$. For each $\mathfrak{p} \in [1, +\infty)$, let $L^{\mathfrak{p}}(\Omega, \mathcal{F}, \mathbb{P})$ denote the Banach space of $\mathbb{R}$-valued and $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$ with finite $L^{\mathfrak{p}}$-norm given by $\|Y\|_{\mathfrak{p}} := \left(\int |Y|^{\mathfrak{p}} \, d\mathbb{P}\right)^{1/\mathfrak{p}}$.

Let $C[0, M]$ denote the set of all continuous functions on $[0, M]$ (and hence bounded), equipped with the supnorm $\|\cdot\|_{sup}$, and consider the set

$$\mathcal{I}_0 := \left\{ I : [0, M] \mapsto [0, M] \;\middle|\; I(0) = 0, 0 \leqslant I(x_1) - I(x_2) \leqslant x_1 - x_2, \forall\, x_2 \leqslant x_1 \in [0, M] \right\}. \tag{2.1}$$

[1] For the special case of convex distortion risk measures, Bowley solutions are constructed in Cheung et al. (2019). It is easy to verify that the constructed Bowley solutions are Pareto optimal and make the insurer indifferent with the status quo.

[2] With asymmetric information about the insurer's risk preferences, one could obtain Bowley solutions in which both the insurer and reinsurer strictly benefit (see Boonen et al., 2021; Boonen and Zhang, 2021). This paper however focuses on the case of symmetric information, in which the preferences of the insurer and reinsurer are common knowledge.

Then $\mathcal{I}_0$ is a convex and uniformly bounded subset of $C[0,M]$ consisting of Lipschitz-continuous functions $[0,M] \to [0,M]$, with common Lipschitz constant $K = 1$. Therefore, $\mathcal{I}_0$ is also equicontinuous and hence (supnorm) compact by the Arzelà-Ascoli Theorem (Dunford and Schwartz, 1958, Theorem IV.6.7).

The set $\mathcal{I}_0$ is the collection of indemnity functions that satisfy the so-called *no-sabotage* condition of Carlier and Dana (2003b). If an indemnity is decreasing on some part of the domain of X, then the insurer has an incentive to underreport the loss, and this hence leads to an *ex post* moral hazard issue. Likewise, if the indemnity increases faster than the underlying loss, then the insurer may have an incentive to create an incremental loss. Any stop-loss or proportional contracts, of the form $I(X) = \max(X - d, 0)$ for $d \geqslant 0$ and $I(X) = aX$ for $a \in [0,1]$, respectively, are elements of the set $\mathcal{I}_0$. Any $I \in \mathcal{I}_0$ is such that the mappings $x \mapsto I(x)$ and $x \mapsto x - I(x)$ are nondecreasing. Moreover, for each $I \in \mathcal{I}_0$, $0 \leqslant I(x) \leqslant x$ and $0 \leqslant x - I(x) \leqslant x$, for each $x \in [0,M]$. In particular, $I(x) \in [0,M]$ and $x - I(x) \in [0,M]$, for all $I \in \mathcal{I}_0$ and all $x \in [0,M]$.

The insurer seeks an arrangement with a reinsurer, whereby the insurer pays a premium to purchase a coverage $I(X)$ against X. We assume that the set of *ex ante* admissible indemnity functions is given by a (norm-)closed, and hence compact subset $\mathcal{I}$ of $\mathcal{I}_0$, such that $0 \in \mathcal{I}$, where we use the notation $c \in \mathbb{R}$ to denote the constant function $I(x) = c$, for all $x \in \mathbb{R}$. The assumption that $0 \in \mathcal{I}$ implies that a no-insurance indemnity is always feasible, and thus the status quo can be retained as feasible. Note that

$$\left\{I(X) : I \in \mathcal{I}\right\} \subset B(\mathcal{F}) \subset L^{\mathfrak{p}}(\Omega, \mathcal{F}, \mathbb{P}), \ \forall \, \mathfrak{p} \in (0, +\infty]. \tag{2.2}$$

All throughout this paper, Id denotes the identity function $Id(x) = x$, for all $x \in \mathbb{R}$.

2.2. **Preferences and Pricing Kernels.** Recall that, by the Riesz Representation Theorem (Aliprantis and Border, 2006, Theorems 13.26 & 13.28), for any $\mathfrak{p} \in [1, +\infty)$ and $\mathfrak{q} \in (1, +\infty]$ such that $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$, the norm dual of $L^{\mathfrak{p}}(\Omega, \mathcal{F}, \mathbb{P})$ is (isometrically isomorphic to) $L^{\mathfrak{q}}(\Omega, \mathcal{F}, \mathbb{P})$, via the duality

$$< \phi, \psi > := \int \phi \, \psi \, \mathrm{d}\mathbb{P}.$$

For a given $\mathfrak{q} \in (1, +\infty]$, the reinsurer prices indemnity functions using a premium principle Π, defined as the functional

$$\begin{aligned} \Pi : L^{\mathfrak{q}}(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}_0 &\to \mathbb{R} \\ (\xi, I) \mapsto \Pi(\xi, I) &:= \int I(X) \, \xi \, \mathrm{d}\mathbb{P}, \end{aligned} \tag{2.3}$$

where ξ is interpreted as a pricing kernel. Note that we do not require that $\int \xi \, \mathrm{d}\mathbb{P} = 1$. By eq. (2.2), and without loss of generality, we assume all throughout that $\mathfrak{p} = 1$ and $\mathfrak{q} = +\infty$. By abuse of notation, we also write $\Pi(\xi, Z)$ for any $(\xi, Z) \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \times L^{1}(\Omega, \mathcal{F}, \mathbb{P})$. Note that $\Pi(\cdot, 0) = \Pi(0, \cdot) = 0$, for all $(\xi, I) \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$, by definition of Π.

For a given $I \in \mathcal{I}$ and $\xi \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, the risk exposure of the insurer is given by

$$X - I(X) + \Pi(\xi, I),$$

and the risk exposure of the reinsurer is given by

$$I(X) - \Pi(\xi, I).$$

We assume that the preferences of the insurer and the reinsurer are respectively represented by finite monotone risk measures $\rho^{In} : L^{1}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ and $\rho^{Re} : L^{1}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, normalized so that

$\rho^{In}(c) = \rho^{Re}(c) = c$, for all $c \in \mathbb{R}$. By monotonicity, this normalization then implies that for all $I \in \mathcal{I}$,

$$\rho^{In}\left(I\left(X\right)\right)\left(\leqslant M\right) < \infty,\ \rho^{In}\left(X - I\left(X\right)\right) < \infty,\ \rho^{Re}\left(I\left(X\right)\right) < \infty,\ \text{and}\ \ \rho^{Re}\left(X - I\left(X\right)\right) < \infty.$$

Define the following auxiliary functionals $\rho_1^{In}, \rho_1^{Re} : \mathbb{R} \times \mathcal{I} \to \mathbb{R}$ and $\rho_2^{In}, \rho_2^{Re} : L^{\infty}\left(\Omega, \mathcal{F}, \mathbb{P}\right) \times \mathcal{I} \to \mathbb{R}$:

$$\begin{aligned}
\rho_1^{In}\left(p, I\right) &:= \rho^{In}\left(X - I\left(X\right) + p\right);\\
\rho_2^{In}\left(\xi, I\right) &:= \rho^{In}\left(X - I\left(X\right) + \Pi\left(\xi, I\right)\right);\\
\rho_1^{Re}\left(p, I\right) &:= \rho^{Re}\left(I\left(X\right) - p\right);\\
\rho_2^{Re}\left(\xi, I\right) &:= \rho^{Re}\left(I\left(X\right) - \Pi\left(\xi, I\right)\right).
\end{aligned}$$

These functionals will be useful later in this paper. Here $p \in \mathbb{R}$ is interpreted as a deterministic premium payment, while ξ is the pricing kernel that is used to calculate the deterministic premium of $I(X)$ via the pricing functional $\Pi(\xi, I)$. Both p and $\Pi(\xi, I)$ are thus the reinsurance premium, and this premium is paid by the insurer to the reinsurer. Later, in the definition of Bowley and Pareto optimality, the difference in determining the premium via p or ξ becomes clearer.

We make the assumption that there exists a large enough pricing kernel ξ_0 such that, for each feasible indemnity $I \in \mathcal{I}$, the insurer's risk exposure $X - I\left(X\right) + \Pi\left(\xi_0, I\right)$ is large enough, in the following sense.

Assumption 2.1. *There exists $\xi_0 \in L^{\infty}\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ such that for each $I \in \mathcal{I}$,*

$$\rho_2^{In}\left(\xi_0, I\right) \geqslant \rho^{In}\left(0\right).$$

Note that Assumption 2.1 implies that

$$0 \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\xi_0, I\right).$$

We recall below some basic properties of risk measures.

Definition 2.2. A risk measure $\rho : L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right) \to \mathbb{R}$ is said to be:

- **Translation-invariant** if $\rho\left(Y + c\right) = \rho\left(Y\right) + c$, for all $\left(Y, c\right) \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right) \times \mathbb{R}$.
- **Convex** if for all $Y, Z \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ and all $\alpha \in [0, 1]$,
$$\rho\left(\alpha Y + \left(1 - \alpha\right) Z\right) \leqslant \alpha \rho\left(Y\right) + \left(1 - \alpha\right) \rho\left(Z\right).$$
- **Comonotonic-additive** if $\rho\left(Y + Z\right) = \rho\left(Y\right) + \rho\left(Z\right)$, for all $Y, Z \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ that are comonotonic, that is, such that
$$\left[Y\left(\omega_1\right) - Y\left(\omega_2\right)\right]\left[Z\left(\omega_1\right) - Z\left(\omega_2\right)\right] \geqslant 0,\ \forall\, \omega_1, \omega_2 \in \Omega.$$
- **Continuous** if it is continuous with respect to the L^1-norm topology.

Remark 2.3. Note that:

- A monotone and comonotonic-additive risk measure ρ is positively homogeneous, that is, it satisfies the property $\rho\left(\lambda Y\right) = \lambda\, \rho(Y)$ for all $Y \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ and $\lambda > 0$ (e.g., Schmeidler, 1986, Remark 1).

- A monotone, translation-invariant, and finite convex risk measure is continuous (e.g., Rüschendorf, 2013, Corollary 7.10).

2.3. **Individually Rational Reinsurance Contracts.**

Definition 2.4 (**Individual Rationality).** A pair $(p, I) \in \mathbb{R} \times \mathcal{I}$ is said to satisfy the individual rationality constraints, if both the insurer and the reinsurer prefer the contract (p, I) to the status quo. That is,

$$\rho_1^{In}(p, I) \leqslant \rho_1^{In}(0,0) = \rho^{In}(X) \quad \text{and} \quad \rho_1^{Re}(p, I) \leqslant \rho_1^{Re}(0,0) = \rho^{Re}(0) = 0.$$

Let $\mathcal{IR} \subset \mathbb{R} \times \mathcal{I}$ denote the collection of all contracts that satisfy the individual rationality constraints. Note that $\mathcal{IR} \neq \varnothing$, since $(0,0) \in \mathcal{IR}$. As an immediate consequence of the above definition, we obtain the following result.

Lemma 2.5 (**Nonnegativity of Premia).** *If the risk measures ρ^{In} and ρ^{Re} are translation-invariant, then for any $(p, I) \in \mathcal{IR}$, we have*

$$\rho^{Re}\left(I(X)\right) \leqslant p \leqslant \rho^{In}(X) - \rho^{In}\left(X - I(X)\right).$$

Moreover, $p \geqslant 0$.

Proof. If the risk measures are translation-invariant, then for any $(p, I) \in \mathcal{IR}$, we have

$$\rho_1^{In}(p, I) = \rho^{In}\left(X - I(X) + p\right) = \rho^{In}\left(X - I(X)\right) + p \leqslant \rho_1^{In}(0,0) = \rho^{In}(X);$$
$$\rho_1^{Re}(p, I) = \rho^{Re}\left(I(X) - p\right) = \rho^{Re}\left(I(X)\right) - p \leqslant \rho_1^{Re}(0,0) = \rho^{Re}(0) = 0.$$

Therefore,

$$\rho^{Re}\left(I(X)\right) = \rho^{Re}\left(I(X)\right) - \rho^{Re}(0) \leqslant p \leqslant \rho^{In}(X) - \rho^{In}\left(X - I(X)\right).$$

Moreover, since the risk measure ρ^{Re} is monotone by assumption, it follows that $\rho^{Re}\left(I(X)\right) \geqslant \rho^{Re}(0)$, and hence $p \geqslant 0$. ☐

Remark 2.6. An immediate implication of Lemma 2.5 is that for monotone and translation-invariant risk measures, premia are nonnegative for any individually rational reinsurance contract. If, in Lemma 2.5 the stronger assumption of comonotonic-additivity is imposed on ρ^{In} instead of translation invariance, then for any $(p, I) \in \mathcal{IR}$, we have (by monotonicity and normalization)

$$0 \leqslant \rho^{Re}\left(I(X)\right) \leqslant p \leqslant \rho^{In}(X) - \rho^{In}\left(X - I(X)\right) = \rho^{In}\left(I(X)\right) \leqslant \rho^{In}(X) \leqslant M.$$

In particular, $p \in \left[\rho^{Re}\left(I(X)\right), \rho^{In}\left(I(X)\right)\right] \subseteq [0, M]$.

2.4. **Optima.** Two economic concepts that we focus on in this paper are defined next.

Definition 2.7.

(1) A pair $(p^*, I^*) \in \mathcal{IR}$ is said to be **Pareto-Optimal (PO)** if there is no other pair $\left(\tilde{p}, \tilde{I}\right) \in \mathcal{IR}$ such that

$$\rho_1^{In}\left(\tilde{p}, \tilde{I}\right) \leqslant \rho_1^{In}(p^*, I^*) \quad \text{and} \quad \rho_1^{Re}\left(\tilde{p}, \tilde{I}\right) \leqslant \rho_1^{Re}(p^*, I^*),$$

with at least one strict inequality. We denote by $\mathcal{PO}$ the set of all PO.

(2) A pair $(\xi^*, I^*) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$ is said to be **Bowley-Optimal (BO)** if

(a) $I^* \in \arg\min_{I \in \mathcal{I}} \rho_2^{In}(\xi^*, I)$;

(b) $\rho_2^{Re}(\xi^*, I^*) \leqslant \rho_2^{Re}\left(\widetilde{\xi}, \widetilde{I}\right)$ for all $\widetilde{\xi} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $\widetilde{I} \in \arg\min_{I \in \mathcal{I}} \rho_2^{In}\left(\widetilde{\xi}, I\right)$.

We denote by $\mathcal{BO}$ the set of all BO.

The following is related to a standard result for translation-invariant risk measures (e.g., Asimit and Boonen, 2018, Theorem 3.1), and we provide a self-contained proof for completeness.

Lemma 2.8 (**Pareto-Optimality**). *Suppose that ρ^{In} and ρ^{Re} are translation-invariant. A pair $(p^*, I^*) \in \mathcal{IR}$ is PO if and only if it is optimal for the following optimization problem:*

$$\min_{(p,I) \in \mathcal{IR}} \left\{\rho_1^{In}(p, I) + \rho_1^{Re}(p, I)\right\}. \tag{2.4}$$

Proof. Suppose first that (p^*, I^*) is optimal for Problem (2.4) but not PO. Then there exists some $\left(\widetilde{p}, \widetilde{I}\right) \in \mathcal{IR}$ such that

$$\rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{In}(p^*, I^*) \text{ and } \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{Re}(p^*, I^*),$$

with at least one strict inequality. Therefore,

$$\rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) + \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) < \rho_1^{In}(p^*, I^*) + \rho_1^{Re}(p^*, I^*),$$

contradicting the optimality of (p^*, I^*) for Problem (2.4).

Conversely, suppose that (p^*, I^*) is PO but not optimal for Problem (2.4). Then there exists some $\left(\widetilde{p}, \widetilde{I}\right) \in \mathcal{IR}$ such that

$$\rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) + \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) < \rho_1^{In}(p^*, I^*) + \rho_1^{Re}(p^*, I^*).$$

Let $\varepsilon := \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) - \rho_1^{Re}(p^*, I^*)$, and $\hat{p} := \widetilde{p} + \varepsilon$. Then, by translation-invariance of the risk measure ρ^{Re}, it follows that

$$\begin{aligned}
\rho_1^{Re}\left(\hat{p}, \widetilde{I}\right) &= \rho^{Re}\left(I(X) - \widetilde{p} - \varepsilon\right) = \rho^{Re}\left(I(X) - \widetilde{p}\right) - \varepsilon \\
&= \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) - \varepsilon = \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) - \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) + \rho_1^{Re}(p^*, I^*) \\
&= \rho_1^{Re}(p^*, I^*).
\end{aligned}$$

Moreover, by translation-invariance of the risk measure ρ^{In}, we obtain

$$\begin{aligned}
\rho_1^{In}\left(\hat{p}, \widetilde{I}\right) &= \rho^{In}\left(X - I(X) + \widetilde{p} + \varepsilon\right) = \rho^{In}\left(X - I(X) + \widetilde{p}\right) + \varepsilon \\
&= \rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) + \varepsilon = \rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) + \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) - \rho_1^{Re}(p^*, I^*) \\
&< \rho_1^{In}(p^*, I^*),
\end{aligned}$$

hence contradicting the fact that (p^*, I^*) is PO. □

Lemma 2.9. *Suppose that ρ^{In} and ρ^{Re} are translation-invariant, and consider the following optimization problem:*

$$\min_{I \in \mathcal{I}} \left\{ \rho_1^{In}(0, I) + \rho_1^{Re}(0, I) : (p, I) \in \mathcal{IR}, \text{ for some } p \in \mathbb{R} \right\}. \tag{2.5}$$

Then I^* is optimal for Problem (2.5) if and only if (p^*, I^*) is optimal for Problem (2.4), for some $p^* \in \mathbb{R}$.

Proof. Suppose that (p^*, I^*) is optimal for Problem (2.4). Then I^* is feasible for Problem (2.5), i.e., $I^* \in \mathcal{I}$. Let I be any other feasible solution for Problem (2.5). Then there exists $p \in \mathbb{R}$ such that $(p, I) \in \mathcal{IR}$, i.e., (p, I) is feasible for Problem (2.4). Consequently,

$$\rho_1^{In}\left(p^*, I^*\right) + \rho_1^{Re}\left(p^*, I^*\right) \leqslant \rho_1^{In}(p, I) + \rho_1^{Re}(p, I).$$

By translation-invariance, this leads to

$$\rho_1^{In}\left(0, I^*\right) + \rho_1^{Re}\left(0, I^*\right) \leqslant \rho_1^{In}\left(0, I^*\right) + \rho_1^{Re}\left(0, I^*\right),$$

and hence I^* is optimal for Problem (2.5).

Conversely, suppose that I^* is optimal for Problem (2.5). Then there exists some $p^* \in \mathbb{R}$ such that (p^*, I^*) is feasible for Problem (2.4). Let $(p, I) \in \mathcal{IR}$ be another feasible solution for Problem (2.4). Then I is feasible for Problem (2.5), and therefore

$$\rho_1^{In}\left(0, I^*\right) + \rho_1^{Re}\left(0, I^*\right) \leqslant \rho_1^{In}(0, I) + \rho_1^{Re}(0, I).$$

By translation-invariance, this leads to

$$\rho_1^{In}\left(p^*, I^*\right) + \rho_1^{Re}\left(p^*, I^*\right) \leqslant \rho_1^{In}(p, I) + \rho_1^{Re}(p, I),$$

and hence (p^*, I^*) is optimal for Problem (2.4). □

The following result provides sufficient conditions for the existence of Pareto and Bowley optima. Its proof is provided in Appendix A.

Theorem 2.10 (Existence of PO and BO)**.**

(1) *If ρ^{In} and ρ^{Re} are continuous and translation-invariant, then $\mathcal{PO} \neq \varnothing$.*

(2) *If, in addition, ρ^{In} is comonotonic-additive and convex, then $\mathcal{BO} \neq \varnothing$.*

2.5. **Pricing Kernels as Subgradients.** Recall that the norm dual of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is (isometrically isomorphic to) $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Using this standard duality, a subgradient of the risk measure ρ^{In} at a given $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is some $\xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\rho^{In}(Z) \geqslant \rho^{In}(Y) + E\left[\xi(Z - Y)\right], \ \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

The subdifferential of ρ^{In} at a given $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, denoted by $\partial\rho^{In}(Y)$, is the collection of all subgradients of ρ^{In} at Y:

$$\partial\rho^{In}(Y) := \left\{ \xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \;\middle|\; \rho^{In}(Z) \geqslant \rho^{In}(Y) + E\left[\xi(Z - Y)\right], \ \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \right\}. \tag{2.6}$$

Hence, for a given $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\xi \in \partial\rho^{In}(Y)$ if and only if

$$\rho^{In}(Z) - \Pi(\xi, Z) \geqslant \rho^{In}(Y) - \Pi(\xi, Y), \ \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.7}$$

The subdifferential of ρ^{Re} at a given $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is defined similarly. We refer to Ruszczyński and Shapiro (2006a,b), and Wozabal (2014) for more information on subdifferentials of risk measures.

Hereafter, we consider collection $\mathcal{Y} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ defined as follows:

$$\mathcal{Y} := \left\{ Y = I(X) \;\middle|\; I \in \mathcal{I} \right\}. \tag{2.8}$$

If a risk measure $\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is convex, continuous, and finite on the set $\mathcal{Y}$, then it follows from Svindland (2009b, Lemma 2.2) or Ekeland and Témam (1999, Prop. 5.2, Chap. 1) that $\partial\rho(Y) \neq \varnothing$ for all $Y \in \mathcal{Y}$. Note that, as stated earlier, ρ^{In} and ρ^{Re} are finite on $\mathcal{Y}$.

Remark 2.11. Suppose ρ^{Re} is translation-invariant and convex, and choose $\widetilde{\xi}^{Re} \in \partial\rho^{Re}\big(I(X)\big) \neq \varnothing$. Then, by translation-invariance of ρ^{Re},

$$\arg\min_{I \in \mathcal{I}} \rho_2^{Re}\left(\widetilde{\xi}^{Re}, I\right) = \arg\min_{I \in \mathcal{I}} \rho^{Re}\left(I(X) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right) = \arg\min_{I \in \mathcal{I}} \left[\rho^{Re}\big(I(X)\big) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right].$$

Now, by definition of $\partial\rho^{Re}\big(I(X)\big)$, we obtain that for all $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho^{Re}(Z) - \Pi\left(\widetilde{\xi}^{Re}, Z\right) \geqslant \rho^{Re}\big(I(X)\big) - \Pi\left(\widetilde{\xi}^{Re}, I(X)\right).$$

Hence, for every $I \in \mathcal{I}$, there exist $\widetilde{\xi}^{Re} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$I \in \arg\min \left[\rho^{Re}(I(X)) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right] = \arg\min\, \rho_2^{Re}\left(\widetilde{\xi}^{Re}, I\right).$$

Lemma 2.12. *If ρ^{In} is translation-invariant and convex, then for all $I^* \in \mathcal{I}$,*

$$I^* \in \arg\min_{I \in \mathcal{I}} \rho_2^{In}(\xi, I), \;\; \forall\, \xi \in \partial\rho^{In}\big(X - I^*(X)\big).$$

Similarly, if ρ^{Re} is translation-invariant and convex, then for all $I^* \in \mathcal{I}$,

$$I^* \in \arg\min_{I \in \mathcal{I}} \rho_2^{Re}(\xi, I), \;\; \forall\, \xi \in \partial\rho^{Re}\big(I^*(X)\big).$$

Proof. First, note that by Remark 2.3, both ρ^{In} and ρ^{Re} are continuous. Therefore, $\partial\rho^{In}(Y) \neq \varnothing$ and $\partial\rho^{Re}(Y) \neq \varnothing$ for all $Y \in \mathcal{Y}$, by convexity and continuity. By translation-invariance of ρ^{In}, it follows that for every $(\xi, I) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$,

$$\rho_2^{In}(\xi, I) = \rho^{In}\big(X - I(X) + \Pi(\xi, I)\big) = \rho^{In}\big(X - I(X)\big) + \Pi(\xi, X) - \Pi\big(\xi, X - I(X)\big).$$

Hence,

$$\begin{aligned}
\arg\min_{I \in \mathcal{I}} \rho_2^{In}(\xi, I) &= \arg\min_{I \in \mathcal{I}} \left[\rho^{In}\big(X - I(X)\big) + \Pi(\xi, X) - \Pi\big(\xi, X - I(X)\big)\right] \\
&= \arg\min_{I \in \mathcal{I}} \left[\rho^{In}\big(X - I(X)\big) - \Pi\big(\xi, X - I(X)\big)\right].
\end{aligned}$$

Now, fix $I^* \in \mathcal{I}$ and choose $\widetilde{\xi}^{In} \in \partial\rho^{In}\big(X - I^*(X)\big) \neq \varnothing$. Then, from eq. (2.7) it follows that for all $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho^{In}(Z) - \Pi\left(\widetilde{\xi}^{In}, Z\right) \geqslant \rho^{In}\big(X - I^*(X)\big) - \Pi\left(\widetilde{\xi}^{In}, X - I^*(X)\right).$$

Therefore,

$$I^* \in \arg\min \left[\rho^{In}(X - I(X)) - \Pi\left(\widetilde{\xi}^{In}, X - I(X)\right)\right] = \arg\min_{I \in \mathcal{I}} \rho_2^{In}\left(\widetilde{\xi}^{In}, I\right).$$

Similarly, fix $I^* \in \mathcal{I}$ and choose $\widetilde{\xi}^{Re} \in \partial\rho^{Re}\left(I^*(X)\right) \neq \varnothing$. Then by translation-invariance of ρ^{Re},

$$\arg\min_{I \in \mathcal{I}} \rho_2^{Re}\left(\widetilde{\xi}^{Re}, I\right) = \arg\min_{I \in \mathcal{I}} \rho^{Re}\left(I\left(X\right) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right) = \arg\min_{I \in \mathcal{I}} \left[\rho^{Re}\left(I\left(X\right)\right) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right].$$

Now, by definition of $\partial\rho^{Re}\left(I(X)\right)$, we obtain that for all $Z \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$,

$$\rho^{Re}\left(Z\right) - \Pi\left(\widetilde{\xi}^{Re}, Z\right) \geqslant \rho^{Re}\left(I^*(X)\right) - \Pi\left(\widetilde{\xi}^{Re}, I^*(X)\right).$$

Hence,

$$I \in \arg\min_{I \in \mathcal{I}} \left[\rho^{Re}(I(X)) - \Pi\left(\widetilde{\xi}^{Re}, I\right)\right] = \arg\min_{I \in \mathcal{I}} \rho_2^{Re}\left(\widetilde{\xi}^{Re}, I\right).$$

□

Lemma 2.13. *If ρ^{In} is comonotonic-additive and convex, then for each $I \in \mathcal{I}$,*

(1) $\Pi\left(\xi, X - I(X)\right) = \rho^{In}\left(X - I(X)\right)$, *for all* $\xi \in \partial\rho^{In}\left(X - I(X)\right)$;

(2) $\Pi\left(\xi, I(X)\right) = \rho^{In}\left(I(X)\right)$, *for all* $\xi \in \partial\rho^{In}\left(I(X)\right)$.

Proof. First, note that by Remark 2.3, ρ^{In} is continuous since comonotonic-additivity implies translation-invariance, by the normalization $\rho^{In}(c) = c$, for $c \in \mathbb{R}$. We only show (1), as the proof of (2) is almost identical. The subdifferential $\partial\rho^{In}(X - I(X))$ is non-empty by convexity and continuity of ρ^{In}. Moreover, $\rho^{In}(0) = 0$, by comonotonic-additivity of ρ^{In}. Fix $\xi \in \partial\rho^{In}\left(X - I(X)\right)$ and suppose first, by way of contradiction, that $\Pi\left(\xi, X - I(X)\right) < \rho^{In}\left(X - I(X)\right)$. Then eq. (2.7) implies that for all $Z \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$,

$$\rho^{In}(Z) - \Pi\left(\xi, Z\right) \geqslant \rho^{In}(X - I(X)) - \Pi\left(\xi, X - I(X)\right) > 0.$$

Hence, for $Z = 0$, $\rho^{In}(Z) - \Pi\left(\xi, Z\right) = \rho^{In}(0) - \Pi\left(\xi, 0\right) = 0 > 0$, a contradiction.

Next, suppose that $\Pi\left(\xi, X - I(X)\right) > \rho^{In}\left(X - I(X)\right)$. Then by comonotonic-additivity of ρ^{In}, it follows that for each $n \in \mathbb{N}$,

$$\rho^{In}\left(n(X - I(X))\right) - \Pi\left(\xi, n\left(X - I(X)\right)\right) = n\left[\rho^{In}\left(X - I(X)\right) - \Pi\left(\xi, X - I(X)\right)\right].$$

In particular, letting $Z = 2\left(X - I(X)\right)$ yields

$$\rho^{In}(Z) - \Pi\left(\xi, Z\right) = 2\left[\rho^{In}(X - I(X)) - \Pi\left(\xi, X - I(X)\right)\right] < \rho^{In}(X - I(X)) - \Pi\left(\xi, X - I(X)\right),$$

which contradicts the fact that for all $Z \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)$,

$$\rho^{In}(Z) - \Pi\left(\xi, Z\right) \geqslant \rho^{In}(X - I(X)) - \Pi\left(\xi, X - I(X)\right),$$

by eq. (2.7). Therefore, $\Pi\left(\xi, X - I(X)\right) = \rho^{In}\left(X - I(X)\right)$. □

Remark 2.14. Fix $I \in \mathcal{I}$. By Lemma 2.13 it follows that if ρ^{In} is comonotonic-additive and convex, and if $\xi \in \partial\rho^{In}\left(X - I(X)\right)$, then

$$\Pi\left(\xi, X\right) - \Pi\left(\xi, I(X)\right) = \rho^{In}\left(X\right) - \rho^{In}\left(I(X)\right).$$

If, moreover, $\xi \in \partial\rho^{In}\left(I(X)\right)$, then $\Pi\left(\xi, I(X)\right) = \rho^{In}\left(I(X)\right)$. Consequently,

$$\Pi\left(\xi, X\right) = \rho^{In}(X) \;\text{ and }\; \Pi\left(\xi, I(X)\right) = \rho^{In}\left(I(X)\right), \;\forall\, \xi \in \partial\rho^{In}\left(X - I(X)\right) \cap \partial\rho^{In}\left(I(X)\right). \tag{2.9}$$

Lemma 2.15. *If ρ^{In} is comonotonic-additive and convex, then for all $I \in \mathcal{I}$,*

$$\varnothing \neq \partial\rho^{In}(X) \subset \partial\rho^{In}(I(X)) \cap \partial\rho^{In}(X - I(X)),$$

and hence $\partial\rho^{In}(I(X)) \cap \partial\rho^{In}(X - I(X)) \neq \varnothing$.

Proof. First, note that by Remark 2.3, ρ^{In} is continuous since comonotonic-additivity implies translation-invariance, by the normalization $\rho^{In}(c) = c$, for $c \in \mathbb{R}$. Thus, $\partial\rho^{In}(X) \neq \varnothing$ by convexity and continuity of ρ^{In}. Let $\xi \in \partial\rho^{In}(X)$. Then, by eq. (2.7),

$$\rho^{In}(Z) - \Pi(\xi, Z) \geqslant \rho^{In}(X) - \Pi(\xi, X), \ \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.10}$$

Moreover, by Lemma 2.13, $\rho^{In}(X) - \Pi(\xi, X) = 0$. Hence, for all $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho^{In}(Z) - \Pi(\xi, Z) \geqslant 0.$$

Furthermore, taking $Z = X - I(X)$, eq. (2.10) implies that

$$\rho^{In}\big(X - I(X)\big) - \Pi\big(\xi, X - I(X)\big) \geqslant 0.$$

Hence, since ρ^{In} is comonotonic-additive, it follows that for all $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\begin{aligned}\rho^{In}(Z) - \Pi(\xi, Z) &\geqslant \rho^{In}(X) - \Pi(\xi, X)\\ &= \rho^{In}\big(I(X)\big) - \Pi\big(\xi, I(X)\big) + \rho^{In}\big(X - I(X)\big) - \Pi\big(\xi, X - I(X)\big)\\ &\geqslant \rho^{In}\big(I(X)\big) - \Pi\big(\xi, I(X)\big),\end{aligned}$$

which implies that $\xi \in \partial\rho^{In}(I(X))$. By a similar proof, one can show that $\xi \in \partial\rho^{In}(X - I(X))$, and thus $\xi \in \partial\rho^{In}(I(X)) \cap \partial\rho^{In}(X - I(X)) \neq \varnothing$. ∎

3. Pareto Vs. Bowley Optima

In this section, we will link PO and BO. First, we show in the next theorem that every BO is associated with a PO.

Theorem 3.1. *Suppose that ρ^{In} is comonotonic-additive and convex, and that ρ^{Re} is translation-invariant. If (ξ^*, I^*) is BO, then $\big(\Pi(\xi^*, I^*), I^*\big)$ is PO and $\rho_2^{In}(\xi^*, I^*) = \rho_2^{In}(\xi^*, 0)$.*

Proof. First, note that by Remark 2.3, ρ^{In} is continuous since comonotonic-additivity implies translation-invariance, by the normalization $\rho^{In}(c) = c$, for $c \in \mathbb{R}$. Suppose that $(\xi^*, I^*) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$ is BO, and let $p^* := \Pi(\xi^*, I^*)$. Then

$$\begin{aligned}&\rho_2^{In}\big(\xi^*, I^*\big) \leqslant \rho_2^{In}\big(\xi^*, I\big), \ \text{for all } I \in \mathcal{I};\\ &\rho_2^{Re}\big(\xi^*, I^*\big) \leqslant \rho_2^{Re}(\xi, I), \ \text{for all } \xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \ \text{and } I \in \arg\min_{\widetilde{I} \in \mathcal{I}} \rho_2^{In}\Big(\xi, \widetilde{I}\Big).\end{aligned}$$

We first show that $(p^*, I^*) \in \mathcal{IR}$. Note that since $\rho_2^{In}(\xi^*, I^*) \leqslant \rho_2^{In}(\xi^*, I), \forall I \in \mathcal{I}$, we have

$$\rho_1^{In}\big(p^*, I^*\big) = \rho_2^{In}\big(\xi^*, I^*\big) \leqslant \rho_2^{In}\big(\xi^*, 0\big) = \rho_1^{In}\Big(\Pi\big(\xi^*, 0\big), 0\Big) = \rho_1^{In}(0, 0).$$

Moreover, since $\rho_2^{Re}(\xi^*, I^*) \leqslant \rho_2^{Re}(\xi, I)$, for all $\xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $I \in \arg\min_{\widetilde{I} \in \mathcal{I}} \rho_2^{In}\Big(\xi, \widetilde{I}\Big)$, we have

$$\rho_1^{Re}\big(p^*, I^*\big) = \rho_2^{Re}\big(\xi^*, I^*\big) \leqslant \rho_2^{Re}(\xi_0, 0) = \rho^{Re}(0) = \rho_1^{Re}(0, 0),$$

where ξ_0 is as in Assumption 2.1, since $0 \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}(\xi_0, I)$. Therefore, $(p^*, I^*) \in \mathcal{IR}$.

Now, suppose, by way of contradiction, that (p^*, I^*) is not PO. Then there exists some $\left(\widetilde{p}, \widetilde{I}\right) \in \mathcal{IR}$ with $\widetilde{I} \in \mathcal{I}$ such that

$$\rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{In}(p^*, I^*) = \rho_2^{In}(\xi^*, I^*);$$
$$\rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{Re}(p^*, I^*) = \rho_2^{Re}(\xi^*, I^*),$$

with at least one strict inequality. Let

$$\delta = \rho_1^{In}(0,0) - \rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) \geqslant 0 \quad \text{and} \quad \bar{p} = \widetilde{p} + \delta.$$

Then, by translation-invariance, we obtain

$$\rho_1^{In}\left(\bar{p}, \widetilde{I}\right) = \rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) + \delta = \rho_1^{In}(0,0) = \rho^{In}(X);$$
$$\rho_1^{Re}\left(\bar{p}, \widetilde{I}\right) = \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) - \delta \leqslant \rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{Re}(p^*, I^*),$$

where one of the two inequalities is strict. Moreover, since $\rho_1^{Re}(p^*, I^*) \leqslant \rho_1^{Re}(0,0)$ (by individual rationality), it follows that $\rho_1^{Re}\left(\bar{p}, \widetilde{I}\right) < \rho_1^{Re}(0,0)$, and hence $\bar{p} > \rho^{Re}(\widetilde{I}(X)) - \rho^{Re}(0)$.

Next, fix $\widetilde{\xi} \in \partial\rho^{In}(X - \widetilde{I}(X)) \cap \partial\rho^{In}(\widetilde{I}(X))$, where $\partial\rho^{In}(X - \widetilde{I}(X)) \cap \partial\rho^{In}(\widetilde{I}(X)) \neq \varnothing$ by Lemma 2.15. Then, by Lemma 2.12, $\widetilde{I} \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\widetilde{\xi}, I\right)$, and by Lemma 2.13, $\Pi\left(\widetilde{\xi}, \widetilde{I}\right) = \rho^{In}\left(\widetilde{I}(X)\right)$. Hence,

$$\begin{aligned}
\rho_2^{In}\left(\widetilde{\xi}, \widetilde{I}\right) &= \rho^{In}\left(X - \widetilde{I}(X) + \Pi\left(\widetilde{\xi}, \widetilde{I}\right)\right) = \rho^{In}(X) - \rho^{In}\left(\widetilde{I}(X)\right) + \Pi\left(\widetilde{\xi}, \widetilde{I}\right) \\
&= \rho^{In}(X) - \rho^{In}\left(\widetilde{I}(X)\right) + \rho^{In}\left(\widetilde{I}(X)\right) = \rho^{In}(X) = \rho_1^{In}(0,0) \\
&= \rho_1^{In}\left(\bar{p}, \widetilde{I}\right),
\end{aligned}$$

and so $\Pi\left(\widetilde{\xi}, \widetilde{I}\right) = \bar{p}$. Thus,

$$\rho_2^{Re}\left(\widetilde{\xi}, \widetilde{I}\right) = \rho_1^{Re}\left(\bar{p}, \widetilde{I}\right) < \rho_1^{Re}(p^*, I^*) = \rho_2^{Re}(\xi^*, I^*).$$

Consequently,

$$\widetilde{I} \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\widetilde{\xi}, I\right) \quad \text{and} \quad \rho_2^{Re}\left(\widetilde{\xi}, \widetilde{I}\right) < \rho_2^{Re}(\xi^*, I^*),$$

contradicting the fact that (ξ^*, I^*) is BO. Hence, (ξ^*, I^*) is PO.

Now, suppose that (ξ^*, I^*) is BO but $\rho_2^{In}(\xi^*, I^*) \neq \rho_2^{In}(\xi^*, 0)$. If $\rho_2^{In}(\xi^*, I^*) > \rho_2^{In}(\xi^*, 0)$, then $\left(\Pi(\xi^*, I^*), I^*\right) \notin \mathcal{IR}$ and hence (ξ^*, I^*) is not BO (as per the first part of this proof), a contradiction. If $\rho_2^{In}(\xi^*, I^*) < \rho_2^{In}(\xi^*, 0)$, then by Lemmata 2.12 and 2.15 choose $\hat{\xi} \in \partial\rho^{In}(X - I^*(X)) \cap \partial\rho^{In}(I^*(X))$, so that $I^* \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\hat{\xi}, I\right)$, and hence

$$I^* \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}(\xi^*, I) \cap \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\hat{\xi}, I\right).$$

By Lemma 2.13-(2), it follows that $\Pi\left(\hat{\xi}, I^*(X)\right) = \rho^{In}\left(I^*(X)\right)$, and therefore

$$\begin{aligned}\rho_2^{In}\left(\hat{\xi}, I^*\right) &= \rho^{In}\left(X - I^*(X) + \Pi\left(\hat{\xi}, I^*\right)\right) = \rho^{In}\left(X - I^*(X)\right) + \Pi\left(\hat{\xi}, I^*\right) \\ &= \rho^{In}(X) - \rho^{In}\left(I^*(X)\right) + \Pi\left(\hat{\xi}, I^*\right) = \rho^{In}(X) = \rho_2^{In}\left(\hat{\xi}, 0\right).\end{aligned}$$

Consequently

$$\rho_2^{In}\left(\xi^*, I^*\right) < \rho_2^{In}\left(\xi^*, 0\right) = \rho_2^{In}\left(\hat{\xi}, 0\right) = \rho^{In}(X) = \rho_2^{In}\left(\hat{\xi}, I^*\right),$$

which implies that $\Pi(\xi^*, I^*) < \Pi(\hat{\xi}, I^*)$, and hence

$$\rho_2^{Re}\left(\xi^*, I^*\right) = \rho^{Re}(I^*(X) - \Pi(\xi^*, I^*)) > \rho^{Re}(I^*(X) - \Pi(\hat{\xi}, I^*)) = \rho_2^{Re}\left(\hat{\xi}, I^*\right).$$

This, together with $I^* \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}\left(\hat{\xi}, I\right)$, implies a contradiction with (ξ^*, I^*) being BO. Consequently, $\rho_2^{In}\left(\xi^*, I^*\right) = \rho_2^{In}\left(\xi^*, 0\right)$. □

Theorem 3.1 states that if (ξ^*, I^*) is BO, then it must hold that $\rho_1^{In}(\Pi(\xi, I^*), I^*) = \rho_1^{In}(0, 0)$; that is, the insurer is indifferent with the status quo. Therefore, BO leads to very peculiar arrangements that are not marketable. The following result provides a partial converse to Theorem 3.1, but PO is then restricted to only those reinsurance contracts that make the insurer indifferent.

Theorem 3.2. *Suppose that ρ^{In} is comonotonic-additive and convex, and that ρ^{Re} is translation-invariant. If (p^*, I^*) is PO and such that $\rho_1^{In}(p^*, I^*) = \rho_1^{In}(0, 0)$, then there exists some $\xi^* \in \partial\rho^{In}\left(I^*(X)\right) \cap \partial\rho^{In}\left(X - I^*(X)\right) \neq \varnothing$ such that (ξ^*, I^*) is BO and $p^* = \Pi(\xi^*, I^*)$.*

Proof. First, note that by Remark 2.3, ρ^{In} is continuous since comonotonic-additivity implies translation-invariance, by the normalization $\rho^{In}(c) = c$, for $c \in \mathbb{R}$. Let (p^*, I^*) be PO and such that $\rho_1^{In}(p^*, I^*) = \rho_1^{In}(0, 0)$. By Lemmata 2.12 and 2.15, choose $\xi^* \in \partial\rho^{In}\left(I^*(X)\right) \cap \partial\rho^{In}\left(X - I^*(X)\right) \neq \varnothing$, so that $I^* \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}(\xi^*, I)$. Then, as in Remark 2.14, $p^* = \Pi(\xi^*, I^*) = \rho^{In}\left(I^*(X)\right)$.

Suppose that (ξ^*, I^*) is not BO. Then, there exists $\left(\widetilde{\xi}, \widetilde{I}\right) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$ such that

$$\begin{aligned}&\rho_2^{Re}\left(\widetilde{\xi}, \widetilde{I}\right) < \rho_2^{Re}\left(\xi^*, I^*\right) = \rho_1^{Re}\left(p^*, I^*\right); \\ &\rho_2^{In}\left(\widetilde{\xi}, \widetilde{I}\right) \leqslant \rho_2^{In}\left(\widetilde{\xi}, I\right), \text{ for all } I \in \mathcal{I}.\end{aligned}$$

Letting $\widetilde{p} := \Pi\left(\widetilde{\xi}, \widetilde{I}\right)$, and recalling that $(p^*, I^*) \in \mathcal{IR}$ since (p^*, I^*) is PO, the first inequality above gives

$$\rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) < \rho_1^{Re}\left(p^*, I^*\right) \leqslant \rho_1^{Re}(0, 0) = \rho^{Re}(0) = 0.$$

Moreover, since $\rho_1^{In}(p^*, I^*) = \rho_1^{In}(0, 0)$, the second inequality implies that

$$\rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) = \rho_2^{In}\left(\widetilde{\xi}, \widetilde{I}\right) \leqslant \rho_2^{In}\left(\widetilde{\xi}, 0\right) = \rho_1^{In}(0, 0) = \rho_1^{In}\left(p^*, I^*\right).$$

Consequently, $\left(\Pi\left(\widetilde{\xi}, \widetilde{I}\right), \widetilde{I}\right) \in \mathcal{IR}$ and is such that

$$\rho_1^{Re}\left(\widetilde{p}, \widetilde{I}\right) < \rho_1^{Re}\left(p^*, I^*\right) \quad \text{and} \quad \rho_1^{In}\left(\widetilde{p}, \widetilde{I}\right) \leqslant \rho_1^{In}\left(p^*, I^*\right),$$

contradicting the fact that (p^*, I^*) is PO. Hence, (ξ^*, I^*) is BO. □

In summary, when ρ^{In} is comonotonic-additive and convex, and ρ^{Re} is translation-invariant, every BO leads to a PO reinsurance contract in which the insurer is indifferent between reinsuring and not reinsuring (Theorem 3.1). Moreover, every PO in which the insurer is indifferent between reinsuring and not reinsuring can constitute a BO in which (by Lemmata 2.12 and 2.13) the reinsurer can select any price in $\partial\rho^{In}\left(I^*(X)\right) \cap \partial\rho^{In}\left(X - I^*(X)\right)$, which is non-empty by Lemma 2.15 (Theorem 3.2).

4. Pareto-Optimality and Competitive Equilibria with Convex Distortion Risk Measures

In this section, our focus is on convex distortion risk measures (DRMs), that is, risk measures of the form

$$\rho_g(Y) = \int_{-\infty}^{0} [g(S_Y(z)) - 1]dz + \int_{0}^{\infty} g(S_Y(z))dz, \ \forall\, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}),$$

where $g : [0,1] \to [0,1]$ is a non-decreasing and concave function satisfying $g(0) = 0$ and $g(1) = 1$, and S_Y denotes the decumulative distribution function of Y with respect to the probability measure $\mathbb{P}$ (also called survival function). A convex DRM is monotone, comonotonic-additive, translation-invariant, and convex (e.g., Marinacci and Montrucchio, 2004b). If, in addition it is finite, then it is also continuous (e.g., Rüschendorf, 2013, Corollary 7.10). Hereafter, let $\rho^{In} = \rho_{g_1}$ and $\rho^{Re} = \rho_{g_2}$, for given concave distortion functions g_1, g_2.

4.1. Competitive Equilibria.

We consider competitive equilibria in two market settings involving n economic agents. First, in a *complete market*, the set of admissible allocations is given by

$$\mathbb{A}(X) := \left\{ (X_1, \ldots, X_n) \in \left(L^1(\Omega, \mathcal{F}, \mathbb{P})\right)^n : \sum_{i=1}^{n} X_i = X \right\}.$$

Second, in a *comonotone market* (a special type of an incomplete market introduced by Boonen et al. (2021)), allocations are confined to the set $C(X)$ of comonotonic allocations, namely,

$$C(X) := \{Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : (Y, X - Y) \text{ is comonotonic}\},$$

and the resulting set of admissible allocations is then given by

$$\mathbb{A}^c(X) := \left\{ (X_1, \ldots, X_n) \in (C(X))^n : \sum_{i=1}^{n} X_i = X \right\}.$$

Note that $n = 2$ in the context of our optimal reinsurance problem. In that case, we use the terminology "*complete reinsurance market*" and "*comonotone reinsurance market*".

Definition 4.1 (Competitive Equilibria).

(1) In a complete reinsurance market, a competitive equilibrium is a pair $\left((X_1, X_2), \xi\right) \in \mathbb{A}(X) \times L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies:

  (a) $\Pi(\xi, X_1) \leqslant \Pi(\xi, X)$.

  (b) $\Pi(\xi, X_2) \leqslant 0 \ \left(= \Pi(\xi, 0)\right)$.

  (c) $\rho^{In}(X_1) = \min\left\{\rho^{In}(Y_1) : \Pi(\xi, Y_1) \leqslant \Pi(\xi, X)\right\}$.

  (d) $\rho^{Re}(X_2) = \min\left\{\rho^{Re}(Y_2) : \Pi(\xi, Y_2) \leqslant 0\right\}$.

Such a competitive equilibrium is called an **Unconstrained Competitive Equilibrium (UCE)**.

(2) In a comonotone reinsurance market, a competitive equilibrium is a pair $\big((X_1, X_2), \xi\big) \in \mathbb{A}^c(X) \times L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies:

(a) $\Pi(\xi, X_1) \leqslant \Pi(\xi, X)$.

(b) $\Pi(\xi, X_2) \leqslant 0 \; \big(= \Pi(\xi, 0)\big)$.

(c) $\rho^{In}(X_1) = \min\Big\{\rho^{In}(Y_1) : Y_1 \in C(X),\, \Pi(\xi, Y_1) \leqslant \Pi(\xi, X)\Big\}$.

(d) $\rho^{Re}(X_2) = \min\Big\{\rho^{Re}(Y_2) : Y_2 \in C(X),\, \Pi(\xi, Y_2) \leqslant 0\Big\}$.

Such a competitive equilibrium is called a **Constrained Competitive Equilibrium (CCE)**.

4.2. **Competitive Equilibria and Pareto Efficiency.** If the distortion functions are strictly concave, then it is known (e.g., Boonen, 2015) that the equilibrium price in UCE exists and is unique.[3] Moreover, it is given by

$$\xi := \frac{d\mathbb{Q}}{d\mathbb{P}},$$

where $\mathbb{Q}$ is defined by

$$\mathbb{Q}(X > z) := \max\big\{g_1(S_X(z)), g_2(S_X(z))\big\}, \;\forall\, z \in \mathbb{R}.$$

Furthermore, for any UCE $\big((X_1, X_2), \xi\big)$, it follows that $\Pi(\xi, X_2) = \Pi(\xi, 0) = 0$ (Boonen, 2015, Proposition 4.1). A direct consequence of this is

$$\rho^{Re}(X_2) = \rho^{Re}(0) = 0.$$

Indeed, if $\rho^{Re}(X_2) < 0$ then $\rho^{Re}(cX_2^*) = c\rho^{Re}(X_2) < \rho^{Re}(X_2)$ for any $c > 1$, while $\Pi(\xi, cX_2) = 0$ is still in the budget set: a contradiction. Also, $\xi \in \partial\rho^{In}(X_1) \cap \partial\rho^{Re}(X_2)$ (e.g., Flåm, 2011). In conclusion, any UCE is PO and such that the *reinsurer* is indifferent between selling reinsurance and not selling reinsurance. This is in sharp contrast with BO solutions, which are PO and such that the *insurer* is indifferent, as per our Theorems 3.1 and 3.2.

Additionally, Boonen et al. (2021, Theorem 1) show that in a comonotone reinsurance market, the pair $((X_1, X_2), \xi)$ is a CCE if and only if the following hold jointly:

(1) $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and $\min\Big\{g_1\big(S_X(z)\big), g_2\big(S_X(z)\big)\Big\} \leqslant \mathbb{Q}(X > z) \leqslant \max\Big\{g_1\big(S_X(z)\big), g_2\big(S_X(z)\big)\Big\}$ for all z;

(2) $X_2 = f(X) - \Pi\big(\xi, f(X)\big)$, a.s., where f satisfies

$$f'(z) = \begin{cases} 1 & \text{if } g_1\big(S_X(z)\big) > g_2\big(S_X(z)\big), \\ 0 & \text{if } g_1\big(S_X(z)\big) < g_2\big(S_X(z)\big), \end{cases} \tag{4.1}$$

and $f(X) \in C(X)$.

[3] Boonen (2015) only shows this result for finite Ω, but the result is extendable to an infinite state space.

Moreover, Boonen et al. (2021) show that a CCE always exists, but equilibrium prices might no longer be unique.

Now, for a given $X_2 := f(X) \in C(X)$, one can construct a one-to-one mapping to a reinsurance contract $(p, I) \in \mathbb{R} \times \mathcal{I}$ as follows:

- Let $p := f(0)$.
- Let $I(X) := f(X) - f(0) = f(X) - p$.

Proposition 4.2. *The following two results hold:*

- *for any CCE* $\left(\left(X_1^*, X_2^*\right), \xi^*\right)$*, the contract* (p^*, I^*) *is PO, where* $f(X) := X_2^*$*,* $p^* := f(0)$*, and* $I^*(X) := f(X) - f(0)$*;*
- *if* (p^*, I^*) *is PO, then there exists some* ξ^* *such that* $\left(\left(X_1^*, X_2^*\right), \xi^*\right)$ *is a CCE, where* $X_1^* := X - I^*(X) + \Pi(\xi^*, I^*)$ *and* $X_2^* := I^*(X) - \Pi(\xi^*, I^*)$*.*

Proof. The proof that any CCE is PO is given in Boonen et al. (2021, Theorem 3(i)). The second result is proven by construction. Let (p^*, I^*) be a PO. By Lemmata 2.8 and 2.9, it then follows that I^* solves

$$\min_{I \in \mathcal{I}} \left\{ \rho_1^{In}(0, I) + \rho_1^{Re}(0, I) : (p, I) \in \mathcal{IR}, \text{ for some } p \in \mathbb{R} \right\}.$$

This implies that $(I^*)'(x)$ coincides with (4.1) (see Asimit and Boonen, 2018, Proposition 4.1). Now, define the function $g : [0, M] \times [0, 1] \to [0, 1]$ by

$$g(z; \gamma) := (1 - \gamma) \min\left\{g_1(S_X(z)), g_2(S_X(z))\right\} + \gamma \max\left\{g_1(S_X(z)), g_2(S_X(z))\right\},$$

for $z \in [0, M]$ and $\gamma \in [0, 1]$. Then by (1) above, for any $\gamma \in [0, 1]$, $\hat{\mathbb{Q}}_\gamma(X > z) := g(z; \gamma)$ defines an equilibrium price ξ^γ given by

$$\xi^\gamma := \frac{d\hat{\mathbb{Q}}_\gamma}{\mathrm{d}\mathbb{P}}.$$

Then, $\rho_1^{In}(\xi^\gamma, I^*)$ is continuous and increasing in γ, and satisfies $\rho_1^{In}(\xi^1, I^*) = \rho_1^{In}(\xi^1, 0)$ and $\rho_1^{In}(\xi^0, I^*) = \rho^{In}(X) - \rho^{In}(X - I^*(X)) - \rho^{Re}(I^*(X))$. Therefore, there exists a $\gamma^* \in [0, 1]$ such that $\rho_1^{In}(\xi^{\gamma^*}, I^*) = \rho_2^{In}(p^*, I^*)$. By (1)-(2) above, letting $\xi^* := \xi^{\gamma^*}$, $X_1^* := X - I^*(X) + \Pi(\xi^*, I^*)$, and $X_2^* := I^*(X) - \Pi(\xi^*, I^*)$ implies that $((X_1^*, X_2^*), \xi^*)$ is a CCE. □

4.3. **Asymmetric Nash Bargaining and Pareto Efficiency.** Asymmetric Nash bargaining solutions are contracts $(p, I) \in \mathcal{IR}$ that are optimal for the following problem:

$$\sup_{(p,I) \in \mathcal{IR}} \left(\rho^{In}(X) - \rho^{In}\left(X - I(X) + p\right)\right)^\gamma \left(\rho^{Re}(0) - \rho^{Re}\left(I(X) - p\right)\right)^{1-\gamma},$$

for some $\gamma \in [0, 1]$. Here, one can interpret γ as the bargaining power of the insurer; $\gamma = 0$ ($\gamma = 1$) represents the case of no (full) bargaining power of the insurer and $\gamma = \frac{1}{2}$ represents the case of equal bargaining power leading to the (symmetric) Nash bargaining solution of Nash (1950). It is shown by Kalai (1977) that asymmetric Nash bargaining solutions are PO. Moreover, Boonen et al. (2016, Proposition 2.7, Eq. (19), Proposition 3.4) show that $(p, I) \in \mathbb{R} \times \mathcal{I}$ is an asymmetric Nash bargaining solution if and only if the following conditions hold jointly:

- The indemnity function $I \in \mathcal{I}$ is such that

$$I'(x) = \begin{cases} 1 & \text{if } g_1\left(S_X(x)\right) < g_2\left(S_X(x)\right), \\ 0 & \text{if } g_1\left(S_X(x)\right) > g_2\left(S_X(x)\right). \end{cases}$$

- The premium p is such that

$$p = \rho^{Re}\left(I(X)\right) + \gamma(\rho^{In}\left(I(X)\right) - \rho^{Re}\left(I(X)\right). \tag{4.2}$$

In fact, eq. (4.2) implies that there is a one-to-one correspondence between the asymmetric Nash bargaining solution and a premium p that lies in between the indifference premiums[4], that is, $p \in \left[\rho^{Re}(I(X)), \rho^{In}(I(X))\right]$. This means that asymmetric Nash bargaining solutions are only consistent with BO solutions if $\gamma = 0$ or $\rho^{In}(I(X)) = \rho^{Re}(I(X))$.

4.4. **PO, BO, UCE, CCE, and Nash Bargaining for Convex DRMs.** To sum up, for convex distortion risk measures, the following holds:

(1) In any UCE, the risk transfer is PO and the *reinsurer* will be indifferent.
(2) In any CCE, the risk transfer is PO and any premium in between the indifference prices will constitute an equilibrium.
(3) The latter set of contracts can also be obtained via asymmetric Nash bargaining solutions.

Recall that by Theorem 3.2, in any BO the risk transfer is PO and the *insurer* will be indifferent. This paper does not claim that UCE, CCE, or asymmetric Nash bargaining solutions are more realistic to occur in bilateral reinsurance risk transfer. In fact, in a market with only two agents (insurer and reinsurer), it may be problematic to assume that a competitive equilibrium holds. In such competitive equilibria, it is namely assumed that individual agents have no bargaining power, and cannot affect the prices in the market. We however find it useful to link our result on BO to this literature.

5. An Example: Convex Distortion Risk Measures

Consider a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a concave (hence a.e. differentiable) distortion function g. By the Fenchel-Moreau theorem, the convex DRM ρ_g admits the dual representation (e.g., Pflug, 2006)

$$\rho_g(Y) = \sup\left\{E(YZ) : Z = g'(U),\ U \text{ has a uniform distribution on } [0,1]\right\}.$$

Moreover, by the concavity of g, it follows from Carlier and Dana (2003a, Corollary 2)[5] and Marinacci and Montrucchio (2004a) that the subdifferential of ρ_g at $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$\partial\rho_g(Y) = \overline{co}\left\{g'(1-U) : U \sim Unif(0,1),\ (U, Y) \text{ is comonotonic}\right\}, \tag{5.1}$$

where $\overline{co}$ denotes the $L^1(\Omega, \mathcal{F}, \mathbb{P})$-closed convex hull.

[4] Note that by Lemma 2.5, for any $(p, I) \in \mathcal{IR}$, we have $\rho^{Re}\left(I(X)\right) \leqslant p \leqslant \rho^{In}(X) - \rho^{In}\left(X - I(X)\right)$. Moreover, by comonotonic-additivity of distortion risk measures, it follows that $\rho^{In}(X) - \rho^{In}\left(X - I(X)\right) = \rho^{In}\left(I(X)\right)$, and so $p \in \left[\rho^{Re}(I(X)), \rho^{In}(I(X))\right]$.

[5] Note that Carlier and Dana (2003a) assume differentiability of the distortion function. However, this assumption can be relaxed, as shown by Marinacci and Montrucchio (2004a).

Next, we propose a parameterization of the concave distortion function for the class of convex DRMs. This parameterization is inspired by Anthropelos and Boonen (2020). For a given $\gamma \in [0,1]$, let $g(\cdot;\gamma) : [0,1] \to [0,1]$ be a concave distortion function. Moreover, for every $s \in (0,1)$, let $g(s;\cdot) : [0,1] \to [0,1]$ be increasing in γ. We define $\rho(Y;\gamma) := \rho_{g(\cdot;\gamma)}(Y)$ as a DRM with concave distortion function $g(\cdot;\gamma)$. Then, we parameterize ρ^{In} and ρ^{Re} as follows:

$$\rho^{In}(Y) := \rho\left(Y;\gamma^{In}\right) \quad \text{and} \quad \rho^{Re}(Y) := \rho\left(Y;\gamma^{Re}\right), \quad \forall\, Y \in L^1(\Omega,\mathcal{F},\mathbb{P}), \tag{5.2}$$

with $\gamma^{In}, \gamma^{Re} \in [0,1]$.

Special cases of such a class are discussed in Anthropelos and Boonen (2020), and they include the case in which the insurer and reinsurer use a TVaR (see Section 6). Also, the function $g(s;\gamma) := s^{1-\gamma}$ is a special case, and this leads to the well-known proportional hazard (PH) transform of Wang (1995).

Proposition 5.1. *Let $\mathcal{I} = \mathcal{I}_0$. Consider the optimization problem given by Problem (2.5) in Lemma 2.9, and assume that ρ^{In} and ρ^{Re} are as in eq. (5.2). Then the indemnity function I^* defined below is optimal for Problem (2.5):*

$$I^* = \begin{cases} 0 & \text{if } \gamma^{In} < \gamma^{Re}, \\ \in \mathcal{I} & \text{if } \gamma^{In} = \gamma^{Re}, \\ Id & \text{if } \gamma^{In} > \gamma^{Re}. \end{cases} \tag{5.3}$$

Proof. Note that by definition of a DRM, it holds that for each $Y \in L^1(\Omega,\mathcal{F},\mathbb{P})$ the function $\gamma \mapsto \rho(Y;\gamma)$ is non-decreasing. Thus, it follows that if $\gamma^{In} < \gamma^{Re}$ then $\rho^{In}(Y) \leqslant \rho^{Re}(Y)$ for all $Y \in L^1(\Omega,\mathcal{F},\mathbb{P})$. Hence, if $\gamma^{In} < \gamma^{Re}$ then the comonotonic-additivity of DRMs implies that for all $I \in \mathcal{I}$,

$$\begin{aligned} \rho_1^{In}(0,0) + \rho_1^{Re}(0,0) &= \rho^{In}(X) = \rho^{In}\big(X - I(X)\big) + \rho^{In}\big(I(X)\big) \\ &\leqslant \rho^{In}\big(X - I(X)\big) + \rho^{Re}\big(I(X)\big) \\ &= \rho_1^{In}(0,I) + \rho_1^{Re}(0,I). \end{aligned}$$

Similarly, if $\gamma^{In} > \gamma^{Re}$ then $\rho_1^{In}(0,X) + \rho_1^{Re}(0,X) \leqslant \rho_1^{In}(0,I) + \rho_1^{Re}(0,I)$ for all $I \in \mathcal{I}$. Finally, if $\gamma^{In} = \gamma^{Re}$, then it follows from comonotonic-additivity that for all $I \in \mathcal{I}$,

$$\begin{aligned} \rho_1^{In}(0,I) + \rho_1^{Re}(0,I) &= \rho^{In}\big(X - I(X)\big) + \rho^{Re}\big(I(X)\big) \\ &= \rho^{In}\big(X - I(X)\big) + \rho^{In}\big(I(X)\big) \\ &= \rho^{In}(X) = \rho_1^{In}(0,0) + \rho_1^{Re}(0,0). \end{aligned}$$

This concludes the proof. ∎

From Lemmata 2.8 and 2.9, Theorems 3.1 and 3.2, as well as Proposition 6.1, we obtain the following result.

Proposition 5.2. *Assume that ρ^{In} and ρ^{Re} are as in eq. (5.2).*

Then, the following holds:

- *If $\gamma^{In} < \gamma^{Re}$, then $(0,0)$ is PO and $(\xi_0, 0)$ is BO, where $\xi_0 \in \partial\rho^{In}(0)$ as in Assumption 2.1.*
- *If $\gamma^{In} = \gamma^{Re}$, then for any $I \in \mathcal{I}$, $\left(\rho^{In}\left(I(X)\right), I\right)$ is PO and (ξ, I) is BO, where $\xi \in \partial\rho^{In}(X)$.*

- *If* $\gamma^{In} > \gamma^{Re}$*, then* $\left(\rho^{In}(X), Id\right)$ *is PO and* (ξ, Id) *is BO, where* $\xi \in \partial\rho^{In}(X)$.

Example 5.3. Let $\mathcal{I} := \left\{I = a\,Id \,:\, a \in [0, \overline{a}]\right\}$ with $0 < \overline{a} \leqslant 1$, which is a closed subset of $\mathcal{I}_0$. Then, for $I \in \mathcal{I}$,

$$\rho_1^{In}(0, I) + \rho_1^{Re}(0, I) = (1-a)\rho^{In}(X) + a\rho^{Re}(X).$$

From this, we readily find that the indemnity function I^* defined below is optimal for Problem (2.5):

$$I^* = \begin{cases} 0 & \text{if } \gamma^{In} < \gamma^{Re}, \\ \in \mathcal{I} & \text{if } \gamma^{In} = \gamma^{Re}, \\ \overline{a}\,Id & \text{if } \gamma^{In} > \gamma^{Re}. \end{cases} \tag{5.4}$$

Moreover, Proposition 5.2 can directly be generalized as follows:

- if $\gamma^{In} < \gamma^{Re}$, then $(0, 0)$ is PO and $(\xi_0, 0)$ is BO, where $\xi_0 \in \partial\rho^{In}(0)$ as in Assumption 2.1.
- If $\gamma^{In} = \gamma^{Re}$, then for any $I \in \mathcal{I}$, $\left(\rho^{In}\left(I(X)\right), I\right)$ is PO and (ξ, I) is BO, where $\xi \in \partial\rho^{In}(X)$.
- If $\gamma^{In} > \gamma^{Re}$, then $\left(\rho^{In}(\overline{a}X), \overline{a}\,Id\right)$ is PO and $(\xi, \overline{a}\,Id)$ is BO, where $\xi \in \partial\rho^{In}(X)$.

6. An Example: PO and BO for TVaR

In this section, we provide an illustrative example for the special case in which the convex DRMs are given by the Tail Value-at-Risk (TVaR) risk measure. The TVaR at level $\alpha \in (0, 1)$ is a continuous DRM (see, e.g., Svindland, 2009a,b) for which the (concave) distortion function is given by (e.g., Dhaene et al., 2006):

$$g_\alpha(t) := \min\left\{\frac{t}{1-\alpha}, 1\right\}, \ \forall t \in [0, 1].$$

First, note that since for each $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ the function $\alpha \mapsto TVaR_\alpha(Y)$ is non-decreasing (e.g., Denuit et al., 2005, Property 2.4.5), it follows that if $\alpha < \beta$ then $\rho^{In}(Y) \leqslant \rho^{Re}(Y)$ for all $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Corollary 6.1. *Let* $\mathcal{I} = \mathcal{I}_0$*. Consider the optimization problem given by Problem* (2.5) *in Lemma 2.9, and assume that* ρ^{In} *and* ρ^{Re} *are TVaR risk measures at respective levels* $\alpha, \beta \in (0, 1)$*:*

$$\rho^{In} = TVaR_\alpha \quad \text{and} \quad \rho^{Re} = TVaR_\beta.$$

Then the indemnity function I^* *defined below is optimal for Problem* (2.5)*:*

$$I^* = \begin{cases} 0 & \textit{if } \alpha < \beta, \\ \in \mathcal{I} & \textit{if } \alpha = \beta, \\ Id & \textit{if } \alpha > \beta. \end{cases} \tag{6.1}$$

The dual representation of TVaR is given by

$$TVaR_\alpha(X) = \sup\left\{E(X\,Z) : E(Z) = 1,\ 0 \leqslant Z \leqslant \frac{1}{1-\alpha}\right\},$$

and thus

$$\partial TVaR_{\alpha}(X) = \arg\max \left\{ E(X\,Z) : E(Z) = 1,\ 0 \leqslant Z \leqslant \frac{1}{1-\alpha} \right\}.$$

By eq. (5.1),

$$\partial TVaR_{\alpha}(X) = \overline{co} \left\{ \left(\frac{1}{1-\alpha} \right) \mathbb{1}_{[U<1-\alpha]} : U \sim Unif(0,1),\ (U,X) \text{ is comonotonic} \right\}.$$

Therefore, if X is a continuous random variable then $F_X(X) \sim Unif(0,1)$ and

$$\partial TVaR_{\alpha}(X) = \left(\frac{1}{1-\alpha} \right) \mathbb{1}_{[X>VaR_{\alpha}(X)]}.$$

More generally (e.g., Föllmer and Schied (2016, Remark 4.53) or Svindland (2009b, Section 6.2)), $\partial TVaR_{\alpha}(X) \neq \varnothing$ for $\alpha \in (0,1)$, and $\xi^* \in \partial TVaR_{\alpha}(X)$, where

$$\xi^* := \left(\frac{1}{1-\alpha} \right) \mathbb{1}_{[X>VaR_{\alpha}(X)]} + \left(\frac{1-\alpha-\mathbb{P}\left(X > VaR_{\alpha}(X)\right)}{\mathbb{P}\left(X \geqslant VaR_{\alpha}(X)\right) - \mathbb{P}\left(X > VaR_{\alpha}(X)\right)} \right) \mathbb{1}_{[X=VaR_{\alpha}(X)]}.$$

7. Conclusions

In this paper, we show that in the context of an optimal reinsurance design problem with translation-invariant, convex, and comonotonic-additive risk measures (e.g., convex distortion risk measures), the set of Bowley-optimal solutions is associated with the set of Pareto optima for which the insurer is indifferent. Specifically, we show that every Bowley optimum leads to a Pareto-efficient reinsurance contract in which the insurer is indifferent between reinsuring and not reinsuring (Theorem 3.1). Moreover, only those Pareto-optimal contracts for which the insurer is indifferent between reinsuring and not reinsuring can constitute a Bowley optimum (Theorem 3.2). Moreover, in such a Bowley optimum, the reinsurer has some flexibility in selecting the pricing kernel.

Our results hence suggest that if the Bowley solution is to be taken as an optimality criterion in problems of optimal reinsurance, then optimal contracts are characterized by the fact that the insurer has *no* incentive to purchase reinsurance, as only the reinsurer strictly benefits from the reinsurance transaction. Thus, we interpret our results as providing a warning about the applicability of the Bowley solution in problems of optimal reinsurance.

Reinsurance contracts in which both the insurer and reinsurer strictly benefit cannot be obtained with Bowley solutions. Alternatively, if one wishes to design a market mechanism that yields equilibria with strictly positive welfare gains from trading for both the insurer and reinsurer, then comonotone market competitive equilibria or asymmetric Nash bargaining solutions could be studied, at least for convex distortion risk measures. Also, other alternatives appear in the literature on optimal reinsurance (Balbás et al., 2022) or optimal risk sharing (Chen and Xie, 2021).

Appendix A. Proof of Theorem 2.10

(1) Let $\widetilde{\mathcal{I}} := \left\{ I \in \mathcal{I} : (p, I) \in \mathcal{IR}, \text{ for some } p \in \mathbb{R} \right\}$. Since $(0,0) \in \mathcal{IR}$, it follows that $\widetilde{\mathcal{I}} \neq \varnothing$. Moreover, since $\mathcal{I}$ is a closed subset of $\mathcal{I}_0$, so is $\widetilde{\mathcal{I}}$. Indeed, let $\{I_n\}_{n\geqslant 1}$ be a sequence in $\widetilde{\mathcal{I}}$ that converges to some I^*. Since $\mathcal{I}$ is closed, $I^* \in \mathcal{I}$. Moreover, since $I_n \in \widetilde{\mathcal{I}}$, for each $n \geqslant 1$, there exists $\{p_n\}_{n\geqslant 1} \subset \mathbb{R}$ such that $(p_n, I_n) \in \mathcal{IR}$, for each $n \geqslant 1$. Since uniform convergence implies L^1 convergence, it follows from the continuity of the risk measures that

$$\lim_{n\to+\infty} \rho^{In}\left(X - I_n(X)\right) = \rho^{In}\left(X - I^*(X)\right) \quad \text{and} \quad \lim_{n\to+\infty} \rho^{Re}\left(I_n(X)\right) = \rho^{Re}\left(I^*(X)\right).$$

Moreover, for each $n \geqslant 1$, individual rationality yields

$$0 = \rho^{Re}(0) \leqslant \rho^{Re}\left(I_n(X)\right) \leqslant p_n \leqslant \rho^{In}(X) - \rho^{In}\left(X - I_n(X)\right) \leqslant \rho^{In}(X) \leqslant M,$$

and so the sequence $\{p_n\}_{n\geqslant 1}$ is a uniformly bounded sequence in $[0, M]$. Hence, by the Bolzano-Weierstrass Theorem, it admits a convergent subsequence $\{p_m\}_{m\geqslant 1}$, with limit $p^* \in [0, M]$. That is, $\lim_{m\to+\infty} p_m = p^*$. Clearly, $(p^*, I^*) \in \mathcal{IR}$, and thus $I^* \in \widetilde{\mathcal{I}}$. Hence, $\widetilde{\mathcal{I}}$ is closed.

Therefore, since $\mathcal{I}_0$ is (supnorm) compact, so is $\widetilde{\mathcal{I}}$. Consequently, in light of Lemmata 2.8 and 2.9, the existence of Pareto optima follows from the continuity of ρ^{In} and ρ^{Re}, since uniform convergence implies L^1 convergence.

(2) To show that the set of $\mathcal{BO} \neq \varnothing$, it suffices by Theorem 3.2 to show that at least one PO makes the insurer indifferent with the status quo. That is, it suffices to show that there exists $(p^*, I^*) \in \mathcal{PO}$ such that $\rho_1^{In}(p^*, I^*) = \rho_1^{In}(0,0)$. Suppose, by way of contradiction, that for any $(p^*, I^*) \in \mathcal{PO}$, we have $\rho_1^{In}(p^*, I^*) \neq \rho_1^{In}(0,0)$, and fix such a PO (p^*, I^*). First note that since $(p^*, I^*) \in \mathcal{PO}$, it follows from Lemmata 2.8 and 2.9 that I^* is optimal for the problem

$$\min_{I\in\mathcal{I}} \left\{ \rho_1^{In}(0, I) + \rho_1^{Re}(0, I) : (p, I) \in \mathcal{IR}, \text{ for some } p \in \mathbb{R} \right\}.$$

Since $(p^*, I^*) \in \mathcal{IR}$, it follows that

$$\rho^{In}\left(X - I^*(X)\right) + p^* = \rho^{In}\left(X - I^*(X) + p^*\right) = \rho_1^{In}\left(p^*, I^*\right) < \rho_1^{In}(0,0) = \rho^{In}(X);$$

and

$$\rho^{Re}\left(I^*(X)\right) - p^* = \rho^{Re}\left(I^*(X) - p^*\right) = \rho_1^{Re}\left(p^*, I^*\right) \leqslant \rho_1^{Re}(0,0) = 0.$$

Hence,

$$p^* < \rho^{In}(X) - \rho^{In}\left(X - I^*(X)\right) = \rho^{In}\left(I^*(X)\right) \quad \text{and} \quad \rho^{Re}\left(I^*(X)\right) - p^* \leqslant 0.$$

Therefore,

$$\rho^{Re}\left(I^*(X)\right) \leqslant p^* < \rho^{In}\left(I^*(X)\right).$$

In particular, $\rho^{Re}\left(I^*(X)\right) - \rho^{In}\left(I^*(X)\right) < 0$. Let $\widetilde{p} := \rho^{In}\left(I^*(X)\right)$. Then

$$\begin{aligned}\rho_1^{In}\left(\widetilde{p}, I^*\right) &= \rho^{In}\left(X - I^*(X) + \rho^{In}\left(I^*(X)\right)\right)\\ &= \rho^{In}(X) - \rho^{In}\left(I^*(X)\right) + \rho^{In}\left(I^*(X)\right) = \rho^{In}(X);\end{aligned}$$

and

$$\rho_1^{Re}\left(\widetilde{p}, I^*\right) = \rho^{Re}\left(I^*(X) - \widetilde{p}\right) = \rho^{Re}\left(I^*(X)\right) - \rho^{In}\left(I^*(X)\right) < 0.$$

Consequently, $(\widetilde{p}, I^*) \in \mathcal{IR}$. Suppose $(\widetilde{p}, I^*) \notin \mathcal{PO}$. Then there exists some $\left(\bar{p}, \bar{I}\right) \in \mathcal{IR}$ such that

$$\begin{aligned}\rho_1^{In}\left(\bar{p}, \bar{I}\right) + \rho_1^{Re}\left(\bar{p}, \bar{I}\right) &= \rho_1^{In}\left(0, \bar{I}\right) + \rho_1^{Re}\left(0, \bar{I}\right) \\ &< \rho_1^{In}\left(0, I^*\right) + \rho_1^{Re}\left(0, I^*\right) = \rho_1^{In}\left(\widetilde{p}, I^*\right) + \rho_1^{Re}\left(\widetilde{p}, I^*\right),\end{aligned}$$

contradicting the optimality of I^* for the problem

$$\min_{I \in \mathcal{I}} \left\{\rho_1^{In}\left(0, I\right) + \rho_1^{Re}\left(0, I\right) : (p, I) \in \mathcal{IR}, \text{ for some } p \in \mathbb{R}\right\}.$$

Hence, $(\widetilde{p}, I^*) \in \mathcal{PO}$ and $\rho_1^{In}\left(\widetilde{p}, I^*\right) = \rho_1^{In}\left(0, 0\right)$, which contradicts the assumption that for all $(p^*, I^*) \in \mathcal{PO}$, $\rho_1^{In}\left(p^*, I^*\right) \neq \rho_1^{In}\left(0, 0\right)$. □

References

Aliprantis, C. D. and K. C. Border (2006). *Infinite Dimensional Analysis - 3^{rd} edition.* Springer-Verlag.

Anthropelos, M. and T. J. Boonen (2020). Nash equilibria in optimal reinsurance bargaining. *Insurance: Mathematics and Economics 93*, 196–205.

Artzner, P., F. Delbaen, J. M. Eber, and D. Heath (1999). Coherent Measures of Risk. *Mathematical Finance 9*(3), 203–228.

Asimit, A. V., V. Bignozzi, K. C. Cheung, J. Hu, and E. S. Kim (2017). Robust and Pareto Optimality of Insurance Contracts. *European Journal of Operational Research 262*(2), 720–732.

Asimit, A. V. and T. J. Boonen (2018). Insurance with Multiple Insurers: a Game-Theoretic Approach. *European Journal of Operational Research 267*(2), 778–790.

Assa, H., H. Sharifi, and A. Lyons (2021). An Examination of the Role of Price Insurance Products in Stimulating Investment in Agriculture Supply Chains for Sustained Productivity. *European Journal of Operational Research 288*(3), 918–934.

Balbás, A., B. Balbás, R. Balbás, and A. Heras (2022). Risk Transference Constraints in Optimal Reinsurance. *Insurance: Mathematics and Economics 103*, 27–40.

Balbás, A., B. Balbás, and A. Heras (2011). Stable Solutions for Optimal Reinsurance Problems Involving Risk Measures. *European Journal of Operational Research 214*(3), 796–804.

Boonen, T. J. (2015). Competitive Equilibria with Distortion Risk Measures. *ASTIN Bulletin 45*(3), 703–728.

Boonen, T. J., K. C. Cheung, and Y. Zhang (2021). Bowley Reinsurance with Asymmetric Information on the Insurer's Risk Preferences. *Scandinavian Actuarial Journal 2021*(7), 623–644.

Boonen, T. J., F. Liu, and R. Wang (2021). Competitive Equilibria in a Comonotone Market. *Economic Theory 72*(4), 1217–1255.

Boonen, T. J., K. S. Tan, and S. C. Zhuang (2016). Pricing in Reinsurance Bargaining with Comonotonic Additive Utility Functions. *ASTIN Bulletin 46*(2), 507–530.

Boonen, T. J. and Y. Zhang (2021). Bowley Reinsurance with Asymmetric Information: A First-best Solution. Forthcoming at *Scandinavian Actuarial Journal*.

Bowley, A. L. (1928). Bilateral Monopoly. *Economic Journal 38*(152), 651–659.

Carlier, G. and R. A. Dana (2003a). Core of Convex Distortions of a Probability. *Journal of Economic Theory 113*(2), 199–222.

Carlier, G. and R. A. Dana (2003b). Pareto Efficient Insurance Contracts when the Insurer's Cost Function is Discontinuous. *Economic Theory 21*(4), 871–893.

Chan, F.-Y. and H. U. Gerber (1985). The Reinsurer's Monopoly and the Bowley Solution. *ASTIN Bulletin 15*(2), 141–148.

Chen, L. and Y. Shen (2018). On a New Paradigm of Optimal Reinsurance: A Stochastic Stackelberg Differential Game between an Insurer and a Reinsurer. *ASTIN Bulletin 48*(2), 905–960.

Chen, Z. and W. Xie (2021). Sharing the Value-at-Risk under Distributional Ambiguity. *Mathematical Finance 31*(1), 531–559.

Cheung, K. C., S. C. P. Yam, and Y. Zhang (2019). Risk-adjusted Bowley Reinsurance under Distorted Probabilities. *Insurance: Mathematics and Economics 86*, 64–72.

Chi, Y., K. S. Tan, and S. C. Zhuang (2020). A Bowley Solution with Limited Ceded Risk for a Monopolistic Reinsurer. *Insurance: Mathematics and Economics 91*, 188–201.

Denuit, M., J. Dhaene, M. J. Goovaerts, and R. Kaas (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. John Wiley & Sons.

Dhaene, J., S. Vanduffel, M. J. Goovaerts, R. Kaas, Q. Tang, and D. Vyncke (2006). Risk Measures and Comonotonicity: A Review. *Stochastic Models 22*(4), 573–606.

Dunford, N. and J. T. Schwartz (1958). *Linear Operators, Part 1: General Theory*. Wiley-Interscience.

Ekeland, I. and R. Témam (1999). *Convex Analysis and Variational Problems*. SIAM.

Flåm, S. D. (2011). Exchanges and Measures of Risks. *Mathematics and Financial Economics 5*(4), 249–267.

Föllmer, H. and A. Schied (2016). *Stochastic Finance: An Introduction in Discrete Time – 4^{th} ed.* Walter de Gruyter.

Gavagan, J., L. Hu, G. Y. Lee, H. Liu, and A. Weixel (2022). Optimal Reinsurance with Model Uncertainty and Stackelberg Game. *Scandinavian Actuarial Journal 2022*(1), 29–48.

Kalai, E. (1977). Nonsymmetric Nash Solutions and Replications of 2-person Bargaining. *International Journal of Game Theory 6*(3), 129–133.

Li, D. and V. R. Young (2021). Bowley Ssolution of a Mean–Variance Game in Insurance. *Insurance: Mathematics and Economics 98*, 35–43.

Marinacci, M. and L. Montrucchio (2004a). A Characterization of the Core of Convex games through Gateaux Derivatives. *Journal of Economic Theory 116*(2), 229–248.

Marinacci, M. and L. Montrucchio (2004b). Introduction to the Mathematics of Ambiguity. In *I. Gilboa (ed.), Uncertainty in Economic Theory: Essays in Honor of David SchmeidlerÕs 65th Birthday*, pp. 46–107. London: Routledge.

Nash, J. F. (1950). The bargaining problem. *Econometrica 18*(2), 155–162.

Pflug, G. C. (2006). Subdifferential Representations of Risk Measures. *Mathematical Programming 108*(2-3), 339–354.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*. Springer.

Ruszczyński, A. and A. Shapiro (2006a). Optimization of convex risk functions. *Mathematics of operations research 31*(3), 433–452.

Ruszczyński, A. and A. Shapiro (2006b). Optimization of risk measures. In *Probabilistic and Randomized Methods for Design under Uncertainty*, pp. 119–157. Springer.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American mathematical society 97*(2), 255–261.

Svindland, G. (2009a). Continuity Properties of Law-Invariant (Quasi-) Convex Risk Functions on L^∞. *Mathematics and Financial Economics 3*(1), 39–43.

Svindland, G. (2009b). Subgradients of Law-Invariant Convex Risk Measures on L1. *Statistics & Decisions 27*, 169–199.

Tan, K. S., P. Wei, W. Wei, and S. C. Zhuang (2020). Optimal Dynamic Reinsurance Policies under a Generalized Denneberg's Absolute Deviation Principle. *European Journal of Operational Research 282*(1), 345–362.

Taylor, G. (1992). Risk Exchange I: A Unification of Some Existing Results. *Scandinavian Actuarial Journal 1992*(1), 15–39.

Torraca, A. P. and B. Fanzeres (2021). Optimal Insurance Contract Specification in the Upstream Sector of the Oil and Gas Industry. *European Journal of Operational Research 295*(2), 718–732.

Wang, S. S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics 17*(1), 43–54.

Wang, S. S., V. R. Young, and H. H. Panjer (1997). Axiomatic Characterization of Insurance Prices. *Insurance: Mathematics and Economics 21*(2), 173–183.

Wozabal, D. (2014). Robustifying Convex Risk Measures for Linear Portfolios: A Nonparametric Approach. *Operations Research 62*(6), 1302–1315.

Yuan, Y., Z. Liang, and X. Han (2022). Robust Reinsurance Contract with Asymmetric Information in a Stochastic Stackelberg Differential Game. *Scandinavian Actuarial Journal 2022*(4), 328–355.

Tim J. Boonen: Amsterdam School of Economics – University of Amsterdam – Roetersstraat 11, 1018 WB – Amsterdam – The Netherlands

Email address: t.j.boonen@uva.nl

Mario Ghossoub: University of Waterloo – Department of Statistics and Actuarial Science – 200 University Ave. W. – Waterloo, ON, N2L 3G1 – Canada

Email address: mario.ghossoub@uwaterloo.ca