On the Existence of a Representative Reinsurer under Heterogeneous Beliefs

Tim J. Boonen* Mario Ghossoub†‡

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Abstract

This paper studies a one-period optimal reinsurance design model with n reinsurers and an insurer. The reinsurers are endowed with expected-value premium principles and with heterogeneous beliefs regarding the underlying distribution of the insurer's risk. Under general preferences for the insurer, a representative reinsurer is characterized. This means that all reinsurers can be treated collectively by means of a hypothetical premium principle in order to determine the optimal total risk that is ceded to all reinsurers. The optimal total ceded risk is then allocated to the reinsurers by means of an explicit solution. This is shown both in the general case and under the no-sabotage condition that avoids possible *ex post* moral hazard on the side of the insurer, thereby extending the results of Boonen et al. (2016). We subsequently derive closed-form optimal reinsurance contracts in case the insurer maximizes expected net wealth. Moreover, under the no-sabotage condition, we derive optimal reinsurance contracts in case the insurer maximizes dual utility, or in case the insurer maximizes a generic objective that preserves second-order stochastic dominance under the assumption of a monotone hazard ratio.

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Key Words: optimal reinsurance design, heterogeneous beliefs, multiple reinsurers, representative reinsurer, deductible.

*Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB, Amsterdam, The Netherlands. Email address: t.j.boonen@uva.nl.

†Department of Statistics and Actuarial Science, University of Waterloo, 200 University Ave. W., Waterloo, ON N2L 3G1, Canada. Email address: mario.ghossoub@uwaterloo.ca.

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1 Introduction

The optimal risk sharing between an insurer and a reinsurer has been widely studied, both in the academic literature and in actuarial practice. This problem is first formally analyzed by Borch (1960) who demonstrates that, under the assumption that the reinsurance premium is calculated by the expected-value principle, the stop-loss reinsurance treaty is the optimal strategy that minimizes the variance of the insurer's retained loss. By maximizing the expected utility of the terminal wealth of a risk-averse insurer, Arrow (1963) similarly shows that the stop-loss reinsurance treaty is optimal. These pioneering results have subsequently been refined to incorporate more sophisticated objective functions and/or more realistic premium principles. For instance, Bernard et al. (2015), Xu et al. (2018), and Ghossoub (2019b) all study the case where the insurer aims to maximize a rank-dependent expected utility function evaluated on its future net wealth.

Recently, Ghossoub (2016, 2017, 2018, 2019a), Boonen (2016), and Chi (2019) studied heterogeneous beliefs. All of these studies focus on the case of one insurer and one reinsurer. In particular, Ghossoub (2016, 2017, 2018, 2019a) focus on the case of a fixed premium budget. That is, the insurer does not decide how much premium to spend on reinsurance, but is only interested in the optimal reinsurance indemnity schedule for the given premium level, treated as a budget constraint. On the other hand, Boonen (2016) and Chi (2019) optimize the utility of the insurer's net wealth given a premium principle of the reinsurer. Boonen (2016) studies the case where the insurer minimizes a distortion risk measure as proposed by Wang (1996). Chi (2019) solves the problem with expected utility maximizing preferences for the insurer, and he determines the optimal reinsurance contract under the condition of a monotone hazard ratio. Here, we extend the results of Boonen (2016) and Chi (2019) to a setting with multiple reinsurers and more general preference for the insurer.

While most of the existing literature on optimal reinsurance has predominantly been confined to an analysis of the optimal risk sharing between two parties, i.e., an insurer and a reinsurer, some progress has recently been made on addressing the optimal reinsurance strategy in the presence of multiple reinsurers. See, for instance, Asimit et al. (2013), Chi and Meng (2014), Cong and Tan (2016), and Boonen et al. (2016). Such formulation is more reasonable since in a well-established reinsurance market, an insurer could have recourse to more than one reinsurer to reinsure its risk. In fact, it may be desirable for the insurer to do so in view of the differences in reinsurers' beliefs and the competitiveness of the reinsurance market. As a result, an insurer that exploits such discrepancy among reinsurers might be able to achieve a more desirable risk profile. We focus on expected-value premium principles, while Asimit et al. (2013), Chi and Meng (2014), Cong and Tan (2016), and Boonen et al. (2016) all assume other premium principles for the reinsurers. Asimit et al. (2013) study the case where all agents use quantile-based risk measures; Chi and Meng (2014) focus on the case where one reinsurer uses an expected-value premium principle and the other reinsurer uses a law-invariant risk measure preserving the stop-loss order; Cong and Tan (2016) study monotonic piecewise premium principles; and Boonen et al. (2016) study generalized Wang's premium principles. All of these papers assume common beliefs and impose *ex ante* that the retained risk be comonotonic with the reinsurance contracts or underlying risk. This is motivated by *ex post* moral hazard on the side of the ceding insurer, and it is also referred to as the *no-sabotage* condition (Carlier and Dana, 2003).

In this paper, we study reinsurance contracts that are allowed to have a more general shape. This will make it possible for us to study the differences between the optimal contracts in this general case and the optimal contract in case we impose this no-sabotage condition. In fact, one of our main contributions to the related literature is that we highlight the consequences of imposing the no-sabotage condition on reinsurance contracts, under general preferences for the insurer. To the best of our knowledge, only Carlier and Dana (2003) studied both cases and compared optimal insurance contracts. Carlier and Dana (2003) focus on bilateral insurance bargaining with a single, risk-neutral reinsurer, whereas we examine the case of multiple reinsurers and belief heterogeneity.

Moreover, we characterize a representative reinsurer for both cases (with and without the no-sabotage condition), under general preferences for the insurer. We determine the set of optimal reinsurance contracts with multiple reinsurers by first focusing on the single representative reinsurer problem: this gives us the aggregate indemnities that will be jointly ceded to the reinsurers. We then explicitly characterize the optimal allocation of the aggregate ceded loss to the n reinsurers.Needless to say, the premium principles of the representative reinsurer with and without the no-sabotage condition do not coincide, and hence optimal reinsurance contracts do not coincide either. When the no-sabotage condition is imposed, we solve the optimal reinsurance problem using the Marginal Indemnification Function approach of Assa (2015); and we show how marginal changes in the loss are allocated to specific reinsurers. In the absence of the no-sabotage condition, we solve the optimal reinsurance problem by means of a statewise optimization approach; and we show how the total reinsured risk is shifted to the reinsurer(s) that have the lowest state-price density, which is quite intuitive.

We then consider three special cases for the insurer's preferences: the case of expected-wealth maximization, the case of dual-utility-of-wealth maximization, and the case of expected-utility-of-wealth maximization. We derive closed-form solutions to the optimal reinsurance problem with multiple reinsurers in each case. Moreover, under the no-sabotage condition and the technical condition of monotone hazard ratios, Chi (2019) derives the optimal reinsurance contract in the case of one reinsurer. This paper provides a similar result in the case of multiple reinsurers, using the existence of a representative reinsurer.

The rest of this paper is organized as follows. The model is defined in Section 2, where we introduce the heterogeneous beliefs, the feasible indemnity functions, and the general optimal reinsurance problem that we study in this paper. For general preferences of the insurer, Section 3 characterizes the representative reinsurer in optimal reinsurance, both in the general case and under the no-sabotage condition. In Section 4, we fully describe optimal reinsurance contracts for three different classes of preferences for the insurer. Finally, Section 5 concludes. Most proofs are presented in the Appendices.

2 The Model

This section provides the model description. First, we introduce the pricing formula of the reinsurers in Subsection 2.1. Second, we define two feasible sets of indemnity functions in Subsection 2.2. Third, we introduce the optimal reinsurance problem with multiple reinsurers in Subsection 2.3.

2.1 Expected Value Premium Principles with Heterogeneous Beliefs

Let (Ω, Σ) be a measurable space, and assume that the insurer is subject to a risk $X : \Omega \to \mathbb{R}^+$ that we interpret as a loss, which is taken as given exogenously throughout this paper. Here, we assume that $\Sigma = \sigma\{X\}$, the σ-algebra generated by the risk X. The insurer is endowed with beliefs given by $\mathbb{P}$ on (Ω, Σ). Under the beliefs $\mathbb{P}$, the insurer maximizes a general objective function that we will specify in the next section. We index the finite set of reinsurers as $1, 2, \ldots, n$. The reinsurers are endowed with beliefs that are given by probability measures $\mathbb{Q}_i$ on (Ω, Σ), for $i \in \{1, 2, \ldots, n\}$. We allow all beliefs, given by $\mathbb{P}, \mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n$, to differ from each other.

Define the probability measure μ on (Ω, Σ) by

$$\mu := \frac{1}{n+1}\left[\mathbb{P} + \sum_{i=1}^{n} \mathbb{Q}_i\right]. \tag{1}$$

By construction of μ, it follows that $\mathbb{P} \ll \mu$ and $\mathbb{Q}_i \ll \mu$, for each $i \in \{1, 2, \ldots, n\}$. Let $L^\infty(\Omega, \Sigma, \mu)$ denote the class of μ-essentially bounded random variables on (Ω, Σ, μ), and let $L^\infty_+(\Omega, \Sigma, \mu)$ denote its positive cone. For brevity, we use the notation L^∞ (resp. L^∞_+) to denote $L^\infty(\Omega, \Sigma, \mu)$ (resp. $L^\infty_+(\Omega, \Sigma, \mu)$), when there is no confusion. All throughout, we assume that the risk $X \in L^\infty_+$. This allows us to define $M := \operatorname{esssup} X = \inf\{a \in \mathbb{R} : \mu(X > a) = 0\} < \infty$.

For $i \in \{1, 2, \ldots, n\}$, the expected value premium principle used by insurer i is given by

$$\hat{\pi}^{\theta_i, \mathbb{Q}_i}(Y) := (1+\theta_i)\mathbb{E}^{\mathbb{Q}_i}[Y] = (1+\theta_i)\int_0^\infty \mathbb{Q}_i(Y > z)dz, \text{ for all } Y \in L^\infty_+, \tag{2}$$

where $\theta_i \geq 0$ is interpreted as a risk-loading charged by insurer i. This premium principle hence extends the expected value premium principle of Arrow (1963) to allow for heterogeneous beliefs. We can then write the premium principle in eq. (2) as

$$\pi^{\zeta_i}(Y) := \hat{\pi}^{\theta_i, \mathbb{Q}_i}(Y) = (1+\theta_i)\mathbb{E}^\mu\left[Y \frac{d\,\mathbb{Q}_i}{d\mu}\right] = \mathbb{E}^\mu[Y\zeta_i], \text{ for all } Y \in L^\infty_+, \tag{3}$$

where

$$\zeta_i := (1+\theta_i)\,\frac{d\,\mathbb{Q}_i}{d\mu}, \tag{4}$$

and $\frac{d\,\mathbb{Q}_i}{d\mu}$ is the Radon-Nikodym derivative of $\mathbb{Q}_i$ with respect to μ, for each $i \in \{1, 2, \ldots, n\}$. Hence, for each $i \in \{1, 2, \ldots, n\}$, it holds that $\mathbb{E}^\mu[\zeta_i] = (1+\theta_i)$. Note that the finite, non-negative measure defined by $\zeta_i d\mu$ is not necessarily a probability measure. We can interpret the heterogeneous beliefs as the existence of different pricing kernels in the market. Therefore, we can also interpret this premium principle similarly to the one studied by Chi et al. (2017) in the setting with one reinsurer, and preferences given by the risk-adjusted value of an insurer's liability.

2.2 Two Feasible Sets of Indemnity Functions

The problem of optimal reinsurance is concerned with the optimal partitioning of X into $f_i(X), i \in \{1, 2, \ldots, n\}$, and $X - \sum_{i=1}^n f_i(X)$, whereby $\sum_{i=1}^n f_i(X)$ represents the aggregate loss that is ceded to all n participating reinsurers, and $X - \sum_{i=1}^n f_i(X)$ captures the loss that is retained by the insurer.

In this paper, we consider two general classes of problems, depending on the *ex ante* requirements used on the set of feasible indemnity functions: the general case and the case of *no-sabotage*. In the general case, the feasible indemnity functions are such that $f_i \in \mathcal{F}^G, i \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^n f_i \in \mathcal{F}^G$, with

$$\mathcal{F}^G := \left\{f : [0, M] \to [0, M] \;\middle|\; 0 \leq f(X) \leq X, \;\mu\text{-a.s.}\right\}, \tag{5}$$

which ensures that the indemnities are non-negative and cannot exceed the amount of the total occurred loss (often referred to as the *principle of indemnity*). This class of indemnity functions is studied by, e.g., Kaluszka (2005); Bernard et al. (2015); and many others.

In the other case, the feasible indemnity functions are such that $f_i \in \mathcal{F}^{NS}, i \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^n f_i \in \mathcal{F}^{NS}$, with:

$$\mathcal{F}^{NS} = \left\{ f \in \mathcal{F}^G \;\middle|\; 0 \leq f'(z) \leq 1, \text{ for a.e. } z \in [0, M] \right\}. \tag{6}$$

That is, the functions $f_i(z), i \in \{1, 2, \ldots, n\}$ and $z - \sum_{i=1}^n f_i(z)$ are all non-decreasing, and any incremental compensation is never larger than the incremental loss. These constraints are referred to as the *no-sabotage* condition (Carlier and Dana, 2003). The assumption that aggregate indemnities $\sum_{i=1}^n f_i$ are in $\mathcal{F}^{NS}$ is meant to prevent *ex post* moral hazard that could otherwise arise from possible misreporting of the loss by the insurer (Huberman et al., 1983; Denuit and Vermandele, 1998; Young, 1999; Chi and Tan, 2011).

In this paper, we study the classes $\mathcal{F}^G$ and $\mathcal{F}^{NS}$ separately, and we provide a comparison of the optimal contracts arising in each case. The solution techniques that we use in the proofs of this paper are noticeably different for the two classes of feasible indemnity functions.

2.3 A Reinsurance Problem with Multiple Reinsurers

We assume that the insurer's preferences over terminal wealth profiles admit a representation in terms of a functional $V^{\mathbb{P}} : L^\infty \to \mathbb{R}$. Hence, the insurer maximizes

$$V^{\mathbb{P}}\left(W_0 - X - \sum_{i=1}^n \left(\pi^{\zeta_i}(f_i(X)) - f_i(X) \right) \right),$$

over admissible functions $\{f_i\}_{i=1}^n$. Here, $W_0 \in L^\infty_+$ is the initial (background) wealth that is allowed to be stochastic. We assume that $V^{\mathbb{P}}(W_0) < \infty$ and that $V^{\mathbb{P}}$ is strictly monotonic in the sense that for all $Z_1 > Z_2$, $\mathbb{P}$-a.s., we have $V^{\mathbb{P}}(Z_1) > V^{\mathbb{P}}(Z_2)$.

We focus on two cases of feasible indemnity functions, namely the general case ($\mathcal{F}^G$) and the case of no-sabotage ($\mathcal{F}^{NS}$). Let $\mathcal{F} \in \{\mathcal{F}^G, \mathcal{F}^{NS}\}$. Then the optimal strategy for the insurer to cede its risk to n reinsures can be determined by solving the following optimization problem:

$$\begin{array}{ll} \sup\limits_{\{f_i\}_{i=1}^n} & V^{\mathbb{P}}\Big(W_0 - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\zeta_i}(f_i(X)) \Big) \\ \text{s.t.} & \{f_i\}_{i=1}^n \subset \mathcal{F}, \; \sum_{i=1}^n f_i \in \mathcal{F}. \end{array} \tag{7}$$

Recall that the term inside the function $V^{\mathbb{P}}$ gives the net worth of the insurer in the presence of n multiple reinsurers. The optimal indemnity profile $\{f_i\}_{i=1}^n$ therefore maximize the objective of the insurer while subject to the condition as stipulated by $\mathcal{F}$. In the special case with only one reinsurer, we define

$$\sup_{f \in \mathcal{F}} \quad V^{\mathbb{P}}\Big(W_0 - X + f(X) - \pi(f(X)) \Big), \tag{8}$$

where π is a given expected value premium principle. This problem has been studied extensively for various functional forms of the preference relation $V^{\mathbb{P}}$ and premium principle π in the literature under homogeneous beliefs and deterministic initial wealth. For instance, if $V^{\mathbb{P}}$ is the expected utility of the insurer and the reinsurance premium principle is the expected-value premium principle, the resulting problem is studied by Arrow (1963), who shows that the stop-loss contract is optimal. Recently, Hong et al. (2011) and Hong (2018) studied the case where W_0 is stochastic. If $V^{\mathbb{P}}$ is an expected utility or dual utility functional and the reinsurance premium is determined by an expected value premium principle, Problem (8) has been studied by Young (1999) and Cui et al. (2013) with $\mathcal{F} = \mathcal{F}^{NS}$.

Throughout this paper, we assume that Problem (8) admits a solution:

Assumption 2.1 *Problem (8) admits a solution, for $\mathcal{F} = \mathcal{F}^{NS}$ or $\mathcal{F} = \mathcal{F}^{G}$.*

Remark 1 *Assumption 2.1 is justified in light of the existing literature on optimal reinsurance with one reinsurer. For instance, Boonen et al. (2016) show the if the functional $V^{\mathbb{P}}$ is continuous in a certain sense, then Problem (8) admits a solution, when $\mathcal{F} = \mathcal{F}^{NS}$. When $\mathcal{F} = \mathcal{F}^{G}$, and assuming in addition to monotonicity, that $V^{\mathbb{P}}$ preserves second-order stochastic dominance (SSD) and is continuous w.r.t. the L^1-norm topology, on every bounded subset of L^∞, the existence of solutions to Problem (8) follows from a proof similar to that of Theorem 5 of Carlier and Dana (2003), and it is therefore omitted. Additionally, Theorem 3 of Carlier and Dana (2005) provides a similar result. Hence, under mild technical conditions, Problem (8) admits a solution for $\mathcal{F} = \mathcal{F}^{G}$, under risk-averse preferences (that is, preferences that preserve SSD). An example of such risk-averse preferences is given by rank-dependent utility, where the distortion function is convex and the utility function concave (Chew et al., 1987).*

We proceed with defining the *representative reinsurer*. Consider a reinsurance market where the set of indemnity functions $\{f_i\}_{i=1}^n$ is a solution to the Problem (7). Suppose further there exists a reinsurer that uses a premium principle π such that:

- $\sum_{i=1}^n f_i$ solves Problem (8) with given premium principle π;
- $\sum_{i=1}^n \pi^{\zeta_i}(f_i(X)) = \pi(\sum_{i=1}^n f_i(X))$.

Then the reinsurer with pricing functional π is referred to as the *representative reinsurer*. We can solve Problem (7) by first focusing on the single reinsurer problem (8). Once we have established the representative reinsurer from Problem (8), this in turn gives us the total losses that will be ceded to the n reinsurers, jointly with the aggregate reinsurance premium for solving Problem (7). The remaining task is then to identify the optimal risk sharing among the n reinsurers, while ensuring that the total reinsurance premium charged by these n reinsurers is the same as the amount charged by the representative reinsurer.

For any given $f \in \mathcal{F}$, we define the following problem in which we minimize the total reinsurance premium given a total reinsurance contract $f(X)$:

$$\begin{array}{ll} \inf\limits_{\{f_i\}_{i=1}^n} & \sum_{i=1}^n \pi^{\zeta_i}(f_i(X)) \\ \text{s.t.} & \{f_i\}_{i=1}^n \subset \mathcal{F}, \text{ and } \sum_{i=1}^n f_i(X) = f(X), \ \mu\text{-a.s.} \end{array} \tag{9}$$

Given the total risk $f(X)$ that will be ceded to n reinsurers, the above optimization problem selects the least expensive reinsurance strategy for allocating the total risk to the n reinsurers.

In this paper, our focus is the multiple reinsurance design problem (7). In particular, we will analyze the construction of the optimal indemnity profile $\{f_i(X)\}_{i=1}^n$. In the next section, we will show that Problem (7) reduces to a case with one reinsurer as in Problem (8) - both with and without the no-sabotage condition. Problem (9) will play a crucial role in these characterizations of a representative reinsurer.

3 Existence of a Representative Reinsurer

In this section, we characterize the representative reinsurer for the case where the feasible set is given by general indemnity functions in $\mathcal{F}^G$ (Subsection 3.1), and for the case where the feasible set is under the no-sabotage condition $\mathcal{F}^{NS}$ (Subsection 3.2). Moreover, Subsection 3.3 provides a discussion of the differences between the two cases.

3.1 The General Case of $\mathcal{F}^G$

In this subsection, we study the case $\mathcal{F} = \mathcal{F}^G$. Before we show the existence of the representative reinsurer, we first specify its characteristics. For each $\omega \in \Omega$, we define

$$\widetilde{\zeta}(\omega) := \min_{1 \le i \le n} \zeta_i(\omega), \tag{10}$$

$$\mathcal{I}(\omega) := \operatorname*{argmin}_{1 \le j \le n} \zeta_j(\omega), \tag{11}$$

where ζ_i is defined in eq. (4). We will show that the premium principle of the representative reinsurer is given by

$$\pi(f(X)) = \pi^{\widetilde{\zeta}}(f(X)), \tag{12}$$

for all $f \in \mathcal{F}^G$. Note that $\mathbb{E}^\mu[\widetilde{\zeta}]$ is not necessarily equal to 1. In other words, $\widetilde{\zeta} d\mu$ is not necessarily a probability measure. The associated expected value premium principle $\hat{\pi}$ has parameters $d\mathbb{Q} = \widetilde{\zeta} d\mu / \mathbb{E}^\mu[\widetilde{\zeta}]$ and $\theta = \mathbb{E}^\mu[\widetilde{\zeta}] - 1$.

Define

$$\mathcal{F}^G(f) := \Big\{ \{f_i\}_{i=1}^n \subset \mathcal{F}^G : \{f_i\}_{i=1}^n \text{ solves (9) on } \mathcal{F} = \mathcal{F}^G \text{ for a given } f \Big\},$$

where $f \in \mathcal{F}^G$. In the following theorem, we characterize all optimal solutions to Problem (9). Its proof is given in Appendix A.

Theorem 3.1 *Let $\mathcal{F} = \mathcal{F}^G$, and fix $f \in \mathcal{F}^G$. Then, $\{f_i\}_{i=1}^n \in \mathcal{F}^G(f)$ if and only if the following two conditions hold simultaneously:*

(i) $\{f_i\}_{i=1}^n$ is such that for each i and for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$;

(ii) $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$ and $\sum_{j=1}^n f_j(X) = f(X)$, μ-a.s.

Theorem 3.1 shows the reinsurance contracts in $\mathcal{F}^G(f)$, that are defined per realization of the total risk $X(\omega)$. The $\mathcal{F}^G(f)$ shows the cheapest possible price that can be obtained in the market, if the insurer aims to reinsure the total indemnity profile $f(X)$. In this way, the loss $f(X)$ is ceded to a representative reinsurer that uses a "representative" premium principle. We show this in the next theorem. Its proof is given in Appendix B.

Theorem 3.2 *Let $\mathcal{F} = \mathcal{F}^G$. The following are equivalent:*

1. *$\{f_i\}_{i=1}^n$ is optimal for Problem (7);*
2. *$\sum_{i=1}^n f_i$ is optimal for Problem (8) with $\pi = \pi^{\tilde{\zeta}}$, and $\{f_i\}_{i=1}^n \in \mathcal{F}^G\left(\sum_{j=1}^n f_j\right)$.*

In Theorem 3.2, we can apply Theorem 3.1 to determine the solutions $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$ after we determined $\sum_{j=1}^n f_j$ as a solution to Problem (8) with $\pi = \pi^{\tilde{\zeta}}$.

Corollary 3.3 *Suppose that f^* is optimal for Problem (8) with $\pi = \pi^{\tilde{\zeta}}$. For each $i \in \{1, 2, \ldots, n\}$, let f_i be defined by:*

$$f_i(X(\omega)) = \begin{cases} 0 & \text{if } i \notin \mathcal{I}(\omega); \\ \frac{f^*(X(\omega))}{|\mathcal{I}(\omega)|} & \text{if } i \in \mathcal{I}(\omega), \end{cases} \tag{13}$$

where $|\mathcal{I}(\omega)|$ denotes the cardinality of the set $\mathcal{I}(\omega)$. Then $\{f_i\}_{i=1}^n$ is optimal for Problem (7).

Proof: First, note that $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$, since f^* is feasible for Problem (8) and $|\mathcal{I}(\omega)| \geq 1$, for each ω. Moreover, for each $\omega \in \Omega$,

$$\begin{aligned}\sum_{i=1}^n f_i(X(\omega)) &= \sum_{i \in \mathcal{I}(\omega)} f_i(X(\omega)) = \sum_{i \in \mathcal{I}(\omega)} \frac{f^*(X(\omega))}{|\mathcal{I}(\omega)|} = f^*(X(\omega)) \sum_{i \in \mathcal{I}(\omega)} \frac{1}{|\mathcal{I}(\omega)|} \\ &= f^*(X(\omega)) \times \frac{1}{|\mathcal{I}(\omega)|} \times |\mathcal{I}(\omega)| = f^*(X(\omega)).\end{aligned}$$

The rest follows from Theorem 3.2. ☐

3.2 The Case of $\mathcal{F}^{NS}$: The No-Sabotage Condition

We assume that $\mathcal{F} = \mathcal{F}^{NS}$, so we must have $\{f_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ and $\sum_{i=1}^n f_i \in \mathcal{F}^{NS}$. Moreover, it turns out to be more insightful to formulate the problem in terms of $\hat{\pi}^{\theta_i, \mathbb{Q}_i}$ instead of π^{ζ_i}. Now, define the set function $\upsilon : \Sigma \to \mathbb{R}^+$ and the collection $\hat{\mathcal{I}}(z)$ by

$$\upsilon(B) := \min_{1 \leq i \leq n} \Big\{(1 + \theta_i)\, \mathbb{Q}_i(B)\Big\}, \quad \forall\, B \in \Sigma, \tag{14}$$

$$\hat{\mathcal{I}}(z) := \operatorname*{argmin}_{1 \leq j \leq n} (1 + \theta_j)\, \mathbb{Q}_j(X > z), \quad \forall\, z \in [0, M], \tag{15}$$

where $(\theta_i, \mathbb{Q}_i)$ is defined in eq. (2). The function υ is also called a *capacity* on the space (Ω, Σ). Then, we define the following premium principle of the representative reinsurer:

$$\pi(f(X)) = \int_0^{f(M)} \upsilon(f(X) > z)\, dz, \tag{16}$$

for all[1] $f \in \mathcal{F}^{NS}$. We also define, for each $f \in \mathcal{F}^{NS}$, the collection

$$\mathcal{F}^{NS}(f) := \Big\{ \{f_i\}_{i=1}^n \subset \mathcal{F}^{NS} : \{f_i\}_{i=1}^n \text{ solves (9) on } \mathcal{F} = \mathcal{F}^{NS} \text{ for a given } f \Big\}.$$

The next theorem describes the set $\mathcal{F}^{NS}(f)$; the proof of which is given in Appendix C.

Theorem 3.4 *Let $\mathcal{F} = \mathcal{F}^{NS}$, and fix $f \in \mathcal{F}^{NS}$. Then, $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$ if and only if the following two conditions hold simultaneously:*

(i) $\{f_i\}_{i=1}^n$ is such that for each $i \in \{1, 2, \ldots, n\}$ and for each $x \in [0, M]$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. $z \in [0, M]$,

$$h_i(z) = 0 \quad \textit{whenever} \quad i \notin \hat{\mathcal{I}}(z), \tag{17}$$

(ii) $\{f_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ and $\sum_{i=1}^n h_i(z) = f'(z)$, for a.e. $z \in [0, M]$.

Under the No-sabotage condition, Theorem 3.4 shows that reinsurance contracts in $\mathcal{F}^{NS}(f)$ are determined by means of a characterization of $\{f_i'(z)\}_{i=1}^n$ for a.e. $z \in [0, M]$. These contracts are different then without the no-sabotage condition (cf. Theorem 3.1). The next theorem shows that an optimal reinsurer also exists under the no-sabotage condition.

Theorem 3.5 *Let $\mathcal{F} = \mathcal{F}^{NS}$. The following are equivalent:*

1. *$\{f_i\}_{i=1}^n$ is optimal for Problem (7);*
2. *$\sum_{i=1}^n f_i$ is optimal for Problem (8) with π as in eq. (16), and $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}\left(\sum_{j=1}^n f_j\right)$.*

Proof: If $f \in \mathcal{F}^{NS}$ and $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$, it follows from Theorem 3.4 that

$$\begin{aligned}
\sum_{i=1}^n \hat{\pi}^{\theta_i, \mathbb{Q}_i}(f_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^{f_i(M)} \mathbb{Q}_i(f_i(X) > z)dz = \sum_{i=1}^n (1+\theta_i) \int_0^M \mathbb{Q}_i(X > z)df_i(z) \\
&= \sum_{i=1}^n \int_0^M (1+\theta_i)\, \mathbb{Q}_i(X > z) f_i'(z)dz = \int_0^M \sum_{i=1}^n \Big[(1+\theta_i)\, \mathbb{Q}_i(X > z) f_i'(z)\Big] dz \\
&= \int_0^M \sum_{i=1}^n \min_{1 \le j \le n} \{(1+\theta_j)\, \mathbb{Q}_j(X > z)\} f_i'(z)dz \\
&= \int_0^M \sum_{i=1}^n \upsilon(X > z) f_i'(z)dz = \int_0^M \upsilon(X > z) \sum_{i=1}^n f_i'(z)dz \\
&= \int_0^M \upsilon(X > z) f'(z)dz = \int_0^M \upsilon(X > z)df(z) \\
&= \int_0^{f(M)} \upsilon(f(X) > z)dz = \pi(f(X)),
\end{aligned}$$

where the second and second-to-last equalities are due to Lemma 2.1 of Zhuang et al. (2016), and π is as in eq. (16). The rest of the proof is identical to the proof of Theorem 3.2, and it is hence omitted. ☐

[1] In this paper, we write $\upsilon(f(X) > z)$ to mean $\upsilon(\{\omega \in \Omega : f(X(\omega)) > z\})$.

3.3 $\mathcal{F}^G$ vs. $\mathcal{F}^{NS}$: A Comparison

In the previous two subsections, we show that the multiple reinsurers case (Problem (7)) reduces to a single reinsurer case (Problem (8)), where given the total reinsurance contract the reinsurers share this risk in an optimal way. This is shown in Theorem 3.2 in case reinsurance contracts can take general functional forms (the class $\mathcal{F}^G$), and in Theorem 3.5 in case the no-sabotage condition is imposed (the class $\mathcal{F}^{NS}$). We now illustrate two special cases of the representative reinsurer, with and without the no-sabotage condition.

3.3.1 Special Case 1

The first special case we examine is a situation in which there exists some event $A \in \Sigma$ such that $\mu(A) > 0$ and $\mathbb{Q}_i(A) = 0$ for some reinsurer i. Then $\int_A \zeta_i d\mu = 0$. Since $\zeta_i \geq 0$, it then follows that $\zeta_i(\omega) = 0$, for μ-a.e. $\omega \in A$. Therefore, for μ-a.e. $\omega \in A$, $\mathcal{I}(\omega) = \operatorname{argmin}_{1 \leq j \leq n} \zeta_j(\omega) = \{i\}$ and $\widetilde{\zeta}(\omega) = \min_{1 \leq j \leq n} \zeta_j(\omega) = 0$.

Assume first that $\mathcal{F} = \mathcal{F}^G$. Then there exist solutions $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$ to Problem (7) such that, for μ-a.e. $\omega \in A$, $f_j(X(\omega)) = 0$ for $j \neq i$ (Theorem 3.1). Moreover, μ-a.e. $\omega \in A$, it holds that $\widetilde{\zeta}(\omega) = 0$. From this and the assumption that $V^{\mathbb{P}}$ is strictly monotonic (with respect to $\mathbb{P}$-a.s. inequality), it follows that a possible solution $\sum_{k=1}^n f_k$ to Problem (8) satisfies $\sum_{k=1}^n f_k(X(\omega)) = X(\omega)$ for $\omega \in A$. Hence, Theorem 3.2 yields the existence of a solution to Problem (9) with $f_i(X(\omega)) = X(\omega)$ for $\omega \in A$. If reinsurer i does not charge any incremental premium to reinsure risk in A, then an optimal solution is to shift all risk to this reinsurer. The insurer will therefore exploit the opportunity to reinsure part of its risk to a reinsurer for no cost.

Now, assume that $\mathcal{F} = \mathcal{F}^{NS}$. The "arbitrage opportunity" described above will not necessarily be exploited when we impose the no-sabotage condition. If we would shift all risk to the insurer i, then in all states where X is above $x^A := \sup\{X(\omega) : \omega \in A\}$, the reinsurer will absorb at least the amount x^A, that is, $f_i(X(\omega)) \geq x^A$ for all $\omega \in \Omega$ such that $X(\omega) \geq x^A$. The indemnity profile at realization $X(\omega^*)$, denoted by $\{f_i(X(\omega^*))\}_{i=1}^n$, depends on the beliefs regarding the events $\{\omega \in \Omega : X(\omega) > z\}$ for all $z \in [0, X(\omega^*)]$ (see Theorem 3.4). Therefore, optimal reinsurance contracts are determined marginally.

3.3.2 Special Case 2

Next, we continue with the second special case. Suppose that there exists a reinsurer i such that $(1+\theta_i)\mathbb{Q}_i(X > z) < (1+\theta_j)\mathbb{Q}_j(X > z)$ for all $j \neq i$ and $z \in [0, M)$. First, we let $\mathcal{F} = \mathcal{F}^{NS}$. We obtain from (15) that $\hat{\mathcal{I}}(z) = \{i\}$ for all $z \in [0, M)$. It follows from Theorem 3.4 that if $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$ then for each $x \in [0, M]$, $f_i(x) = \int_0^x h_i(z)dz$ with $h_j(z) = 0$ for all $j \neq i$ and $z \in [0, M)$. Hence, $f_j(X) = 0$ for all $j \neq i$, and thus $f_i(X) = f(X)$. The optimal insurance problem with multiple reinsurers reduces to a single-reinsurer problem, with reinsurer i. Note that this does not hold for the case with $\mathcal{F} = \mathcal{F}^G$, which is illustrated in the following example.

Example 3.1 *Let $\mathcal{F} = \mathcal{F}^G$ and $\theta_i = 0$, for each $1 \leq i \leq n$. Suppose that the beliefs $\mathbb{Q}_i$ are such* *that, for each $1 \leq i \leq n$,*

$$\mathbb{Q}_i(X > z) = \begin{cases} \exp(-iz) & \text{if } z \in [0, M), \\ 0 & \text{if } z \geq M. \end{cases}$$

Consequently, this is a special case of the situation in which there exists a reinsurer i (namely Reinsurer $i = n$) such that $(1+\theta_i)\mathbb{Q}_i(X > z) < (1+\theta_j)\mathbb{Q}_j(X > z)$ for all $j \neq i$ and $z \in [0, M)$. Note that $\zeta_i = \frac{d\mathbb{Q}_i}{d\mu} = \frac{d\mathbb{Q}_i}{d\mathbb{Q}_1}\frac{d\mathbb{Q}_1}{d\mu} = i\exp((1-i)X)\frac{d\mathbb{Q}_1}{d\mu}$, whenever $X \in [0, M)$. Therefore, (11) implies that

$$\mathcal{I}(\omega) = \underset{1\leq j\leq n}{\operatorname{argmin}}\, \zeta_j(\omega) = \underset{1\leq j\leq n}{\operatorname{argmin}}\, j\exp((1-j)X(\omega)),$$

for all $\omega \in \Omega$ such that $X(\omega) \in [0, M)$. For instance, if $X(\omega) = 0$ then $\mathcal{I}(\omega) = \{1\}$. More generally, if $X(\omega) \in \left[0, \frac{1}{n}\right]$ then $\mathcal{I}(\omega) = \{1\}$, and if $X(\omega) \in [1, M)$ then $\mathcal{I}(\omega) = \{n\}$. Moreover, if $X(\omega) = M$ we find that $\mathcal{I}(\omega) = \operatorname{argmin}_{1\leq j\leq n}\widetilde{\zeta}(\omega) = \operatorname{argmin}_{1\leq j\leq n}\exp(-jM)/\mu(\omega) = \{n\}$.

Theorem 3.1 then provides the optimal reinsurance contracts $\{f_i\}_{i=1}^n \in \mathcal{F}^G(f)$ for a given $f \in \mathcal{F}^G$. For instance, if $n = 3$ and $M > \ln(3)/2$, then it follows that for each $\omega \in \Omega$, Reinsurer 1 reinsures the risk $f_1(X(\omega)) = f(X(\omega))1_{X(\omega)<\ln(3)/2}$, Reinsurer 2 reinsures no risk ($f_2(X(\omega)) = 0$), and Reinsurer 3 reinsures the risk $f_3(X(\omega)) = f(X(\omega))1_{X(\omega)>\ln(3)/2}$. If $X(\omega) = \ln(3)/2$, then Reinsurers 1 and 3 share the risk $f(X)$ in any way that is feasible.

4 Three Examples for $V^{\mathbb{P}}$

In this section, we return to Problem (8), where there is one (representative) reinsurer. By means of our representative reinsurer results (Theorems 3.1, 3.2, 3.4, and 3.5), this then leads to a full description of the solutions to the original problem, Problem (7). We study three special choices for the preference relation $V^{\mathbb{P}}$ of the insurer: the expectation, dual utility, and expected utility.

4.1 The Case of an Expected-Value-Maximizing Insurer

A classical approach to model preferences of a firm is given by maximizing expected net worth. Let $V^{\mathbb{P}}(Y) = \mathbb{E}^{\mathbb{P}}[Y]$ for any $Y \in L^\infty$. This preference relation is also chosen because it allows us to obtain a full description of the optimal reinsurance contracts in Problem (8), and thus Problem (7), both with and without the no-sabotage condition.

4.1.1 Without the No-Sabotage Condition

Let $\mathcal{F} = \mathcal{F}^G$. Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$ is given by

$$\sup_{f\in\mathcal{F}^G} \mathbb{E}^{\mathbb{P}}\Big[W_0 - X + f(X) - \pi^{\widetilde{\zeta}}(f(X))\Big]. \tag{18}$$

The objective function in Problem (18) can be simplified as follows:

$$\mathbb{E}^{\mathbb{P}}\Big[W_0 - X + f(X) - \pi^{\widetilde{\zeta}}(f(X))\Big] = \mathbb{E}^{\mathbb{P}}[W_0 - X] + \mathbb{E}^{\mathbb{P}}\Big[f(X)\Big] - \pi^{\widetilde{\zeta}}(f(X)), \tag{19}$$

since $\pi^{\widetilde{\zeta}}(f(X))$ is a constant. We can ignore $\mathbb{E}^{\mathbb{P}}[W_0 - X]$ in eq. (19) since it does not depend on $f(X)$. Define $\zeta_0 := d\mathbb{P}/d\mu$. Then, the objective function in eq. (19) can then be replaced by

$$\mathbb{E}^{\mathbb{P}}[f(X)] - \pi^{\widetilde{\zeta}}(f(X)) = \mathbb{E}^{\mu}[f(X)\zeta_0] - \mathbb{E}^{\mu}[f(X)\widetilde{\zeta}] = \mathbb{E}^{\mu}[f(X)(\zeta_0 - \widetilde{\zeta})].$$

Consequently, an equivalent formulation of Problem (18) is given by

$$\sup_{f \in \mathcal{F}^G} \int_{\Omega} \left(\zeta_0 - \widetilde{\zeta}\right) f(X) d\mu. \tag{20}$$

This yields the following result, and the proof of which is given in Appendix D.

Proposition 4.1 *Let $\mathcal{F} = \mathcal{F}^G$ and $V^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}$. Then, $\{f_i\}_{i=1}^n$ solves Problem (7) if and only if the following three conditions hold simultaneously:*

(i) $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$;

(ii) $\sum_{i=1}^n f_i \in \mathcal{F}^G$; *and,*

(iii) $\{f_i\}_{i=1}^n$ *is such that for each i and for μ-a.e. $\omega \in \Omega$, we have*

$$\begin{cases} f_i(X(\omega)) = 0 & \text{whenever } \ i \notin \mathcal{I}(\omega) \text{ or } \zeta_0(\omega) < \zeta_i(\omega); \\ \sum_{i=1}^n f_i(X(\omega)) = X(\omega) & \text{whenever } \zeta_0(\omega) > \widetilde{\zeta}(\omega). \end{cases}$$

4.1.2 With the No-Sabotage Condition

We now solve Problem (7) with $\mathcal{F} = \mathcal{F}^{NS}$: under the no-sabotage condition. Then, Problem (8) with pricing function $\pi(f(X)) = \int_0^{f(M)} \upsilon(f(X) > z)\}dz$ is given by

$$\sup_{f \in \mathcal{F}^{NS}} \ \mathbb{E}^{\mathbb{P}}\Big[W_0 - X + f(X) - \pi(f(X))\Big]. \tag{21}$$

This objective function can be simplified as follows:

$$\mathbb{E}^{\mathbb{P}}\Big[W_0 - X + f(X) - \pi(f(X))\Big] = \mathbb{E}^{\mathbb{P}}[W_0 - X] + \mathbb{E}^{\mathbb{P}}\Big[f(X)\Big] - \pi(f(X)), \tag{22}$$

since $\pi(f(X))$ is deterministic. We ignore $\mathbb{E}^{\mathbb{P}}[W_0 - X]$ in eq. (22) since it does not depend on $f(X)$. Then, by Lemma 2.1 of Zhuang et al. (2016), eq. (22) can be written as

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[f(X)] - \pi(f(X)) &= \int_0^{f(M)} \mathbb{P}(f(X) > z)dz - \int_0^{f(M)} \upsilon(f(X) > z)dz \\ &= \int_0^{M} \mathbb{P}(X > z)df(z) - \int_0^{M} \upsilon(X > z)df(z) \\ &= \int_0^{M} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] df(z) = \int_0^{M} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] f'(z)dz. \end{aligned}$$

Consequently, an equivalent formulation of Problem (21) is given by

$$\sup_{f\in\mathcal{F}^{NS}} \int_0^M \left[\mathbb{P}(X>z)-\upsilon(X>z)\right] f'(z)dz. \tag{23}$$

This yields the following result, and the proof of which is given in Appendix E.

Proposition 4.2 *Let $\mathcal{F}=\mathcal{F}^{NS}$ and $V^{\mathbb{P}}=\mathbb{E}^{\mathbb{P}}$. Then, $\{f_i\}_{i=1}^n$ solves Problem (7) if and only if the following three conditions hold simultaneously:*

(i) $\{f_i\}_{i=1}^n \subset \mathcal{F}^{NS}$;

(ii) $\sum_{i=1}^n f_i \in \mathcal{F}^{NS}$; *and,*

(iii) For each $x\in[0,M]$, $f_i(x)=\int_0^x h_i(z)dz$, where for a.e. $z\in[0,M]$, we have

$$\begin{cases} h_i(z)=0 & \text{whenever } \; i\notin\hat{\mathcal{I}}(z) \text{ or } (1+\theta_i)\,\mathbb{Q}_i(X>z)>\mathbb{P}(X>z); \\ \sum_{i=1}^n h_i(z)=1 & \text{whenever } \upsilon(X>z)<\mathbb{P}(X>z). \end{cases}$$

4.2 The Case of a Dual-Utility-Maximizing Insurer under the No-Sabotage Condition

Dual utility (Yaari, 1987) is given by a Choquet integral:

$$V^{\mathbb{P}}(Y)=\int Y d\left(g\circ\mathbb{P}\right) := \int_{-\infty}^{0}\left(g(\mathbb{P}(Y>z))-1\right)dz+\int_0^\infty g(\mathbb{P}(Y>z))dz, \text{ for any } Y\in L^\infty, \tag{24}$$

where g is right-continuous and increasing, $g(0)=0$, and $g(1)=1$. So, $V^{\mathbb{P}}(Y)=\mathbb{E}^{\mathbb{Q}_Y}[Y]$ with $\mathbb{Q}_Y(Y>z):=g(\mathbb{P}(Y>z))$, $z\in[0,M]$: a distortion of the probability measure $\mathbb{P}$. Hence, for all $Y\in L^\infty$ that are comonotonic[2] with X, we have $V^{\mathbb{P}}(Y)=\mathbb{E}^{\mathbb{Q}_X}[Y]$. Assume that the initial wealth of the insurer W_0 is a constant. For $f\in\mathcal{F}^{NS}$, the risk $W_0-X+f(X)-\pi(f(X))$ is comonotonic with $-X$, and thus Problem (8) with $\mathcal{F}=\mathcal{F}^{NS}$ writes as

$$\sup_{f\in\mathcal{F}^{NS}} \mathbb{E}^{\mathbb{Q}_{-X}}\Big(W_0-X+f(X)-\pi(f(X))\Big). \tag{25}$$

This objective function is identical to the one in Section 4.1.2, but with a different probability measure. Thus, the optimal reinsurance contracts under the no-sabotage condition are given in Proposition 4.2, where we replace $\mathbb{P}$ by the distorted probability measure $\mathbb{Q}_{-X}$.

Remark 2 *If $n=1$ (i.e., one reinsurer), W_0 is constant, $\mathbb{Q}_1(X>z)=(1+\theta)(1-g_1(1-\mathbb{P}(X>z)))$ for $\theta\geq 0$, g is continuous and non-decreasing with $g(0)=0$ and $g(1)=1$, and $V^{\mathbb{P}}$ is a dual utility functional, then the solution of Problem (7) with feasible set $\mathcal{F}^{NS}$ is given by Cui et al. (2013), Assa (2015), and Zhuang et al. (2016).*

[2] Random variables $Z_1,Z_2\in L^\infty$ are said to be comonotonic if $\left(Z_1(\omega)-Z_1(\omega')\right)\left(Z_2(\omega)-Z_2(\omega')\right)\geq 0$ for $\mu\otimes\mu$-a.e. $(\omega,\omega')\in\Omega\times\Omega$.

4.3 The Case of an Expected-Utility-Maximizing Insurer under the No-Sabotage Condition

Another classical approach to modelling the ceding insurer's preferences is given by expected utility of net worth. Let $\mathcal{F} = \mathcal{F}^{NS}$, W_0 is a constant, and $V^{\mathbb{P}}(Y) = \mathbb{E}^{\mathbb{P}}\left[u\left(Y\right)\right]$ for any $Y \in L^{\infty}$, where u is a given increasing and concave utility function. Then, Problem (8) with premium principle π as in eq. (16) is given by

$$\sup_{f \in \mathcal{F}^{NS}} \mathbb{E}^{\mathbb{P}}\Big[u\Big(W_0 - X + f(X) - \pi(f(X))\Big)\Big]. \tag{26}$$

The function υ is defined in eq. (14) by $\upsilon\left(B\right) = \min\limits_{1 \le i \le n}\Big\{\left(1+\theta_i\right)\mathbb{Q}_i(B)\Big\}$, for all $B \in \Sigma$, and it is a convex (supermodular) capacity on the space (Ω, Σ). Now, recall that

$$M = \inf\{a \in \mathbb{R} : \mu(X > a) = 0\} < \infty,$$

and let $M_i := \inf\{a \in \mathbb{R} : \mathbb{Q}_i(X > a) = 0\}$, for each $i = 1, \ldots, n$. Then $M_i \le M$, for each $i = 1, \ldots, n$; and

$$M_\upsilon := \inf\{a \in \mathbb{R} : \upsilon(X > a) = 0\} = \min_{1 \le i \le n} M_i.$$

Similarly, $M_{\mathbb{P}} := \inf\{a \in \mathbb{R} : \mathbb{P}(X > a) = 0\} \le M$.

Define the Hazard Ratio HR by

$$HR\left(z\right) = \frac{\upsilon\left(X > z\right)}{\mathbb{P}\left(X > z\right)}, \quad \text{for all } z \in [0, \max\left(M_{\mathbb{P}}, M_\upsilon\right)). \tag{27}$$

The following result shows that when the hazard ratio is non-increasing on its domain[3], the optimal solution to Problem (26) is a linear deductible, and hence an optimal solution to Problem (7) is given by the splitting of this linear deductible indemnity function between the n reinsurers in a way that minimizes the cost of reinsurance. Its proof is given in Appendix F.

Theorem 4.3 *Let $\mathcal{F} = \mathcal{F}^{NS}$ and W_0 is a constant. For each $d \ge 0$, define the function $\hat{f}_d \in \mathcal{F}^{NS}$ by $\hat{f}_d(x) := (x-d)^+$, for each $x \ge 0$. If the hazard rate HR is non-increasing on its domain, then there exists some $d^* \ge 0$ such that an optimal solution for Problem (7) is given by the collection $\{f_i\}_{i=1}^n$ defined as follows:*

(i) For each $i \in \{1, 2, \ldots, n\}$ and for each $x \ge 0$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. z, $h_i(z) = 0$ whenever $i \notin \hat{\mathcal{I}}(z)$; and,

(ii) $\sum_{i=1}^n h_i(z) = \hat{f}'_{d^*}(z)$, for a.e. z.

Remark 3 *Inspired by Example 3.1, we provide an example that illustrates a simple special case in which HR is non-increasing on its domain. For each $i \in \{1, 2, \ldots, n\}$, we assume $\theta_i = 0$, $\lambda_i > 0$,*

$$\mathbb{Q}_i(X > z) = \begin{cases} \exp(-\lambda_i z) & \text{if } z \in [0, M), \\ 0 & \text{if } z \ge M, \end{cases}$$

[3] Note that if then hazard ratio is non-increasing on its domain, then $M_\upsilon \le M_{\mathbb{P}}$.

and, moreover, for $\lambda_I > 0$,

$$\mathbb{P}(X > z) = \begin{cases} \exp(-\lambda_I z) & \text{if } z \in [0, M), \\ 0 & \text{if } z \geq M. \end{cases}$$

Let $i^* \in \{1, 2, \ldots, n\}$ *be the reinsurer with the largest exponential parameter, that is,* $\lambda_{i^*} = \max_{1 \leq j \leq n} \lambda_j$. *Then, it follows that for all* $z \in [0, M)$,

$$\upsilon(X > z) = \min_{1 \leq j \leq n} (1 + \theta_j)\, \mathbb{Q}_j(X > z) = \min_{1 \leq j \leq n} \exp(-\lambda_j z) = \exp(-\lambda_{i^*} z) = \mathbb{Q}_{i^*}(X > z).$$

Hence, if $\lambda_{i^*} \geq \lambda_I$, *then for all* $z \in [0, M)$,

$$HR(z) = \frac{\exp(-\lambda_{i^*} z)}{\exp(-\lambda_I z)} = \exp((\lambda_I - \lambda_{i^*})z).$$

Since $\lambda_I - \lambda_{i^*} \leq 0$, *the function* HR *is non-increasing.*

More generally, the result of Theorem 4.3 still holds if the functional $V^{\mathbb{P}} : L^\infty \to \mathbb{R}$ preserves second-order stochastic dominance (w.r.t. $\mathbb{P}$), but is not necessarily an expected utility functional. Recall that a mapping $\rho : L^\infty \to \mathbb{R}$ is said to preserve second-order stochastic dominance (SSD) w.r.t. $\mathbb{P}$ if for all $Z_1, Z_2 \in L^\infty$, $Z_1 \succcurlyeq_{ssd} Z_2 \Longrightarrow \rho(Z_1) \geq \rho(Z_2)$; where $Z_1 \succcurlyeq_{ssd} Z_2$ if and only if $\mathbb{E}^{\mathbb{P}}[\eta(Z_1)] \geq \mathbb{E}^{\mathbb{P}}[\eta(Z_2)]$, for all non-decreasing and concave functions η for which the expectations exist. The following result provides such an extension of Theorem 4.3, as well as a converse in the general case. The proof is similar to that of Theorem 4.1 in Chi (2019), and it is therefore omitted.

Lemma 4.4 *Let* W_0 *be a constant. The following are equivalent:*

(i) For any functional $V^{\mathbb{P}}$ *that preserves SSD w.r.t.* $\mathbb{P}$, *any indemnity function in* $\mathcal{F}^{NS}$ *for Problem (26) is suboptimal to a linear deductible with the same premium;*

(ii) The hazard ratio HR *is non-increasing on its domain.*

As a result of Proposition 4.4 and Theorems 3.4 and 3.5, we obtain the following result, the proof of which is omitted.

Proposition 4.5 *Let* $\mathcal{F} = \mathcal{F}^{NS}$, W_0 *a constant, and* $\hat{f}_d \in \mathcal{F}^{NS}$ *as in Theorem 4.3. If the hazard rate* HR *is non-increasing on its domain, and if the functional* $V^{\mathbb{P}} : L^\infty \to \mathbb{R}$ *preserves SSD w.r.t.* $\mathbb{P}$, *then there exists some* $d^* \geq 0$ *such that an optimal solution for Problem (7) is given by the collection* $\{f_i\}_{i=1}^n$ *defined as follows:*

(a) For each $i \in \{1, 2, \ldots, n\}$ *and for each* $x \geq 0$, $f_i(x) = \int_0^x h_i(z)dz$, *where for a.e.* z, $h_i(z) = 0$ *whenever* $i \notin \hat{\mathcal{I}}(z)$; *and,*

(b) $\sum_{i=1}^n h_i(z) = \hat{f}'_{d^*}(z)$, *for a.e.* z.

5 Conclusion

This paper studied optimal reinsurance contract design with multiple reinsurers and heterogeneous beliefs. The existence of a representative reinsurer is shown in case we allow for general reinsurance contracts, and in case we impose the no-sabotage condition (Carlier and Dana, 2003). To the best of knowledge, this is one of the first papers to explicitly derive optimal reinsurance contracts with and without the no-sabotage condition, and for general preferences of the insurer.

For instance in the general case, if a reinsurer assigns a state-price of zero to an event, then it is optimal to shift all risk to this reinsurer. This is natural, as it can be seen as an arbitrage opportunity. However, under the no-sabotage condition, this is not necessarily the case: allocating a specific risk to a reinsurer may affect the entire reinsurance contract.

In this paper, we also provide a complete characterization of optimal reinsurance contracts in case the insurer is risk-neutral expected utility maximizer, both with and without the no-sabotage condition. In the case of dual utility preferences for the insurer, we explicitly derive the optimal reinsurance contracts under the no-sabotage condition. Moreover, under the no-sabotage condition and the technical condition of monotone hazard ratios, Chi (2019) derives the optimal reinsurance contract in the case of one reinsurer. This paper provides a similar result in the case of multiple reinsurers, using the existence of a representative reinsurer. Extending these two results on dual and expected utilities to the case without the no-sabotage condition is an important open question that is quite relevant to the understanding of the effect of belief heterogeneity. We leave this open for further research.

Appendices

A Proof of Theorem 3.1

We start with the "if" part. Fix $f \in \mathcal{F}^G$. Suppose that $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$ is such that for each $i = 1, 2, \ldots, n$ and for μ-a.e. $\omega \in \Omega$ we have

$$f_i(X(\omega)) = \begin{cases} 0 & \text{if } i \notin \mathcal{I}(\omega); \\ \lambda_i(X(\omega)) & \text{if } i \in \mathcal{I}(\omega), \end{cases} \tag{28}$$

where $\{\lambda_i\} \subset \mathcal{F}^G$ is such that $\sum_{i=1}^n f_i = f$, μ-a.s. Then $\{f_i\}_{i=1}^n$ is feasible for Problem (9), by construction. To show optimality of $\{f_i\}_{i=1}^n$ for Problem (9), let $\{g_i\}_{i=1}^n$ be also feasible for Problem (9). Then,

$$\sum_{i=1}^n f_i(X) = \sum_{i=1}^n g_i(X) = f(X), \ \mu\text{-a.s.}$$

With $\widetilde{\zeta}$ as in eq. (10), it follows that for μ-a.e. $\omega \in \Omega$,

$$\begin{aligned} \sum_{i=1}^n f_i(X(\omega))\, \zeta_i(\omega) &= \sum_{i \in \mathcal{I}(\omega)} \lambda_i(X(\omega))\, \widetilde{\zeta}(\omega) = \widetilde{\zeta}(\omega) \sum_{i \in \mathcal{I}(\omega)} \lambda_i(X(\omega)) = \widetilde{\zeta}(\omega) \sum_{i=1}^n f_i(X(\omega)) \\ &= \widetilde{\zeta}(\omega) \sum_{i=1}^n g_i(X(\omega)) = \sum_{i=1}^n g_i(X(\omega))\, \widetilde{\zeta}(\omega) \leq \sum_{i=1}^n g_i(X(\omega))\, \zeta_i(\omega). \end{aligned}$$

Consequently,

$$\sum_{i=1}^n \pi^{\zeta_i}(f_i(X)) = \int_\Omega \sum_{i=1}^n f_i(X)\, \zeta_i \, d\mu \leq \int_\Omega \sum_{i=1}^n g_i(X)\, \zeta_i \, d\mu = \sum_{i=1}^n \pi^{\zeta_i}(g_i(X)),$$

that is, $\{f_i\}_{i=1}^n \in \mathcal{F}^G(f)$.

We proceed with the "only if" part. Let $\{f_i\}_{i=1}^n \in \mathcal{F}^G(f)$. Then $\sum_{j=1}^n f_j(X) = f(X)$, μ-a.s., by definition of $\mathcal{F}^G(f)$. For each $\omega \in \Omega$, define the set $\mathcal{J}(\omega)$ by

$$\mathcal{J}(\omega) := \Bigg\{ i \in \{1, 2, \ldots, n\} : i \notin \mathcal{I}(\omega),\, f_i(X(\omega)) > 0 \Bigg\}.$$

Suppose, by way of contradiction, that there exists $A^* \in \Sigma$ such that $\mathcal{J}(\omega) \neq \varnothing$, for each $\omega \in A^*$, and $\mu(A^*) > 0$. Fix $\omega^* \in A^*$. Then $\mathcal{J}(\omega^*) \neq \varnothing$, and for each $i \in \mathcal{J}(\omega^*)$, we have $i \notin \mathcal{I}(\omega^*)$ and $f_i(X(\omega^*)) > 0$.

Define the collection $\{g_i^{\omega^*}(X(\omega^*))\}_{i=1}^n \subset \mathbb{R}^+$ by

$$g_i^{\omega^*}(X(\omega^*)) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(\omega^*); \\ f_i(X(\omega^*)) = 0 & \text{for all } i \in \{1,2,\dots,n\} \setminus (\mathcal{J}(\omega^*) \bigcup \mathcal{I}(\omega^*)); \\ f_i(X(\omega^*)) + \frac{\sum_{j\in\mathcal{J}(\omega^*)} f_j(X(\omega^*))}{|\mathcal{I}(\omega^*)|} & \text{for all } i \in \mathcal{I}(\omega^*), \end{cases} \tag{29}$$

where $|\mathcal{I}(\omega^*)|$ denotes the cardinality of the set $\mathcal{I}(\omega^*)$. Then, by construction, $g_i^{\omega^*}(X(\omega^*)) = 0$ for all $i \notin \mathcal{I}(\omega^*)$, and

$$\sum_{i=1}^n g_i^{\omega^*}(X(\omega^*)) = \sum_{i\in\mathcal{I}(\omega^*)} \left(f_i(X(\omega^*)) + \frac{\sum_{j\in\mathcal{J}(\omega^*)} f_j(X(\omega^*))}{|\mathcal{I}(\omega^*)|} \right) = \sum_{i=1}^n f_i(X(\omega^*)).$$

Moreover, by eq. (29), recalling $\widetilde{\zeta}(\omega^*)$ from eq. (10), we have

$$\begin{aligned} \sum_{i=1}^n g_i^{\omega^*}(X(\omega^*))\zeta_i(\omega^*) &= \sum_{i\in\mathcal{I}(\omega^*)} g_i^{\omega^*}(X(\omega^*))\zeta_i(\omega^*) = \sum_{i\in\mathcal{I}(\omega^*)} g_i^{\omega^*}(X(\omega^*))\widetilde{\zeta}(\omega^*) \\ &= \sum_{i\in\mathcal{I}(\omega^*)} f_i(X(\omega^*))\widetilde{\zeta}(\omega^*) + \sum_{i\in\mathcal{I}(\omega^*)} \left(\frac{\sum_{j\in\mathcal{J}(\omega^*)} f_j(X(\omega^*))}{|\mathcal{I}(\omega^*)|} \right) \widetilde{\zeta}(\omega^*) \\ &= \sum_{i\in\mathcal{I}(\omega^*)} f_i(X(\omega^*))\widetilde{\zeta}(\omega^*) + \sum_{j\in\mathcal{J}(\omega^*)} f_j(X(\omega^*))\widetilde{\zeta}(\omega^*) \\ &< \sum_{i\in\mathcal{I}(\omega^*)} f_i(X(\omega^*))\widetilde{\zeta}(\omega^*) + \sum_{j\in\mathcal{J}(\omega^*)} f_j(X(\omega^*))\zeta_j(\omega^*) \\ &= \sum_{i=1}^n f_i(X(\omega^*))\zeta_i(\omega^*). \end{aligned}$$

For each $\omega \in A^*$, construct the collection $\{g_i^{\omega}(X(\omega))\}_{i=1}^n \subset \mathbb{R}^+$ as in eq. (29), so that in particular

$$\sum_{i=1}^n g_i^{\omega}(X(\omega))\zeta_i(\omega) < \sum_{i=1}^n f_i(X(\omega))\zeta_i(\omega), \quad \text{for all} \ \ \omega \in A^*. \tag{30}$$

Moreover, since $\sum_{j=1}^n f_j(X) = f(X)$, μ-a.s., since $\sum_{i=1}^n g_i^{\omega}(X(\omega)) = \sum_{i=1}^n f_i(X(\omega))$ for each $\omega \in A^*$, and since each $g_i^{\omega}(X(\omega)) \geq 0$ for each $\omega \in A^*$, by construction, it follows that $0 \leq g_i^{\omega}(X(\omega)) \leq X(\omega)$, for μ-a.e. $\omega \in A^*$.

Let $\{h_i\}_{i=1}^n$ be defined by, for each $i \in \{1,2,\dots,n\}$,

$$h_i(X(\omega)) = \begin{cases} f_i(X(\omega)) & \text{if } \omega \notin A^*; \\ g_i^{\omega}(X(\omega)) & \text{if } \omega \in A^*. \end{cases} \tag{31}$$

Then for each $\omega \in A^*$,

$$\sum_{i=1}^{n} h_i(X(\omega)) = \sum_{i=1}^{n} g_i^{\omega}(X(\omega)) = \sum_{i=1}^{n} f_i(X(\omega)),$$

and for each $\omega \notin A^*$

$$\sum_{i=1}^{n} h_i(X(\omega)) = \sum_{i=1}^{n} f_i(X(\omega)).$$

Consequently, since $\sum_{j=1}^{n} f_j(X) = f(X)$, μ-a.s., it follows that $\sum_{j=1}^{n} h_j(X) = f(X)$, μ-a.s. Moreover, by construction, we have $0 \leq h_i(X) \leq X$, μ-a.s., for each $i \in \{1, 2, \ldots, n\}$. Finally, by eq. (30), we have

$$\begin{cases} \displaystyle\sum_{i=1}^{n} h_i(X(\omega))\zeta_i(\omega) < \sum_{i=1}^{n} f_i(X(\omega))\zeta_i(\omega), & \text{for all } \ \omega \in A^*; \\ \displaystyle\sum_{i=1}^{n} h_i(X(\omega))\zeta_i(\omega) = \sum_{i=1}^{n} f_i(X(\omega))\zeta_i(\omega), & \text{for all } \ \omega \notin A^*. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} \pi^{\zeta_i}(h_i(X)) &= \int_{\Omega} \sum_{i=1}^{n} h_i\,(X)\,\zeta_i \; d\mu = \int_{A^*} \sum_{i=1}^{n} h_i\,(X)\,\zeta_i \; d\mu + \int_{\Omega\setminus A^*} \sum_{i=1}^{n} h_i\,(X)\,\zeta_i \; d\mu \\ &= \int_{A^*} \sum_{i=1}^{n} h_i\,(X)\,\zeta_i \; d\mu + \int_{\Omega\setminus A^*} \sum_{i=1}^{n} f_i\,(X)\,\zeta_i \; d\mu \\ &< \int_{A^*} \sum_{i=1}^{n} f_i\,(X)\,\zeta_i \; d\mu + \int_{\Omega\setminus A^*} \sum_{i=1}^{n} f_i\,(X)\,\zeta_i \; d\mu \\ &= \int_{\Omega} \sum_{i=1}^{n} f_i\,(X)\,\zeta_i \; d\mu = \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X)), \end{aligned}$$

which contradicts the assumption that $\{f_i\}_{i=1}^{n} \in \mathcal{F}^G(f)$.

B Proof of Theorem 3.2

First, suppose that $\sum_{i=1}^{n} f_i$ is optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$, and that $\{f_i\}_{i=1}^{n} \in \mathcal{F}\left(\sum_{i=1}^{n} f_i\right)$. Then $\{f_i\}_{i=1}^{n}$ is clearly feasible for Problem (7). To show optimality of $\{f_i\}_{i=1}^{n}$ for Problem (7), suppose by way of contradiction that $\{f_i\}_{i=1}^{n}$ is not optimal for Problem (7). Then there exists some collection $\{\hat{f}_i\}_{i=1}^{n} \subset \mathcal{F}^G$ such that

$$V^{\mathbb{P}}\Big(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(\hat{f}_i(X))\Big) > V^{\mathbb{P}}\Big(W_0 - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X))\Big).$$

Now,

$$\sum_{i=1}^{n} \pi^{\zeta_i}(\hat{f}_i(X)) = \sum_{i=1}^{n} \mathbb{E}^{\mu}\left[\hat{f}_i(X)\zeta_i\right] \geq \sum_{i=1}^{n} \mathbb{E}^{\mu}\left[\hat{f}_i(X)\widetilde{\zeta}\right] = \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} \hat{f}_i(X)\right).$$

Consequently,

$$\begin{aligned} V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} \hat{f}_i(X)\right)\right) &\geq V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(\hat{f}_i(X))\right) \\ &> V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X))\right). \end{aligned}$$

Now, for each ω, recall $\mathcal{I}(\omega)$ from eq. (11). Then, since $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{i=1}^n f_i\right)$, Theorem 3.1 implies that $\{f_i\}_{i=1}^n$ is such that for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$. Therefore, for μ-a.e. $\omega \in \Omega$,

$$\sum_{i=1}^{n} f_i(X(\omega))\zeta_i(\omega) = \sum_{i \in \mathcal{I}(\omega)} f_i(X(\omega))\zeta_i(\omega) = \sum_{i \in \mathcal{I}(\omega)} f_i(X(\omega))\widetilde{\zeta}(\omega) = \sum_{i=1}^{n} f_i(X(\omega))\widetilde{\zeta}(\omega).$$

Consequently,

$$\sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X)) = \sum_{i=1}^{n} \mathbb{E}^{\mu}\left[f_i(X)\zeta_i\right] = \sum_{i=1}^{n} \mathbb{E}^{\mu}\left[f_i(X)\widetilde{\zeta}\right] = \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} f_i(X)\right).$$

Hence,

$$\begin{aligned} V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} \hat{f}_i(X)\right)\right) &> V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X))\right) \\ &= V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} f_i(X)\right)\right), \end{aligned}$$

contradicting the optimality of $\sum_{i=1}^n f_i$ for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$. Hence, $\{f_i\}_{i=1}^n$ is optimal for Problem (7).

Conversely, let $\{f_i\}_{i=1}^n$ be optimal for Problem (7), and let $f^* := \sum_{j=1}^n f_j$. Suppose, by way of contradiction, that either f^* is not optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$, or that $\{f_i\}_{i=1}^n \notin \mathcal{F}(f^*)$:

- First, we assume that f^* is not optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$. Then there exists some $\hat{f}^*$ that is optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$. Also, by Theorem 3.1, there exists some $\{\hat{f}_i\}_{i=1}^n$ such that $\{\hat{f}_i\}_{i=1}^n \in \mathcal{F}^G(\hat{f}^*)$. It then follows from the first part of this proof that $\{\hat{f}_i\}_{i=1}^n$ is optimal for Problem (7). Hence, as in the first part of this proof,

$$V^{\mathbb{P}}\left(W_0 - X + \hat{f}^*(X) - \pi^{\widetilde{\zeta}}\left(\hat{f}^*(X)\right)\right) = V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} \hat{f}_i(X)\right)\right)$$

$$
\begin{aligned}
&= V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(\hat{f}_i(X))\right) \\
&= V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X))\right) \\
&\leq V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \pi^{\widetilde{\zeta}}\left(\sum_{i=1}^{n} f_i(X)\right)\right) \\
&= V^{\mathbb{P}}\left(W_0 - X + f^*(X) - \pi^{\widetilde{\zeta}}\left(f^*(X)\right)\right),
\end{aligned}
$$

where the third equality follows from the assumption that $\{f_i\}_{i=1}^n$ is optimal for Problem (7). However, this contradicts the assumption that f^* is not optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$.

- Second, suppose that $\{f_i\}_{i=1}^n \notin \mathcal{F}(f^*)$, and, by Theorem 3.1, choose $\{\hat{f}_i\}_{i=1}^n \in \mathcal{F}^G(f^*)$. Then, by strict monotonicity of $V^{\mathbb{P}}$,

$$
V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} \hat{f}_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(\hat{f}_i(X))\right) > V^{\mathbb{P}}\left(W_0 - X + \sum_{i=1}^{n} f_i(X) - \sum_{i=1}^{n} \pi^{\zeta_i}(f_i(X))\right),
$$

which contradicts the assumption that $\{f_i\}_{i=1}^n$ is optimal for Problem (7).

Consequently, f^* is optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$, and $\{f_i\}_{i=1}^n \in \mathcal{F}^G(f^*)$.

C Proof of Theorem 3.4

We start with the "if" part. Fix $f \in \mathcal{F}^{NS}$. Suppose that $\{f_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ is such that for each $i \in \{1, 2, \ldots, n\}$ and for each $x \in [0, M]$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. $z \in [0, M]$,

$$
h_i(z) = \begin{cases} 0 & \text{if } i \notin \hat{\mathcal{I}}(z); \\ \lambda_i(z) & \text{if } i \in \hat{\mathcal{I}}(z), \end{cases} \tag{32}
$$

and where $\{\lambda_i\}_{i=1}^n$ is such that $\sum_{i=1}^n h_i(z) = f'(z)$, for a.e. $z \in [0, M]$. Then, in particular, for μ-a.e. $\omega \in \Omega$,

$$
\sum_{i=1}^{n} f_i\left(X(\omega)\right) = \sum_{i=1}^{n} \int_0^{X(\omega)} h_i(z)dz = \int_0^{X(\omega)} \sum_{i=1}^{n} h_i(z)dz \int_0^{X(\omega)} f'(z)dz = f\left(X(\omega)\right),
$$

where the last equality follows from the fact that $f \in \mathcal{F}^{NS}$ is absolutely continuous and is such that $f(0) = 0$, so that we can write $f\left(X(\omega)\right) = \int_0^{X(\omega)} f'(z)dz$, for each $\omega \in \Omega$. Hence $\{f_i\}_{i=1}^n$ is feasible for Problem (9), by construction. To show optimality of $\{f_i\}_{i=1}^n$ for Problem (9), let $\{g_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ be also feasible for Problem (9). Then, in particular

$$
\sum_{i=1}^{n} f_i\left(X\right) = \sum_{i=1}^{n} g_i\left(X\right) = f\left(X\right), \ \mu\text{-a.s.}
$$

Moreover, since for each i, $g_i \in \mathcal{F}^{NS}$ is absolutely continuous and is such that $g_i(0) = 0$, we can write $g_i(X(\omega)) = \int_0^{X(\omega)} g_i'(z)dz$, for each $\omega \in \Omega$, where $g_i'(z) \in [0,1]$, for a.e. $z \geq 0$. Hence, it follows that for μ-a.e. $\omega \in \Omega$,

$$\int_0^{X(\omega)} f'(z)dz = f\left(X(\omega)\right) = \sum_{i=1}^n g_i\left(X(\omega)\right) = \sum_{i=1}^n \int_0^{X(\omega)} g_i'(z)dz = \int_0^{X(\omega)} \sum_{i=1}^n g_i'(z)dz.$$

Consequently,

$$\sum_{i=1}^n g_i'(z) = f'(z), \text{ for a.e. } z \in [0,M]. \tag{33}$$

Now, we have

$$\begin{aligned}
\sum_{i=1}^n \hat{\pi}^{\theta_i,\mathbb{Q}_i}(f_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^M \mathbb{Q}_i(f_i(X) > z)dz = \sum_{i=1}^n (1+\theta_i) \int_0^M \mathbb{Q}_i(X > z)df_i(z) \\
&= \sum_{i=1}^n \int_0^M (1+\theta_i)\, \mathbb{Q}_i(X > z)h_i(z)dz = \int_0^M \sum_{i=1}^n \Big[(1+\theta_i)\, \mathbb{Q}_i(X > z)h_i(z)\Big]dz \\
&= \int_0^M \sum_{i \in \hat{\mathcal{I}}(z)} \Big[(1+\theta_i)\, \mathbb{Q}_i(X > z)\lambda_i(z)\Big]dz = \int_0^M \sum_{i \in \hat{\mathcal{I}}(z)} \Big[\upsilon(X > z)\lambda_i(z)\Big]dz \\
&= \int_0^M \upsilon(X > z) \sum_{i \in \hat{\mathcal{I}}(z)} \lambda_i(z)dz = \int_0^M \upsilon(X > z) \sum_{i=1}^n h_i(z)dz \\
&= \int_0^M \upsilon(X > z)f'(z)dz = \int_0^M \upsilon(X > z) \sum_{i=1}^n g_i'(z)dz = \int_0^M \sum_{i=1}^n \Big[\upsilon(X > z)g_i'(z)\Big]dz \\
&\leq \int_0^M \sum_{i=1}^n \Big[(1+\theta_i)\, \mathbb{Q}_i(X > z)g_i'(z)\Big]dz = \sum_{i=1}^n \hat{\pi}^{\theta_i,\mathbb{Q}_i}(g_i(X)),
\end{aligned}$$

where the second and last equalities are due to Lemma 2.1 of Zhuang et al. (2016). Thus, $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$.

We proceed with the "only if" part. Let $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$ be such that $f_i \in \mathcal{F}^{NS}$, for each i. Then $\sum_{j=1}^n f_j(X) = f(X)$, μ-a.s., by definition of $\mathcal{F}^{NS}(f)$. Moreover, since for each i, $f_i \in \mathcal{F}^{NS}$ is absolutely continuous and is such that $f_i(0) = 0$, we can write $f_i(X(\omega)) = \int_0^{X(\omega)} h_i(z)dz$, for each $\omega \in \Omega$, where $h_i(z) = f_i'(z) \in [0,1]$, for a.e. $z \geq 0$. Hence, for a given $f \in \mathcal{F}^{NS}$, it follows that for μ-a.e. $\omega \in \Omega$,

$$\int_0^{X(\omega)} f'(z)dz = f\left(X(\omega)\right) = \sum_{i=1}^n f_i\left(X(\omega)\right) = \sum_{i=1}^n \int_0^{X(\omega)} h_i(z)dz = \int_0^{X(\omega)} \sum_{i=1}^n h_i(z)dz.$$

Consequently, $\sum_{i=1}^n h_i(z) = f'(z)$ for a.e. $z \in [0,M]$.

Now, for each $z \in [0, M]$, define

$$\mathcal{J}(z) := \left\{ i \in \{1, 2, \ldots, n\} : i \notin \hat{\mathcal{I}}(z), h_i(z) > 0 \right\}.$$

Suppose, by way of contradiction, that there exists $B^* \subset [0, M]$ such that $\mathcal{J}(z) \neq \varnothing$, for each $z \in B^*$, and $\int_{B^*} dz > 0$. Fix $z^* \in B^*$. Then $\mathcal{J}(z^*) \neq \varnothing$, and for each $i \in \mathcal{J}(z^*)$, we have $i \notin \hat{\mathcal{I}}(z^*)$ and $h_i(z^*) > 0$.
Define the collection $\{\kappa_i^{z^*}(z^*)\}_{i=1}^n \subset \mathbb{R}^+$ by

$$\kappa_i^{z^*}(z^*) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(z^*); \\ h_i(z^*) = 0 & \text{for all } i \in \{1, 2, \ldots, n\} \setminus \left(\mathcal{J}(z^*) \bigcup \hat{\mathcal{I}}(z^*) \right); \\ h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\hat{\mathcal{I}}(z^*)|} & \text{for all } i \in \hat{\mathcal{I}}(z^*), \end{cases} \tag{34}$$

where $|\hat{\mathcal{I}}(z^*)|$ denotes the cardinality of the set $\hat{\mathcal{I}}(z^*)$. Then, by construction, $\kappa_i^{z^*}(z^*) = 0$ for all $i \notin \hat{\mathcal{I}}(z^*)$, and

$$\sum_{i=1}^n \kappa_i^{z^*}(z^*) = \sum_{i \in \hat{\mathcal{I}}(z^*)} \left(h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\hat{\mathcal{I}}(z^*)|} \right) = \sum_{i=1}^n h_i(z^*).$$

Moreover, by eq. (34), we have

$$\begin{aligned} \sum_{i=1}^n (1+\theta_i)\, \mathbb{Q}_i(X > z^*) \kappa_i^{z^*}(z^*) &= \sum_{i \in \hat{\mathcal{I}}(z^*)} (1+\theta_i)\, \mathbb{Q}_i(X > z^*) \kappa_i^{z^*}(z^*) = \sum_{i \in \hat{\mathcal{I}}(z^*)} \upsilon(X > z^*) \kappa_i^{z^*}(z^*) \\ &= \upsilon(X > z^*) \sum_{i \in \hat{\mathcal{I}}(z^*)} \left[h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\hat{\mathcal{I}}(z^*)|} \right] \\ &= \sum_{i \in \hat{\mathcal{I}}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{i \in \hat{\mathcal{I}}(z^*)} \upsilon(X > z^*) \left(\frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\hat{\mathcal{I}}(z^*)|} \right) \\ &= \sum_{i \in \hat{\mathcal{I}}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{j \in \mathcal{J}(z^*)} \upsilon(X > z^*) h_j(z^*) \\ &< \sum_{i \in \hat{\mathcal{I}}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{j \in \mathcal{J}(z^*)} (1+\theta_j)\, \mathbb{Q}_j(X > z^*) h_j(z^*) \\ &= \sum_{i=1}^n (1+\theta_i)\, \mathbb{Q}_i(X > z^*) h_i(z^*). \end{aligned}$$

For each $z \in B^*$, construct the collection $\{\kappa_i^z(z)\}_{i=1}^n \subset \mathbb{R}^+$ as in eq. (29), so that in particular

$$\sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)\kappa_i^z(z) < \sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)h_i(z), \quad \text{for all} \quad z \in B^*. \tag{35}$$

Moreover, since $\sum_{i=1}^n h_i(z) = f'(z)$ for a.e. $z \in [0, M]$, since $\sum_{i=1}^n \kappa_i^z(z) = \sum_{i=1}^n h_i(z)$ for each $z \in B^*$, and since each $\kappa_i^z(z) \geq 0$ for each $z \in B^*$, by construction, it follows that $\kappa_i^z(z) \in [0,1]$, for all $z \in B^*$. Now, let $\{\phi_i\}_{i=1}^n$ be defined by, for each $i \in \{1, 2, \ldots, n\}$,

$$\phi_i(z) = \begin{cases} h_i(z) & \text{if } z \notin B^*; \\ \kappa_i^z(z) & \text{if } z \in B^*. \end{cases} \tag{36}$$

Then

$$\begin{cases} \displaystyle\sum_{i=1}^n \phi_i(z) = \sum_{i=1}^n \kappa_i^z(z) = \sum_{i=1}^n h_i(z) & \text{for each } z \in B^*; \\ \displaystyle\sum_{i=1}^n \phi_i(z) = \sum_{i=1}^n h_i(z) & \text{for each } z \notin B^*. \end{cases}$$

Consequently, since $\sum_{j=1}^n h_j(z) = f'(z)$ for a.e. $z \in [0, M]$, it follows that $\sum_{j=1}^n \phi_j(z) = f'(z)$ for a.e. $z \in [0, M]$. Moreover, by construction, we have $\phi_i(z) \in [0,1]$, for a.e. $z \geq 0$, for each $i \in \{1, 2, \ldots, n\}$. Now, by eq. (35), we have

$$\begin{cases} \displaystyle\sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)\phi_i(z) < \sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)h_i(z), & \text{for all} \quad z \in B^*; \\ \displaystyle\sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)\phi_i(z) = \sum_{i=1}^n (1+\theta_i)\,\mathbb{Q}_i(X > z)h_i(z), & \text{for all} \quad z \notin B^*. \end{cases}$$

Therefore, letting the collection $\{\Psi_i\}_{i=1}^n$ be defined by $\Psi_i(X(\omega)) := \int_0^{X(\omega)} \phi_i(z)dz$, for each $\omega \in \Omega$ and for each $i \in \{1, 2, \ldots, n\}$, it follows that for μ-a.e. $\omega \in \Omega$,

$$\begin{aligned} f(X(\omega)) &= \sum_{i=1}^n f_i(X(\omega)) = \sum_{i=1}^n \int_0^{X(\omega)} h_i(z)dz = \int_0^{X(\omega)} \sum_{i=1}^n h_i(z)dz \\ &= \int_0^{X(\omega)} \sum_{i=1}^n \phi_i(z)dz = \sum_{i=1}^n \int_0^{X(\omega)} \phi_i(z)dz = \sum_{i=1}^n \Psi_i(X(\omega)). \end{aligned}$$

Moreover, since for each $i \in \{1, 2, \ldots, n\}$, $\phi_i(z) \in [0,1]$, for a.e. $z \geq 0$, it follows that $\Psi_i \in \mathcal{F}^{NS}$, for each $i \in \{1, 2, \ldots, n\}$. Hence $\{\Psi\}_{i=1}^n$ is feasible for Problem (9). Finally,

$$\begin{aligned} \sum_{i=1}^n \hat{\pi}^{\theta_i, \mathbb{Q}_i}(\Psi_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^{\Psi_i(M)} \mathbb{Q}_i(\Psi_i(X) > z)dz = \sum_{i=1}^n (1+\theta_i) \int_0^M \mathbb{Q}_i(X > z)d\Psi_i(z) \\ &= \sum_{i=1}^n \int_0^M (1+\theta_i)\,\mathbb{Q}_i(X > z)\phi_i(z)dz = \int_0^M \sum_{i=1}^n \Big[(1+\theta_i)\,\mathbb{Q}_i(X > z)\phi_i(z)\Big]dz \end{aligned}$$

$$
\begin{aligned}
&= \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)\phi_i(z)\Big] dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)\phi_i(z)\Big] dz \\
&= \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)\kappa_i^z(z)\Big] dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)h_i(z)\Big] dz \\
&< \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)h_i(z)\Big] dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)h_i(z)\Big] dz \\
&= \int_0^M \sum_{i=1}^{n} \Big[(1+\theta_i)\, \mathbb{Q}_i(X>z)h_i(z)\Big] dz = \sum_{i=1}^{n} \int_0^M (1+\theta_i)\, \mathbb{Q}_i(X>z)h_i(z) dz \\
&= \sum_{i=1}^{n} \int_0^M (1+\theta_i)\, \mathbb{Q}_i(X>z) df_i(z) = \sum_{i=1}^{n} (1+\theta_i) \int_0^M \mathbb{Q}_i(X>z) df_i(z) \\
&= \sum_{i=1}^{n} (1+\theta_i) \int_0^M \mathbb{Q}_i(f_i(X)>z) dz = \sum_{i=1}^{n} \hat{\pi}^{\theta_i,\mathbb{Q}_i}(f_i(X)),
\end{aligned}
$$

where the second and second-last equalities are due to Lemma 2.1 of Zhuang et al. (2016). This contradicts with the assumption that $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}(f)$.

D Proof of Proposition 4.1

First, suppose that $\{f_i\}_{i=1}^n$ solves Problem (7). Then, by Theorem 3.2, $\{f_i\}_{i=1}^n \subset \mathcal{F}^G$, $\sum_{i=1}^n f_i \in \mathcal{F}^G$, $\sum_{i=1}^n f_i$ is optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$, and $\{f_i\}_{i=1}^n \in \mathcal{F}^G\left(\sum_{j=1}^n f_j\right)$. Hence, $\sum_{i=1}^n f_i$ is optimal for Problem (20), and $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$. Consequently, Theorem 3.1 implies that $\{f_i\}_{i=1}^n$ is such that for each i and for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$, where $\mathcal{I}(\omega)$ is defined in eq. (11). It remains to show that, for μ-a.e. $\omega \in \Omega$, $f_i(X(\omega)) = 0$ whenever $\zeta_0(\omega) < \zeta_i(\omega)$ and $\sum_{j=1}^n f_j(X(\omega)) = X(\omega)$ whenever $\zeta_0(\omega) > \widetilde{\zeta}(\omega)$. For each $\omega \in \Omega$, let

$$
\mathcal{J}(\omega) := \Big\{ i \in \{1,2,\ldots,n\} : \zeta_0(\omega) < \zeta_i(\omega),\ f_i(X(\omega)) > 0 \Big\}.
$$

Suppose, by way of contradiction, that there exists $A^* \in \Sigma$ such that $\mathcal{J}(\omega) \neq \varnothing$, for each $\omega \in A^*$, and $\mu(A^*) > 0$. Fix $\omega^* \in A^*$. Then $\mathcal{J}(\omega^*) \neq \varnothing$, and for each $i \in \mathcal{J}(\omega^*)$, we have $\zeta_0(\omega^*) < \zeta_i(\omega^*)$. Define the collection $\{g_i^{\omega^*}(X(\omega^*))\}_{i=1}^n$ by

$$
g_i^{\omega^*}(X(\omega^*)) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(\omega^*); \\ f_i(X(\omega^*)) & \text{for all } i \notin \mathcal{J}(\omega^*). \end{cases} \tag{37}
$$

For each $\omega \in A^*$, construct the collection $\{g_i^{\omega}(X(\omega))\}_{i=1}^n$ as in eq. (37). Then, in particular, for μ-a.e. $\omega \in A^*$, we have $0 \le g_i^{\omega}(X(\omega)) \le f_i(X(\omega)) \le X(\omega)$. Let $\{h_i\}_{i=1}^n$ be defined by, for each $i \in \{1,2,\ldots,n\}$,

$$
h_i(X(\omega)) = \begin{cases} f_i(X(\omega)) & \text{if } \omega \notin A^*; \\ g_i^{\omega}(X(\omega)) & \text{if } \omega \in A^*. \end{cases} \tag{38}
$$

Then $\{h_i\}_{i=1}^n \subset \mathcal{F}^G$. Since $\widetilde{\zeta}(\omega) \leq \zeta_i(\omega)$, for each $\omega \in \Omega$ and each $i \in \{1, 2, \ldots, n\}$, it follows that for each $\omega \in A^*$,

$$\begin{aligned}
\sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) h_i(X(\omega)) &\geq \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) h_i(X(\omega)) = \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) g_i^\omega(X(\omega)) \\
&= \sum_{i \notin \mathcal{J}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) g_i^\omega(X(\omega)) = \sum_{i \notin \mathcal{J}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) \\
&> \sum_{i \notin \mathcal{J}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) + \sum_{i \in \mathcal{J}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) = \sum_{i \in \mathcal{I}(\omega)} \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)).
\end{aligned}$$

Moreover, for each $\omega \notin A^*$

$$\begin{aligned}
\sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) h_i(X(\omega)) &\geq \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) h_i(X(\omega)) = \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i \in \mathcal{I}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) = \sum_{i \in \mathcal{I}(\omega)} \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_\Omega \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu &= \int_{A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu + \int_{\Omega \setminus A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu \\
&> \int_{A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n f_i(X) d\mu + \int_{\Omega \setminus A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n f_i(X) d\mu \\
&= \int_\Omega \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n f_i(X) d\mu,
\end{aligned}$$

contradicting the optimality of $\sum_{i=1}^n f_i$ for Problem (20). Hence, $\{f_i\}_{i=1}^n$ is such that for each i and for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$ or $\zeta_0(\omega) < \zeta_i(\omega)$.

Next, we show that, for μ-a.e. $\omega \in \Omega$, $\sum_{j=1}^n f_j(X(\omega)) = X(\omega)$ whenever $\zeta_0(\omega) > \widetilde{\zeta}(\omega)$. For each $\omega \in \Omega$, let

$$\widetilde{\mathcal{J}}(\omega) := \left\{ i \in \{1, 2, \ldots, n\} : \zeta_0(\omega) > \widetilde{\zeta}(\omega), \ \sum_{j=1}^n f_j(X(\omega)) < X(\omega) \right\}.$$

Suppose, by way of contradiction, that there exists $A^* \in \Sigma$ such that $\widetilde{\mathcal{J}}(\omega) \neq \varnothing$, for each $\omega \in A^*$, and $\mu(A^*) > 0$. Fix $\omega^* \in A^*$, and define the collection $\{g_i^{\omega^*}(X(\omega^*))\}_{i=1}^n$ by

$$g_i^{\omega^*}(X(\omega^*)) = \begin{cases} 0 & \text{for all } i \notin \mathcal{I}(\omega^*); \\ \frac{X(\omega^*)}{|\mathcal{I}(\omega^*)|} & \text{for all } i \in \mathcal{I}(\omega^*). \end{cases} \tag{39}$$

Let $\{h_i\}_{i=1}^n$ be defined by, for each $i \in \{1, 2, \ldots, n\}$,

$$h_i(X(\omega)) = \begin{cases} f_i(X(\omega)) & \text{if } \omega \notin A^*; \\ g_i^{\omega}(X(\omega)) & \text{if } \omega \in A^*. \end{cases} \tag{40}$$

Then $\{h_i\}_{i=1}^n \subset \mathcal{F}^G$, and for each $\omega \in A^*$,

$$\begin{aligned}
\sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) h_i(X(\omega)) &\geq \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) h_i(X(\omega)) = \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) g_i^{\omega}(X(\omega)) \\
&= \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) g_i^{\omega}(X(\omega)) = \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) \frac{X(\omega)}{|\mathcal{I}(\omega)|} \\
&= \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) \frac{X(\omega)}{|\mathcal{I}(\omega)|} = \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) X(\omega) \\
&> \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) \sum_{i=1}^n f_i(X(\omega)) = \sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)).
\end{aligned}$$

Moreover, for each $\omega \notin A^*$

$$\begin{aligned}
\sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) h_i(X(\omega)) &\geq \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) h_i(X(\omega)) = \sum_{i=1}^n \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \zeta_i(\omega)\right) f_i(X(\omega)) = \sum_{i\in\mathcal{I}(\omega)} \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)) \\
&= \sum_{i=1}^n \left(\zeta_0(\omega) - \widetilde{\zeta}(\omega)\right) f_i(X(\omega)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_\Omega \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu &= \int_{A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu + \int_{\Omega\setminus A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n h_i(X) d\mu \\
&> \int_{A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n f_i(X) d\mu + \int_{\Omega\setminus A^*} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^n f_i(X) d\mu
\end{aligned}$$

$$= \int_{\Omega} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^{n} f_i(X) d\mu,$$

contradicting the optimality of $\sum_{i=1}^{n} f_i$ for Problem (20). Hence, $\{f_i\}_{i=1}^{n}$ is such that for each i and for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$ or $\zeta_0(\omega) < \zeta_i(\omega)$, and for μ-a.e. $\omega \in \Omega$, $\sum_{j=1}^{n} f_j\left(X\left(\omega\right)\right) = X\left(\omega\right)$ whenever $\zeta_0\left(\omega\right) > \widetilde{\zeta}\left(\omega\right)$.

Now, to show the converse, suppose that $\{f_i\}_{i=1}^{n} \subset \mathcal{F}^G$ satisfies $\sum_{i=1}^{n} f_i \in \mathcal{F}^G$ and is such that for each $i \in \{1, 2, \ldots, n\}$ and for μ-a.e. $\omega \in \Omega$, we have $f_i(X(\omega)) = 0$ whenever $i \notin \mathcal{I}(\omega)$ or $\zeta_0(\omega) < \zeta_i(\omega)$, and $\sum_{i=1}^{n} f_i(X(\omega)) = X(\omega)$ whenever $\zeta_0(\omega) > \widetilde{\zeta}(\omega)$. Then, in particular, $\{f_i\}_{i=1}^{n} \in \mathcal{F}\left(\sum_{j=1}^{n} f_j\right)$, by Theorem 3.1. Hence, by Theorem 3.2, it remains to show that $\sum_{i=1}^{n} f_i$ is optimal for Problem (8) with $\pi = \pi^{\widetilde{\zeta}}$, that is, that $\sum_{i=1}^{n} f_i$ is optimal for Problem (20). Suppose, by way of contradiction, that $\sum_{i=1}^{n} f_i$ is not optimal for Problem (20). Then, there exists $g \in \mathcal{F}^G$ such that

$$\int_{\Omega} \left(\zeta_0 - \widetilde{\zeta}\right) g(X) d\mu > \int_{\Omega} \left(\zeta_0 - \widetilde{\zeta}\right) \sum_{i=1}^{n} f_i(X) d\mu. \tag{41}$$

Now, since $g \in \mathcal{F}^G$, there exists $B \in \Sigma$ such that $\mu\left(B\right) = 1$ and $0 \leq g\left(X\left(\omega\right)\right) \leq X\left(\omega\right)$, for each $\omega \in B$. Consequently, eq. (41) implies that there exists $A \subset B$ such that $\mu(A) > 0$ and

$$\left(\zeta_0\left(\omega\right) - \widetilde{\zeta}\left(\omega\right)\right) g\left(X\left(\omega\right)\right) > \left(\zeta_0\left(\omega\right) - \widetilde{\zeta}\left(\omega\right)\right) \sum_{i=1}^{n} f_i\left(X\left(\omega\right)\right), \quad \text{for all } \omega \in A;$$

or, equivalently,

$$\left(\zeta_0\left(\omega\right) - \widetilde{\zeta}\left(\omega\right)\right) \left(g\left(X\left(\omega\right)\right) - \sum_{i=1}^{n} f_i\left(X\left(\omega\right)\right)\right) > 0, \quad \text{for all } \omega \in A.$$

Thus, $\zeta_0\left(\omega\right) - \widetilde{\zeta}\left(\omega\right) \neq 0$, for all $\omega \in A$. Fix $\omega^* \in A$. If $\zeta_0\left(\omega^*\right) - \widetilde{\zeta}\left(\omega^*\right) > 0$, then $g\left(X\left(\omega^*\right)\right) > \sum_{i=1}^{n} f_i\left(X\left(\omega^*\right)\right) = X\left(\omega^*\right)$, contradicting the fact that $g\left(X\left(\omega^*\right)\right) \leq X\left(\omega^*\right)$. If $\zeta_0\left(\omega^*\right) - \widetilde{\zeta}\left(\omega^*\right) < 0$, then $\zeta_0(\omega^*) < \zeta_i(\omega^*)$, for each $i \in \{1, 2, \ldots, n\}$, and hence $f_i(X(\omega)) = 0$, for each $i \in \{1, 2, \ldots, n\}$. Consequently, $g\left(X\left(\omega^*\right)\right) < \sum_{i=1}^{n} f_i\left(X\left(\omega^*\right)\right) = 0$, contradicting the fact that $g\left(X\left(\omega^*\right)\right) \geq 0$. This concludes the proof.

E Proof of Proposition 4.2

First, suppose that $\{f_i\}_{i=1}^{n}$ solves Problem (7). Then, by Theorem 3.5, $\{f_i\}_{i=1}^{n} \subset \mathcal{F}^{NS}$, $\sum_{i=1}^{n} f_i \in \mathcal{F}^{NS}$, $\sum_{i=1}^{n} f_i$ is optimal for Problem (8) with $\pi(f(X)) = \int_0^M \upsilon(f(X) > z) dz$, and $\{f_i\}_{i=1}^{n} \in \mathcal{F}^{NS}\left(\sum_{j=1}^{n} f_j\right)$. Hence, $\sum_{i=1}^{n} f_i$ is optimal for Problem (23) and $\{f_i\}_{i=1}^{n} \in \mathcal{F}^{NS}\left(\sum_{j=1}^{n} f_j\right)$. Consequently, Theorem 3.4 implies that $\{f_i\}_{i=1}^{n}$ is such that for each $i \in \{1, 2, \ldots, n\}$ and for each $x \in [0, M]$, $f_i(x) = \int_0^x h_i(z) dz$, where for a.e. $z \in [0, M]$,

$$h_i(z) = 0 \quad \text{whenever} \quad i \notin \hat{\mathcal{I}}(z), \quad \text{and} \quad 0 \leq \sum_{i=1}^{n} h_i(z) = f'(z) \leq 1.$$

Then, in particular, for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M]$, $0 \le h_i(z) \le 1$.

It remains to show that, for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M]$, $h_i(z) = 0$ whenever $(1 + \theta_i)\, \mathbb{Q}_i(X > z) > \mathbb{P}(X > z)$, and that for a.e. $z \in [0, M]$, $\sum_{i=1}^{n} h_i(z) = 1$ whenever $\upsilon(X > z) < \mathbb{P}(X > z)$. For each $z \in [0, M]$, let

$$\mathcal{J}\,(z) := \Big\{ i \in \{1, 2, \ldots, n\} : (1 + \theta_i)\, \mathbb{Q}_i(X > z) > \mathbb{P}(X > z),\ h_i(z) > 0 \Big\}.$$

Suppose, by way of contradiction, that there exists $B^* \subset [0, M]$ such that $\mathcal{J}(z) \neq \varnothing$, for each $z \in B^*$, and $\int_{B^*} dz > 0$. Fix $z^* \in B^*$. Then $\mathcal{J}(z^*) \neq \varnothing$, and for each $i \in \mathcal{J}(z^*)$, we have

$$(1 + \theta_i)\, \mathbb{Q}_i(X > z^*) > \mathbb{P}(X > z^*) \ \text{ and } \ h_i(z^*) > 0.$$

Define the collection $\{\kappa_i^{z^*}(z^*)\}_{i=1}^{n}$ by

$$\kappa_i^{z^*}(z^*) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(z^*); \\ h_i(z^*) & \text{for all } i \notin \mathcal{J}(z^*). \end{cases} \tag{42}$$

Hence, $\kappa_i^{z^*}(z^*) < h_i(z^*)$ for each $i \in \mathcal{J}(z^*)$.

Now, for each $z \in B^*$, construct the collection $\{\kappa_i^{z}(z)\}_{i=1}^{n}$ as in eq. (42). Then, in particular, for each $i \in \{1, 2, \ldots, n\}$ and $z \in B^*$, we have $0 \le \kappa_i^z(z) \le h_i(z) \le 1$. Let $\{\phi_i\}_{i=1}^{n}$ be defined by, for each $i \in \{1, 2, \ldots, n\}$,

$$\phi_i(z) = \begin{cases} h_i(z) & \text{if } z \notin B^*; \\ \kappa_i^z(z) & \text{if } z \in B^*. \end{cases} \tag{43}$$

Then for each $z \in B^*$,

$$0 \le \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} \kappa_i^z(z) \le \sum_{i=1}^{n} h_i(z),$$

and for each $z \notin B^*$

$$0 \le \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} h_i(z).$$

Consequently, since $0 \le \sum_{i=1}^{n} h_i(z) = f'(z) \le 1$, for a.e. $z \in [0, M]$, it follows that $0 \le \sum_{j=1}^{n} \phi_j(z) \le 1$ for a.e. $z \in [0, M]$. Moreover, by construction, we have $\phi_i(z) \in [0, 1]$, for a.e. $z \ge 0$, for each $i \in \{1, 2, \ldots, n\}$. Now, for each $z \in B^*$,

$$\begin{aligned} \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z) &\ge \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - (1 + \theta_i)\, \mathbb{Q}_i(X > z)\right] \phi_i(z) \\ &= \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - (1 + \theta_i)\, \mathbb{Q}_i(X > z)\right] \kappa_i^z(z) = \sum_{i \notin \mathcal{J}(z)} \left[\mathbb{P}(X > z) - (1 + \theta_i)\, \mathbb{Q}_i(X > z)\right] h_i(z) \\ &> \sum_{i \notin \mathcal{J}(z)} \left[\mathbb{P}(X > z) - (1 + \theta_i)\, \mathbb{Q}_i(X > z)\right] h_i(z) + \sum_{i \in \mathcal{J}(z)} \left[\mathbb{P}(X > z) - (1 + \theta_i)\, \mathbb{Q}_i(X > z)\right] h_i(z) \end{aligned}$$

$$= \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z) = \sum_{i \in \hat{\mathcal{I}}(z)} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z)$$

$$= \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z).$$

Moreover, for each $z \notin B^*$

$$\sum_{i=1}^{n} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z) \geq \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] \phi_i(z)$$

$$= \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z) = \sum_{i \in \hat{\mathcal{I}}(z)} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z)$$

$$= \sum_{i=1}^{n} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z).$$

Now, define the collection $\{\Psi_i\}_{i=1}^n$ by $\Psi_i\left(X(\omega)\right) := \int_0^{X(\omega)} \phi_i(z)dz$, for each $\omega \in \Omega$ and for each $i \in \{1, 2, \ldots, n\}$. Since $0 \leq \sum_{j=1}^n \phi_j(z) \leq 1$ for a.e. $z \in [0, M]$, and since $\phi_i(z) \in [0, 1]$, for a.e. $z \geq 0$ and for each $i \in \{1, 2, \ldots, n\}$, it follows that $\{\Psi_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ and $\sum_{i=1}^n \Psi_i \in \mathcal{F}^{NS}$. Hence $\Psi := \sum_{i=1}^n \Psi_i$ is feasible for Problem (23). Moreover, by construction, we have

$$\int_0^M \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \Psi'(z)dz = \int_0^M \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \sum_{i=1}^n \phi_i(z)dz$$

$$= \int_0^M \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z)dz$$

$$= \int_{B^*} \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z)dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z)dz$$

$$> \int_{B^*} \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z)dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z)dz$$

$$= \int_0^M \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z)dz = \int_0^M \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \sum_{i=1}^n h_i(z)dz$$

$$= \int_0^M \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \left(\sum_{i=1}^n f_i\right)'(z)\ dz,$$

contradicting the optimality of $\sum_{i=1}^n f_i$ for Problem (23). Hence, for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M]$, $h_i(z) = 0$ whenever $(1+\theta_i)\,\mathbb{Q}_i(X > z) > \mathbb{P}(X > z)$.

Next, we show that for a.e. $z \in [0, M]$, $\sum_{i=1}^n h_i(z) = 1$ whenever $\upsilon(X > z) < \mathbb{P}(X > z)$. For each $z \in [0, M]$, let

$$\widetilde{\mathcal{J}}\,(z) := \Big\{i \in \{1, 2, \ldots, n\} : (1+\theta_i)\,\mathbb{Q}_i(X > z) < \mathbb{P}(X > z),\ \sum_{j=1}^n h_j(z) < 1\Big\}.$$

Suppose, by way of contradiction, that there exists $B^* \subset [0, M]$ such that $\widetilde{\mathcal{J}}(z) \neq \varnothing$, for each $z \in B^*$, and $\int_{B^*} dz > 0$. Fix $z^* \in B^*$. Then $\widetilde{\mathcal{J}}(z^*) \neq \varnothing$, and for each $i \in \mathcal{J}(z^*)$, we have $(1+\theta_i)\,\mathbb{Q}_i(X > z^*) < \mathbb{P}(X > z^*)$ and $\sum_{j=1}^n h_j(z^*) < 1$. Define the collection $\{\kappa_i^{z^*}(z^*)\}_{i=1}^n$ by

$$\kappa_i^{z^*}(z^*) = \begin{cases} 0 & \text{for all } i \notin \hat{\mathcal{I}}\,(z^*); \\ \frac{1}{|\hat{\mathcal{I}}(z^*)|} & \text{for all } i \in \hat{\mathcal{I}}\,(z^*). \end{cases} \tag{44}$$

Now, for each $z \in B^*$, construct the collection $\{\kappa_i^z(z)\}_{i=1}^n$ as in eq. (44). Then, in particular, for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M]$, we have $0 \leq \kappa_i^z(z) \leq 1$. Let $\{\phi_i\}_{i=1}^n$ be defined by, for each $i \in \{1, 2, \ldots, n\}$,

$$\phi_i(z) = \begin{cases} h_i(z) & \text{if } z \notin B^*; \\ \kappa_i^z(z) & \text{if } z \in B^*. \end{cases} \tag{45}$$

Then for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M]$, we have $0 \leq \phi_i(z) \leq 1$. Moreover, for each $z \in B^*$,

$$\sum_{i=1}^n \phi_i(z) = \sum_{i=1}^n \kappa_i^z(z) = \sum_{i \in \hat{\mathcal{I}}(z)} \kappa_i^z(z) = \sum_{i \in \hat{\mathcal{I}}(z)} \frac{1}{|\hat{\mathcal{I}}\,(z)\,|} = 1 > \sum_{i=1}^n h_i(z),$$

and for each $z \notin B^*$

$$0 \leq \sum_{i=1}^n \phi_i(z) = \sum_{i=1}^n h_i(z) \leq 1.$$

Hence, $0 \leq \sum_{j=1}^n \phi_j(z) \leq 1$ for a.e. $z \in [0, M]$. Now, for each $z \in B^*$,

$$\begin{aligned}
\sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z) &\geq \sum_{i=1}^n \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] \phi_i(z) \\
&= \sum_{i=1}^n \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] \kappa_i^z(z) \\
&= \sum_{i \in \hat{\mathcal{I}}(z)} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] \frac{1}{|\hat{\mathcal{I}}\,(z^*)\,|} = \sum_{i \in \hat{\mathcal{I}}(z)} \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \frac{1}{|\hat{\mathcal{I}}\,(z^*)\,|} \\
&= \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] = \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z) > \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z).
\end{aligned}$$

Moreover, for each $z \notin B^*$

$$\begin{aligned}
\sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] \phi_i(z) &\geq \sum_{i=1}^n \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] \phi_i(z) \\
&= \sum_{i=1}^n \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z) = \sum_{i \in \hat{\mathcal{I}}(z)} \left[\mathbb{P}(X > z) - (1+\theta_i)\,\mathbb{Q}_i(X > z)\right] h_i(z) \\
&= \sum_{i=1}^n \left[\mathbb{P}(X > z) - \upsilon(X > z)\right] h_i(z).
\end{aligned}$$

Now, define the collection $\{\Psi_i\}_{i=1}^n$ by $\Psi_i\left(X(\omega)\right) := \int_0^{X(\omega)} \phi_i(z)dz$, for each $\omega \in \Omega$ and for each $i \in \{1,2,\dots,n\}$. Since $0 \le \sum_{j=1}^n \phi_j(z) \le 1$ for a.e. $z \in [0,M]$, and since $\phi_i(z) \in [0,1]$, for a.e. $z \ge 0$ and for each $i \in \{1,2,\dots,n\}$, it follows that $\{\Psi_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ and $\sum_{i=1}^n \Psi_i \in \mathcal{F}^{NS}$. Hence $\Psi := \sum_{i=1}^n \Psi_i$ is feasible for Problem (23). Moreover, by construction, we have

$$
\begin{aligned}
\int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \Psi'(z)dz &= \int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \sum_{i=1}^n \phi_i(z)dz \\
&= \int_0^M \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \phi_i(z)dz \\
&= \int_{B^*} \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \phi_i(z)dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \phi_i(z)dz \\
&> \int_{B^*} \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] h_i(z)dz + \int_{[0,M]\setminus B^*} \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] h_i(z)dz \\
&= \int_0^M \sum_{i=1}^n \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] h_i(z)dz = \int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \sum_{i=1}^n h_i(z)dz \\
&= \int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \left(\sum_{i=1}^n f_i\right)'(z)\, dz,
\end{aligned}
$$

contradicting the optimality of $\sum_{i=1}^n f_i$ for Problem (23). Hence, for each $i \in \{1,2,\dots,n\}$ and for a.e. $z \in [0,M]$, $h_i(z) = 0$ whenever $(1+\theta_i)\,\mathbb{Q}_i(X>z) > \mathbb{P}(X>z)$, and for a.e. $z \in [0,M]$, $\sum_{i=1}^n h_i(z) = 1$ whenever $\upsilon(X>z) < \mathbb{P}(X>z)$.

Now, to show the converse, suppose that $\{f_i\}_{i=1}^n \subset \mathcal{F}^{NS}$ satisfies $\sum_{i=1}^n f_i \in \mathcal{F}^{NS}$ and is such that for each $i \in \{1,2,\dots,n\}$ and each $x \in [0,M]$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. $z \in [0,M]$, $h_i(z) = 0$ whenever $i \notin \hat{\mathcal{I}}(z)$ or $(1+\theta_i)\,\mathbb{Q}_i(X>z) > \mathbb{P}(X>z)$, and for a.e. $z \in [0,M]$, $\sum_{i=1}^n h_i(z) = 1$ whenever $\upsilon(X>z) < \mathbb{P}(X>z)$. Then, in particular, $\{f_i\}_{i=1}^n \in \mathcal{F}^{NS}\left(\sum_{j=1}^n f_j\right)$, by Theorem 3.4. Hence, by Theorem 3.5, it remains to show that $\sum_{i=1}^n f_i$ is optimal for Problem (8) with $\pi(f(X)) = \int_0^M \upsilon(f(X)>z)\}dz$, that is, that $\sum_{i=1}^n f_i$ is optimal for Problem (23). Suppose, by way of contradiction, that $\sum_{i=1}^n f_i$ is not optimal for Problem (23). Then, there exists $g \in \mathcal{F}^{NS}$ such that

$$
\int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] g'(z)dz > \int_0^M \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \left(\sum_{i=1}^n f_i\right)'(z)dz. \tag{46}
$$

Now, since $g \in \mathcal{F}^{NS}$, there exists $B \subset [0,M]$ such that $\int_B dz = 1$ and $0 \le g'(z) \le 1$, for each $z \in B$. Consequently, eq. (46) implies that there exists $A \subset B$ such that $\int_A dz > 0$ and for each $z \in A$,

$$
\left[\mathbb{P}(X>z) - \upsilon(X>z)\right] g'(z) > \left[\mathbb{P}(X>z) - \upsilon(X>z)\right] \left(\sum_{i=1}^n f_i\right)'(z).
$$

Equivalently,

$$[\mathbb{P}(X > z) - \upsilon(X > z)] \left(g'(z) - \left(\sum_{i=1}^{n} f_i \right)' (z) \right) > 0,$$

for each $z \in A$. Thus, $\mathbb{P}(X > z) - \upsilon(X > z) \neq 0$, for all $z \in A$. Fix $z^* \in A$. If $\mathbb{P}(X > z^*) - \upsilon(X > z^*) > 0$, then $g'(z^*) > \left(\sum_{i=1}^{n} f_i\right)'(z^*) = \sum_{i=1}^{n} h_i(z^*) = 1$, contradicting the fact that $g'(z^*) \leq 1$. If $\mathbb{P}(X > z^*) - \upsilon(X > z^*) < 0$, then $g'(z^*) < \left(\sum_{i=1}^{n} f_i\right)'(z^*) = \sum_{i=1}^{n} h_i(z^*)$, and $h_i(z^*) = 0$, for each $i \in \{1, 2, \ldots, n\}$. Consequently, $\sum_{i=1}^{n} h_i(z^*) = 0$, and hence $g'(z^*) < 0$, contradicting the fact that $g'(z^*) \geq 0$. This concludes the proof.

F Proof of Theorem 4.3

Let the set function $\upsilon : \Sigma \to \mathbb{R}^+$ be defined as in eq. (14) by $\upsilon(B) = \min\limits_{1 \leq i \leq n} \left\{ (1 + \theta_i) \mathbb{Q}_i(B) \right\}$, for all $B \in \Sigma$, and let the hazard ratio HR be defined as in eq. (27). Then υ is a (non-normalized) finite capacity on (S, Σ), that is, a monotone set function, in the sense that $\upsilon(A) \leq \upsilon(B)$, whenever $A, B \in \Sigma$ are such that $A \subseteq B$; $\upsilon(\varnothing) = 0$; and $\upsilon(\Omega) < +\infty$. Moreover, since $\mathbb{Q}_i$ is a probability measure, for each $i \in \{1, 2, \ldots, n\}$, it follows that υ is a continuous capacity (i.e., both inner- and outer-continuous).

Lemma F.1 *If the hazard ratio HR is non-increasing on its domain, then any admissible indemnity function for Problem (26) is suboptimal to a linear deductible with the same premium.*

Proof: The proof is similar to that of Theorem 4.1 in Chi (2019), but adapted to the present setting. Suppose that HR is non-increasing on its domain. Then it follows that $M_\upsilon \leq M_{\mathbb{P}}$, HR is non-increasing on $[0, M_{\mathbb{P}})$, and HR is positive on $[0, M_\upsilon)$. Let u be any increasing and concave utility function, and let $f \in \mathcal{F}^{NS}$ be any feasible indemnity function. Let $\Pi := \pi(f(X)) \in [0, \pi(X)]$, and for each $d \geq 0$, let $\hat{f}_d(X) := (X - d)^+$. Then $\hat{f}_d \in \mathcal{F}^{NS}$ for each $d \geq 0$. Moreover, since the capacity υ is monotone, finite, and continuous, it follows from Theorem 11.9 of Wang and Klir (2009) that the function $d \mapsto \pi\left(\hat{f}_d(X)\right)$ is decreasing and continuous on $[0, M_\upsilon]$. Hence, for each $d \in [0, M_\upsilon]$,

$$0 = \pi\left(\hat{f}_M(X)\right) \leq \pi\left(\hat{f}_d(X)\right) \leq \pi\left(\hat{f}_0(X)\right) = \pi(X).$$

Hence, by the Intermediate Value Theorem, there exists some $d^* \in [0, M_\upsilon]$ such that $\Pi = \pi\left(\hat{f}_{d^*}(X)\right)$. Consequently,

$$\Pi = \pi(f(X)) = \int_0^{f(M_\upsilon)} \upsilon\left(f(X) > z\right) dz \quad = \int_0^{M_\upsilon} \upsilon\left(X > z\right) df(z) = \int_0^{M_\upsilon} \upsilon\left(X > z\right) f'(z) dz,$$

where the second-to-last equality follows from Lemma 2.1 of Zhuang et al. (2016). Moreover,

$$\Pi = \pi(\hat{f}_{d^*}(X)) = \int_0^{M_\upsilon} \upsilon\left((X - d^*)^+ > z\right) dz = \int_0^{+\infty} \upsilon\left((X - d^*)^+ > z\right) dz$$
$$= \int_{d^*}^{+\infty} \upsilon\left(X > z\right) dz = \int_0^{M_\upsilon} \upsilon\left(X > z\right) \mathbb{1}_{[z > d^*]} dz.$$

Consequently,

$$\int_0^{M_\upsilon} \upsilon\left(X > z\right) \mathbb{1}_{[z>d^*]} dz = \int_0^{M_\upsilon} \upsilon\left(X > z\right) f'(z) dz. \tag{47}$$

Now, since $0 \le f'(z) \le 1$, for a.e. z, and since $M_\upsilon \le M_\mathbb{P}$,

$$\begin{aligned}
\mathbb{E}^\mathbb{P}\left[f\left(X\right)\right] - \mathbb{E}^\mathbb{P}\left[\hat{f}_{d^*}\left(X\right)\right] &= \int_0^{M_\mathbb{P}} \mathbb{P}\left(X > z\right)\left(f'(z) - \mathbb{1}_{[z>d^*]}\right) dz \\
&= \int_0^{d^*} \mathbb{P}\left(X > z\right)\left(f'(z) - \mathbb{1}_{[z>d^*]}\right) dz + \int_{d^*}^{M_\mathbb{P}} \mathbb{P}\left(X > z\right)\left(f'(z) - \mathbb{1}_{[z>d^*]}\right) dz \\
&= \int_0^{d^*} \mathbb{P}\left(X > z\right) f'(z) dz + \int_{d^*}^{M_\mathbb{P}} \mathbb{P}\left(X > z\right)\left(f'(z) - 1\right) dz \\
&\le \int_0^{d^*} \mathbb{P}\left(X > z\right) f'(z) dz + \int_{d^*}^{M_\upsilon} \mathbb{P}\left(X > z\right)\left(f'(z) - 1\right) dz \\
&= \int_0^{d^*} \frac{\upsilon\left(X > z\right)}{HR(z)} f'(z) dz + \int_{d^*}^{M_\upsilon} \frac{\upsilon\left(X > z\right)}{HR(z)}\left(f'(z) - 1\right) dz.
\end{aligned}$$

Moreover, since HR is non-increasing, it follows that

$$\begin{aligned}
\mathbb{E}^\mathbb{P}\left[f\left(X\right)\right] - \mathbb{E}^\mathbb{P}\left[\hat{f}_{d^*}\left(X\right)\right] &\le \int_0^{d^*} \frac{\upsilon\left(X > z\right)}{HR(z)} f'(z) dz + \int_{d^*}^{M_\upsilon} \frac{\upsilon\left(X > z\right)}{HR(z)}\left(f'(z) - 1\right) dz \\
&\le \frac{1}{HR(d^*)}\left[\int_0^{d^*} \upsilon\left(X > z\right) f'(z) dz + \int_{d^*}^{M_\upsilon} \upsilon\left(X > z\right)\left(f'(z) - 1\right) dz\right] \\
&= \frac{1}{HR(d^*)} \int_0^{M_\upsilon} \upsilon\left(X > z\right)\left(f'(z) - \mathbb{1}_{[z>d^*]}\right) dz = 0,
\end{aligned}$$

where the last equality follows from eq. (47). Hence,

$$\mathbb{E}^\mathbb{P}\left[X - f\left(X\right)\right] \ge \mathbb{E}^\mathbb{P}\left[X - \hat{f}_{d^*}\left(X\right)\right] = \mathbb{E}^\mathbb{P}\left[\min\left(X, d^*\right)\right]. \tag{48}$$

Moreover, since $0 \le f(X) \le X$, μ-a.s., and hence $\mathbb{P}$-a.s., one can easily verify that

$$\mathbb{P}\left(\min\left(X, d^*\right) > z\right) - \mathbb{P}\left(X - f\left(X\right) > z\right) \begin{cases} \le 0 & \text{if } z \ge d^*; \\ \ge 0 & \text{if } z < d^*. \end{cases} \tag{49}$$

Therefore, by Proposition 3.4.19 of Denuit et al. (2005),

$$\mathbb{E}^\mathbb{P}\left[v\left(\min\left(X, d^*\right)\right)\right] \le \mathbb{E}^\mathbb{P}\left[v\left(X - f\left(X\right)\right)\right],$$

for any non-decreasing and convex function v for which the expectations exist. Letting $v(x) = -u(W_0 - x - \Pi)$, it follows that

$$\begin{aligned}
\mathbb{E}^\mathbb{P}\left[v\left(\min\left(X, d^*\right)\right)\right] &= -\mathbb{E}^\mathbb{P}\left[u\left(W_0 - \min\left(X, d^*\right) + \Pi\right)\right] = -\mathbb{E}^\mathbb{P}\left[u\left(W_0 - X + \hat{f}_{d^*}\left(X\right) + \pi\left(\hat{f}_{d^*}\left(X\right)\right)\right)\right] \\
&\le \mathbb{E}^\mathbb{P}\left[v\left(X - f\left(X\right)\right)\right] = -\mathbb{E}^\mathbb{P}\left[u\left(W_0 - X + f\left(X\right) + \Pi\right)\right] \\
&= -\mathbb{E}^\mathbb{P}\left[u\left(W_0 - X + f\left(X\right) + \pi(f(X))\right)\right],
\end{aligned}$$

and hence $\mathbb{E}^{\mathbb{P}}\left[u\left(W_0 - X + \hat{f}_{d^*}(X) + \pi\left(\hat{f}_{d^*}(X)\right)\right)\right] \geq \mathbb{E}^{\mathbb{P}}\left[u\left(W_0 - X + f(X) + \pi(f(X))\right)\right]$. $\square$

Therefore, since HR is decreasing, by assumption, Lemma F.1 implies that an optimal solution for Problem (26) is given by $\hat{f}_{d^*}$, for some $d^* \geq 0$. The rest follows from Theorems 3.4 and 3.5.

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