Bowley Reinsurance with Asymmetric Information on the Insurer's Risk Preferences

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Abstract

The Bowley solution refers to the optimal pricing density for the reinsurer and optimal ceded loss for the insurer when there is a monopolistic reinsurer. In a sequential game, the reinsurer first set the pricing kernel, and thereafter the insurer selects the reinsurance contract given the pricing kernel. In this article, we study Bowley solutions under asymmetric information on the insurer's risk preferences where the identity of the insurer is unknown to the reinsurer. By assuming that the insurer adopts a Value-at-Risk measure or a convex distortion risk measure, the optimal pricing kernel for the insurer and the optimal ceded loss function for the reinsurer are determined. Numerical examples are presented to illustrate the results.

Key words: Bowley reinsurance; Asymmetric information; General premium principle; Distortion risk measure; Value-at-Risk.

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1 Introduction

This paper studies Bowley reinsurance solutions. Bowley reinsurance contracts are formed in a sequential game-theoretic framework that is similar to the one of Stackelberg equilibria. Bowley solutions are first studied by Chan and Gerber (1985), Cheung et al. (2019), and Chi et al. (2020). In the reinsurance market, we assume that there are one reinsurer and one insurer. In contrast to Chan and Gerber (1985), Cheung et al. (2019), and Chi et al. (2020), we assume that there is asymmetric information between the two parties. The insurer is endowed with a Value-at-Risk (VaR) or a convex distortion risk measure, and the reinsurer does not know the preferences of the insurer. The reinsurer communicates the premium principle to the insurers. The insurer observes this premium principle, and optimally purchases the reinsurance indemnity. So, the demand for reinsurance is a function of the premium principle. The monopolistic reinsurer will choose the principle that maximizes its expected profit, and so the reinsurer takes into account that the type of the insurer is random. Because the reinsurer cannot observe the preferences of the insurer, the preferences may only be revealed by the reinsurance contract that the insurer chose. This paper provides a theoretical framework to explain why both stop-loss and layer-type reinsurance contracts coexist in reinsurance markets.

Under the expected utility framework, Chan and Gerber (1985) derived the Bowley solutions in closed-form for many special cases under the expected value premium principle. Furthermore, Chi et al. (2020) continued to study a Bowley solution when the reinsurer's risk appetite is controlled by imposing upper limits on the first two moments of the coverage. In their model, the reinsurer imposes a reinsurance budget to the insurer, and the insurer selects its optimal indemnity given this premium budget. Cheung et al. (2019) extended the work of Chan and Gerber (1985) to the setting of general premium principle for the reinsurer and distortion risk measure for the insurer. The solution of this optimal reinsurance problem is reached by solving two sub-problems in order. The first step is to minimize the distortion risk measure of the retained loss of the insurer with concave distortion function for a given premium functional. This problem is closely related to optimal reinsurance with distortion risk measures; see Cui et al. (2013) and Assa (2015). The second step then for the reinsurer is to select the premium functional that maximizes the net expected gain of the reinsurer. This paper extends the literature in this second step, where we include asymmetric information of the preferences of the insurer.

While there are many economic papers on the effects of asymmetric information in insurance (e.g., Chiappori and Salenie, 2000; Finkelstein and Poterba, 2004; Cheung et

al., 2020), it is not frequently applied in actuarial science and optimal reinsurance. This paper aims to fill this gap in the literature. In actuarial science, Stackelberg equilibria with symmetric information are further studied by Albrecher and Dalit (2017), Chen and Shen (2018), and Anthropelos and Boonen (2020). Albrecher and Dalit (2017) studied a price competition between two insurers in one-period situation, while Chen and Shen (2018) studied Stackelberg equilibria in a dynamic differential game with one insurer and one reinsurer. Anthropelos and Boonen (2020) utilized the concept of Nash bargaining solutions, and assumed that the insurer and reinsurer can pose their risk aversion parameters in a strategic manner. In the Stackelberg equilibria, the agents mimic the risk aversion of the counterparty.

If the reinsurer can observe the risk preferences of the insurer, the reinsurer can exploit this information and offer a reinsurance contract to the insurer that makes the insurer indifferent between buying and not buying. This finding indeed appears in the papers above. In this paper, on the other hand, we show that asymmetric information regarding the preferences of the insurer yields a possible profit for the insurer. It is possible that both the insurer and the reinsurer strictly benefit from the reinsurance contract. Consistent with the literature on optimal reinsurance, we find that the insurer selects a layer-type indemnity profile if the insurer minimizes a VaR risk measure. Moreover, stop-loss profiles may be optimal when the insurer minimizes a convex distortion risk measure.

This paper differs from Cheung et al. (2019) in three fundamental aspects. Firstly, we consider the Bowley reinsurance contracts under asymmetric information on the insurer's risk preferences, where the identity of the insurer is unknown to the reinsurer. The reinsurer only has knowledge on the proportions of the insurer adopting the two types of distortion risk measures. This is different in nature from the symmetric information framework of Cheung et al. (2019). Secondly, for the results in Cheung et al. (2019), the optimal pricing kernel depends only on the distortion function and the cost of signing the contracts. As a contrast, the optimal pricing function derived in this paper can be flat in some claim interval and highly relies on the proportions of the types of the insurer. Therefore, the optimal ceded loss functions may be the shut-down policy, the stop-loss form, or the stop-loss contract with an upper limit depending on the risk appetite of the insurer. Thirdly, we show that the insurer may also strictly benefit from buying reinsurance in contrast with the results under symmetric information developed in Cheung et al. (2019). This leads to reinsurance contracts that may yield strictly positive welfare improvement for both the insurer and reinsurer. This does not hold in Bowley solutions or Stackelberg equilibria under symmetric information.

This paper is set out as follows. Section 2 states the asymmetric information problem

that we study in this paper. Section 3 solves this problem in case the insurer uses a VaR risk measure. Moreover, Section 4 solves this problem in case the insurer uses a VaR or a convex distortion risk measure. Section 5 further investigates this problem in case the insurer uses a convex distortion risk measure. Section 6 concludes the paper.

2 Problem formulation

Throughout, we assume all random variables are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let μ be the Lebesgue measure on $\mathbb{R}_+ := [0, \infty)$. Let $\mathbf{1}_A(s)$ be the indicator function such that $\mathbf{1}_A(s) = 1$ for $s \in A$ and $\mathbf{1}_A(s) = 0$ for $s \notin A$. The total loss faced by the insurer is denoted by the non-negative random variable X with a finite mean. The cumulative distribution function and survival function of X are denoted by F_X and $\overline{F}_X$, respectively. Henceforward, it is assumed that the distribution function F_X is known by both the insurer and the reinsurer. The quantile of X at level $p \in [0, 1]$ is denoted by $F_X^{-1}(p) := \inf\{x \in \mathbb{R}_+ | F_X(x) \geq p\}$. Let ν_X be the Radon measure on $[0, 1)$ such that $\nu_X([a, b)) = \mu\{x : a \leq F_X(x) < b\} = \mu\{x : F_X^{-1}(a) \leq x < F_X^{-1}(b)\} = (-F_X^{-1}(1-b)) - (-F_X^{-1}(1-a))$ for $0 \leq a < b < 1$ (cf. Cui et al., 2013; Cheung et al., 2019). The relevance of ν will become apparent later.

2.1 Admissible indemnity functions

The admissible set of ceded loss functions is defined as

$$\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ | f(0) = 0, 0 \leq f(x) - f(y) \leq x - y \text{ for } 0 \leq y \leq x\}.$$

The purpose of restricting the ceded loss functions to $\mathcal{F}$ is to avoid moral hazard or insurance swindles; see for instance Huberman et al. (1983), Denuit and Vermandele (1998), and Cheung et al. (2019). Any ceded loss function in $\mathcal{F}$ is increasing and satisfies the slow-growth property that an additional unit of loss cannot result in more than a unit increment of indemnity claim. Note that every function $f \in \mathcal{F}$ is 1-Lipschitz and hence absolutely continuous, which implies that f is almost everywhere differentiable on $\mathbb{R}_+$ and its derivative is bounded by one. More precisely, there exists a Lebesgue integrable function $h : \mathbb{R}_+ \to [0, 1]$ such that

$$f(x) = \int_0^x h(z)dz, \quad x \in \mathbb{R}_+, \tag{1}$$

where h is the slope of the ceded loss function f.

2.2 Distortion risk measure and general premium principle

A distortion function $g : [0,1] \to [0,1]$ is a non-decreasing and left-continuous function such that $g(0) = 0$ and $g(1) = 1$. The set of all distortion functions is denoted by $\mathcal{G}_d$, and the subset of distortion functions that are concave on [0, 1] is denoted as $\mathcal{G}_{cv}$. Let

$$\mathcal{G} = \{g : [0,1] \to \mathbb{R}_+ \mid g(0) = 0, g \text{ is non-decreasing, left-continuous and bounded}\}.$$

Obviously, it holds that $\mathcal{G}_{cv} \subset \mathcal{G}_d \subset \mathcal{G}$.

Definition 2.1 *A distortion risk measure ρ_g of a non-negative random variable Z with a distortion function $g \in \mathcal{G}_d$ is defined as*

$$\rho_g(Z) = \int_0^\infty g(\overline{F}_Z(t))dt, \tag{2}$$

provided that the integral exists, where $\overline{F}_Z(t) := 1 - F_Z(t)$ is the survival function of Z.

A distortion risk measure ρ_g is called a *convex distortion risk measure* when $g \in \mathcal{G}_{cv}$. Convex distortion risk measures preserve second order stochastic dominance (Wang et al., 1997), and imply aversion to mean-preserving spreads (Yaari, 1987). A prominent example of a distortion risk measure that is not convex is the VaR.

Definition 2.2 *The VaR of a non-negative random variable Z at a confidence level $\alpha \in (0,1)$ is defined as*

$$VaR_\alpha(Z) = \inf\{z \in \mathbb{R}_+ : \mathbb{P}(Z \le z) \ge \alpha\} = F_Z^{-1}(\alpha).$$

The VaR is a distortion risk measure with distortion function $g(t) = \mathbf{1}_{(1-\alpha,1]}(t)$.

We assume that the reinsurance premium principle adopted by the reinsurer is comonotonic additive and law invariant. In particular, for any ceded loss function $f \in \mathcal{F}$ purchased by the insurer, the reinsurance premium charged is determined by the following general premium principle

$$\Pi_{g_r}(f(X)) = \int_0^\infty g_r(\overline{F}_{f(X)}(z))dz = \int_0^\infty g_r(\overline{F}_X(z))h(z)dz, \tag{3}$$

where $g_r \in \mathcal{G}$, $h = f'$, and the second equality follows from Cheung and Lo (2017). If the probability space is non-atomic and g_r is continuously differentiable, then the premium principle (3) can be written via an underlying pricing kernel: $\Pi_{g_r}(f(X)) = \mathbb{E}[\zeta f(X)]$, where $\zeta = g_r'(\overline{F}_X(X))$ (e.g., Boonen and Ghossoub, 2020). Moreover, the assumption

that g_r is non-decreasing implies that the underlying pricing kernel is non-negative. It is obvious that the expected premium principle, Wang's premium principle (Wang, 1996), generalized percentile principle (Kaluszka, 2005) and VaR are special cases of the general premium principle in (3).

2.3 Bowley reinsurance solutions

We assume that insurer is endowed with a distortion function g_i that is in the class $\mathcal{G}_{cv} \cup \{\mathbf{1}_{(1-\alpha,1]}(t)\}$. In other words, the risk attitude of the insurer is measured by either a convex distortion risk measure or the VaR. Since the nature and properties of a convex distortion risk measure are significantly different from that of the VaR, they require different treatments and yield different results. More precisely, stop-loss policies are usually optimal with convex distortion risk measures, while stop-loss policies with an upper limit are usually optimal for the VaR.

To model the information asymmetry between the insurer and the reinsurer, we assume that the explicit characteristics of the insurer is hidden information for the reinsurer, and the reinsurer only knows that there are only two possible types of insurers. The reinsurer holds the opinion that insurer may minimize either $\rho_{g_{i1}}$ or $\rho_{g_{i2}}$ with probability p and $1-p$, respectively, where $p \in [0,1]$, and g_{i1} and g_{i2} are the two possible distortion functions adopted the insurer. We further assume that $\rho_{g_{i1}} = \mathrm{VaR}_{\alpha_1}$, i.e., the reinsurer believes with probability p that the insurer uses a VaR risk measure. Extension to the general case in which there are more than two types of distortion functions corresponding to VaR risk measures will be discussed in Section 3.

When the ceded loss function chosen by the insurer is given by $f \in \mathcal{F}$, then the total retained loss is equal to $X - f(X) + \Pi_{g_r}(f(X))$, where $g_r \in \mathcal{G}$ and $\Pi_{g_r}(f(X))$ is the corresponding reinsurance premium given by (3).

The pair of optimal decision problems faced by the insurer and the reinsurer we study in this paper is formalized as follows:

- (Decision problem faced by the insurer) For any given $g_r \in \mathcal{G}$ provided by the reinsurer, the insurer chooses the optimal ceded loss function $f \in \mathcal{F}$ by solving

$$\min_{f \in \mathcal{F}} \rho_{g_i}(X - f(X) + \Pi_{g_r}(f(X))), \tag{4}$$

  where $g_i = g_{i1}$ or $g_i = g_{i2}$ depending on the type of the insurer.

- (Decision problem faced by the reinsurer) The reinsurer is uncertain about the type of the insurer, but knowing the optimal decision functionals $f_{\{g_r;g_{i1}\}}$ or $f_{\{g_r;g_{i2}\}}$ of

the insurer corresponding to these two possible types of insurers for every $g_r \in \mathcal{G}$. The reinsurer chooses the optimal reinsurance pricing kernel g_r^* by maximizing the net profit. Therefore, the optimization problem of interest is

$$\begin{aligned}
&\max_{g_r \in \mathcal{G}} W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}}) \\
&:= \max_{g_r \in \mathcal{G}} \{p\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i1}\}}(X)) - f_{\{g_r;g_{i1}\}}(X)] - C(f_{\{g_r;g_{i1}\}})\} \\
&\qquad +(1-p)\{\mathbb{E}[\Pi_{g_r}(f_{\{g_r;g_{i2}\}}(X)) - f_{\{g_r;g_{i2}\}}(X)] - C(f_{\{g_r;g_{i2}\}})\}\}, \qquad (5)
\end{aligned}$$

where $C(f_{\{g_r;g_{ij}\}})$ denotes the aggregate administrative cost paid by the reinsurer if the insurer chooses the ceded loss function $f_{\{g_r;g_{ij}\}}$, for $j = 1, 2$.

Thus, the problem of this paper is summarized as follows:

$$\begin{aligned}
&\max_{g_r \in \mathcal{G}} W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}}) \\
&\text{s.t. } f_{\{g_r;g_{ij}\}} \in \operatorname*{argmin}_{f \in \mathcal{F}} \rho_{g_{ij}}(X - f(X) + \Pi_{g_r}(f(X))), \quad j = 1, 2,
\end{aligned}$$

where $W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}})$ is the net profit in (5) for given ceded loss functions $f_{\{g_r;g_{i1}\}}$ and $f_{\{g_r;g_{i2}\}}$. We refer to solutions $(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}})$ of the above problem as Bowley solutions.

Remark 2.3 *Because the reinsurer is assumed to be risk-neutral, the Bowley reinsurance solutions will be the same as in the following setting. There is one reinsurer and n insurers. There are then pn insurers that minimize $\rho_{g_{i1}}$ and $(1-p)n$ insurers that minimize $\rho_{g_{2i}}$, where $\{n, np\} \subset \mathbb{N}$. The objective function of the reinsurer is then equal to $nW(g_r)$, where W is defined in (5).*

Remark 2.4 *Note that there is only one pricing kernel in the market. Suppose that the reinsurer issues multiple pricing kernels to the insurer. By asymmetric information, the reinsurer cannot identify the type of the insurer, and thus all pricing kernels will be available to the insurer. Then, by taking advantage of this information asymmetry, the insurer may have an incentive to buy multiple reinsurance contracts in order to obtain the cheapest aggregate premium. This decision problem faced by the reinsurer has been solved by Boonen et al. (2016), who show that this problem can be reduced to a problem with a single, representative pricing kernel. Thus, letting the reinsurer issue a single pricing kernel is without loss of generality.*

Remark 2.5 *An alternative way to tackle the issue of information asymmetry faced by the reinsurer mentioned in Remark 2.4 is to develop a principal-agent model with the use*

of individual compatibility and individual rationality constraints. Such model has been adopted by Cheung et al. (2020) to analyze how an optimal menu of reinsurance contracts can be derived in case there is asymmetric information with respect to the underlying distribution of the loss of the insurer. However, as this kind of principal-agent models are not pricing kernel-specific, they are not useful to tackle our present problem; thus, we do not pursue this direction in this paper.

Remark 2.6 *Instead of minimizing a risk measure, the insurer might be interested in the expected wealth, while incurring a cost of holding risk capital. Then, instead of the objective function for the insurer in (4), we can adjust the objective of the insurer to:*

$$\begin{aligned}&\min_{f\in\mathcal{F}} \mathbb{E}[T_f(X)] + \delta(\rho_{g_i}(T_f(X)) + (1-\delta)\mathbb{E}[T_f(X)])\\ &= \min_{f\in\mathcal{F}} \delta\rho_{g_i}(T_f(X)) + (1-\delta)\mathbb{E}[T_f(X)],\end{aligned} \tag{6}$$

where $\{g_i, g_r\} \subset \mathcal{G}$, $T_f(X) := X - f(X) + \Pi_{g_r}(f(X))$ is the total retained loss of the insurer corresponding to ceded loss function f and $\delta \in [0,1]$ is a constant known as the cost-of-capital rate (cf. Chi, 2012; Cheung and Lo, 2017). The objective function in (6), known as the insurer's risk-adjusted liability, boils down to the objective function (4) when $\delta = 1$. Then, the preference relation in (6) is a distortion risk measure:

$$\delta\rho_{g_i}(T_f(X)) + (1-\delta)\mathbb{E}[T_f(X)] = \rho_{g_i^*}(T_f(X)),$$

with $g_i^*(t) = \delta g_i(t) + (1-\delta)t$, for $t \in [0,1]$ (see, e.g., Boonen et al. (2016)). If $g_i \in \mathcal{G}_{cv}$, then $g_i^* \in \mathcal{G}_{cv}$. Thus, our main results developed on convex distortion risk measures in Sections 4 and 5 can be naturally adapted to the alternative performance measure in (6).

The solutions to Problem (4) are stated in the following lemma.

Lemma 2.1 *For any $g_r \in \mathcal{G}$, the optimal ceded loss function $f^*_{(g_r,g_{ij})}$ that solves problem (4) is given by*

$$\begin{aligned}f^*_{\{g_r,g_{ij}\}}(x) &= \mu(\{z\in[0,x] \mid \psi_j(F_X(z)) > 0\})\\ &\quad + \int_0^x h_j(z)\mathbf{1}_{\{\psi_j(F_X(z))=0\}}(z)\mu(\mathrm{d}z), \quad x\in\mathbb{R}_+,\end{aligned} \tag{7}$$

where the function ψ_j is defined as

$$\psi_j(t) := g_{ij}(1-t) - g_r(1-t), \quad t\in[0,1],$$

and h_j could be any measurable function with $0 \le h_j(z)\mathbf{1}_{\{\psi_j(F_X(z))=0\}}(z) \le 1$, for $j = 1, 2$.

Proof. This is a direct consequence of applying Theorem 3.1 of Zhuang et al. (2016). ■

Lemma 2.1 states that the optimal ceded loss functions corresponding to g_{ij} depends on g_r through (7), for $j = 1, 2$. For $z \in \mathbb{R}_+$ such that $\psi_j(F_X(z)) = 0$, the insurer with distortion function g_{ij} is indifferent regarding the choice of $h_j(z)$, while the reinsurer may still make a profit by setting $h_j(z) = 1$. As mentioned in Laffont and Martimort (2009), under such indifference circumstance, it is common in the literature to achieve definiteness assuming that $h_j(z) = 1$, which means that the insurer is "willing to" act in favor of the reinsurer. In this way, we shall set $h_j(z) = 1$ in the sequel. We assume that the administrative cost of offering the compensation is proportional to the expectation of the ceded loss, i.e., $C(f) =: \gamma\mathbb{E}(f(X))$ for any $f \in \mathcal{F}$, where $\gamma \ge 0$ is a fixed constant standing for the cost coefficient. Under this setup, problem (5) boils down to solving

$$\begin{aligned}\max_{g_r\in\mathcal{G}} W(g_r) &:= \max_{g_r\in\mathcal{G}} W(g_r; f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}}) \\ &= \max_{g_r\in\mathcal{G}} \int_0^1 [g_r(t) - (1+\gamma)t]\,[p\mathbf{1}_{\{g_r(t)\le g_{i1}(t)\}}(t) \\ &\qquad +(1-p)\mathbf{1}_{\{g_r(t)\le g_{i2}(t)\}}(t)]\nu_X(\mathrm{d}t). \end{aligned} \tag{8}$$

This decision problem faced by the reinsurer is the main focus of this paper. Cheung et al. (2019) studied problem (8) when $p = 0$ or $p = 1$, i.e., the type of the insurer is known by the reinsurer so that there is no information asymmetry. Subsequent sections are devoted to finding complete characterization of the optimal solution of problem (8) for a general value of $p \in [0, 1]$.

3 Solutions for the case when the insurer is endowed with a VaR risk measure

In this section, we study problem (8) for the case where both g_{i1} and g_{i2} are the distortion functions for two different VaR risk measures. The solution is given in the following theorem, where $x_+ := \max\{x, 0\}$.

Theorem 3.1 *Suppose that $\gamma \ge 0$ and the two types of distortion functions of the insurer are $g_{i1}(t) = \mathbf{1}_{(1-\alpha_1,1]}(t)$ and $g_{i2}(t) = \mathbf{1}_{(1-\alpha_2,1]}(t)$ for $t \in [0, 1]$, with $1 > \alpha_1 \ge \alpha_2 > 0$. The solution of problem (8) is given as follows:*

- *Case 1:* $\gamma \geq \frac{1}{1-\alpha_1} - 1$. *The solution set to problem (8) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy* $\nu_X \{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$, *and* $\nu_X \{t \in [1-\alpha_1, 1) : g_r^*(t) \leq 1\} = 0$. *Moreover, for any optimal* g_r^*, *we have* $f^*_{\{g_r^*, g_{i1}\}}(X) = f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, *and* $W(g_r^*) = 0$.

- *Case 2:* $\frac{1}{1-\alpha_2} - 1 \leq \gamma < \frac{1}{1-\alpha_1} - 1$. *The solution set to problem (8) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy (i)* $g_r^*(t) = 1$ *for* $t \in (1-\alpha_1, \frac{1}{1+\gamma}]$, *and (ii)* $\nu_X \{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$ *and* $\nu_X \left\{t \in \left[\frac{1}{1+\gamma}, 1\right) : g_r^*(t) \leq 1\right\} = 0$. *Moreover, for any optimal* g_r^*, *we have* $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$,

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\left\{\left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_1) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right\},$$

for $x \in \mathbb{R}_+$, *and*

$$W(g_r^*) = pF_X^{-1}(\alpha_1) - pF_X^{-1}\left(1 - \frac{1}{1+\gamma}\right) - p(1+\gamma)\int_{1-\alpha_1}^{\frac{1}{1+\gamma}} t\nu_X(\mathrm{d}t).$$

- *Case 3:* $0 \leq \gamma < \frac{1}{1-\alpha_2} - 1$. *The solution set to problem (8) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy (i)* $g_r^*(t) = 1$ *for* $t \in (1-\alpha_1, \frac{1}{1+\gamma}]$, *and (ii)* $\nu_X \{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$ *and* $\nu_X \left\{t \in \left[\frac{1}{1+\gamma}, 1\right) : g_r^*(t) \leq 1\right\} = 0$. *Moreover, for any optimal* g_r^*, *we have*

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \min\left\{\left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_2) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right\}, x \in \mathbb{R}_+,$$

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\left\{\left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_1) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right\}, x \in \mathbb{R}_+,$$

and

$$W(g_r^*) = (1-p)\int_{1-\alpha_1}^{1-\alpha_2} [1 - (1+\gamma)t]\, \nu_X(\mathrm{d}t) + \int_{1-\alpha_2}^{\frac{1}{1+\gamma}} [1 - (1+\gamma)t]\, \nu_X(\mathrm{d}t).$$

Proof. Case 1: $\gamma \geq \frac{1}{1-\alpha_1} - 1$. For this case, it holds that

$$(1+\gamma)t > g_{i1}(t) \geq g_{i2}(t), \quad \text{for } t \in (0, 1).$$

Then, from the expression of $W(g_r)$ in (8), we know that $W(g_r) \leq 0$ for any $g_r \in \mathcal{G}$. Thus, the maximum value of $W(g_r)$ is 0, and it is reached by selecting those $g_r^* \in \mathcal{G}$ satisfying $\nu_X \{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$ and $\nu_X \{t \in [1-\alpha_1, 1) : g_r^*(t) \leq 1\} = 0$.

Case 2: $\frac{1}{1-\alpha_2} - 1 \leq \gamma < \frac{1}{1-\alpha_1} - 1$. For this case, it follows that (cf. Figure 1)

$$(1+\gamma)t > g_{i2}(t), \quad \text{for } t \in (0,1),$$

$$(1+\gamma)t > g_{i1}(t), \quad \text{for } t \in (0, 1-\alpha_1) \bigcup \left(\tfrac{1}{1+\gamma}, 1\right),$$

and

$$(1+\gamma)t < g_{i1}(t), \quad \text{for } t \in \left(1-\alpha_1, \tfrac{1}{1+\gamma}\right).$$

Thus, for the interval $\left[\frac{1}{1+\gamma}, 1\right)$ it must hold that

$$\nu_X \left\{ t \in \left[\frac{1}{1+\gamma}, 1\right) : g_r^*(t) \leq 1 \right\} = 0.$$

Now, we consider the value of g_r^* on the interval $\left(1-\alpha_1, \frac{1}{1+\gamma}\right)$. By using similar arguments as in Case 1, we should ensure the value of $W(g_r)$ as large as possible. Thus, we have

$$g_r^*(t) = 1, \quad \text{for } t \in \left(1-\alpha_1, \tfrac{1}{1+\gamma}\right).$$

Finally, on the interval $[0, 1-\alpha_1]$ the optimal g_r^* should be such that

$$\nu_X \left\{ t \in [0, 1-\alpha_1) : g_r^*(t) \leq 0 \right\} = \nu_X \left\{ t \in [0, 1-\alpha_1) : g_r^*(t) = 0 \right\} = 0.$$

Case 3: $0 \leq \gamma < \frac{1}{1-\alpha_2} - 1$. The proof is similar with that of Case 2, and thus omitted for brevity. ■

It should be noted that while the optimal pricing functions g_r^* described in Theorem 3.1 are exhaustive, uniqueness of g_r^* is not warranted as it is only characterized via the measure ν. It is possible to have multiple pricing functions that yield the same premium and the same net profit of the reinsurer for any choice of $f \in \mathcal{F}$ when F_X^{-1} is not strictly increasing, which is the case for instance when X is discrete. In other words, while g_r^* may be non-unique, for all Bowley solutions $(g_r, f_{\{g_r;g_{i1}\}}, f_{\{g_r;g_{i2}\}})$ with the same ceded loss function $f_{\{g_r;g_{ij}\}}$, the corresponding premium $\Pi_{g_r}(f_{\{g_r^*;g_{ij}\}}(X))$ is the same as well, where $j = 1, 2$.

From Theorem 3.1, we get that if the cost of contracting the reinsurance is very expensive for the reinsurer, there will be no business between the reinsurer and the insurer. If the cost is not that expensive but still rather substantial, then the reinsurer only signs the contract when the insurer's VaR is determined at a higher confidence level, and the insurer will be compensated by a stop-loss policy with an upper limit. For the situation

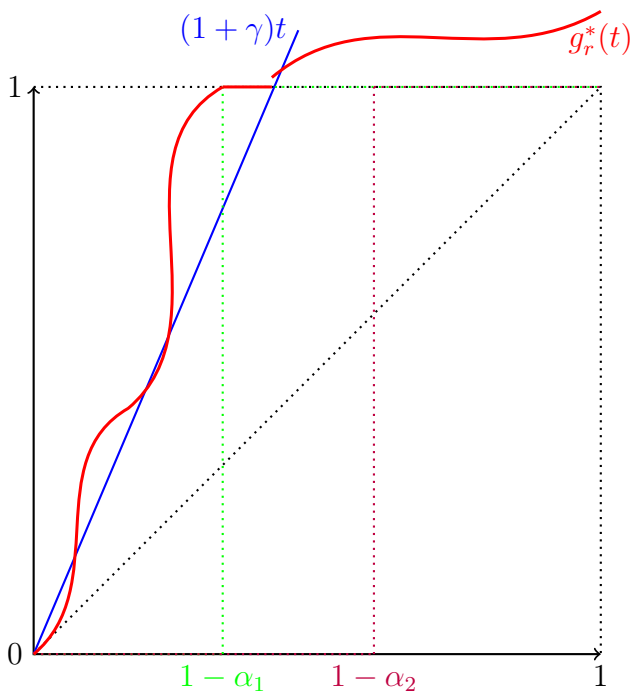

Figure 1: Illustration of g_r^* in Case 2 of Theorem 3.1.

where the cost is cheap for the reinsurer, there will be business between the reinsurer and the insurer no matter what the type of the insurer is. As a result, a two-layer reinsurance contract will be provided for the insurer, and the upper limit is changed according to the probability level of the insurer.

The next example illustrates the findings of Theorem 3.1.

Example 3.2 *Let $p = 0.9$, $\alpha_1 = 0.9$, $\alpha_2 = 0.8$ and the loss variable X is exponentially distributed with parameter $\lambda = 1$, i.e. $\overline{F}_X(x) = e^{-x}$ for $x \geq 0$.*

- *Case 1: $\gamma = 10$. Since $\gamma > \frac{1}{1-\alpha_1} - 1$, according to Theorem 3.1 we have $f^*_{\{g_r^*, g_{i1}\}}(X) = f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, and $W(g_r^*) = 0$. In other words, there is no business between the reinsurer and the insurer.*

- *Case 2: $\gamma = 6$. As per Theorem 3.1, we can conclude that a shutdown policy is provided for the insurer with distortion function g_{i2}, i.e., $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, and the optimal Bowley reinsurance policy for the insurer with distortion function g_{i1} is defined as*

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 1.946)_+, 0.357\}.$$

As a result, the net profit of the reinsurer is $W(g_r^*) = 0.051$.

- *Case 3:* $\gamma = 2$. *For this case, it can be calculated that*

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \min\{(x - 1.099)_+, 0.511\},$$

and

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 1.099)_+, 1.204\}.$$

This means that a higher limit is set up for the reinsurer with larger confidence level, which agrees with intuition. Furthermore, the net profit achieved by the reinsurer is given by $W(g_r^*) = 0.301$.

Following the the same line of reasoning as in the proof of Theorem 3.1, the Bowley solutions can also be derived when the reinsurer believes that the insurer uses one out of m possible VaR measures. Let $g_{ij}(t) = \mathbf{1}_{(1-\alpha_j,1]}(t)$ for $\alpha_j \in (0,1)$, and the reinsurer believes that the insurer is endowed with distortion function g_{ij} with probability p_j for $j = 1, \ldots, m$ and $m \geq 2$. By construction, $\sum_{j=1}^m p_j = 1$. Without loss of generality, we assume that $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. Then, problem (8) can be written as

$$\max_{g_r \in \mathcal{G}} \int_0^1 [g_r(t) - (1+\gamma)t] \left[\sum_{j=1}^m p_j \mathbf{1}_{\{g_r(t) \leq g_{ij}(t)\}}(t) \right] \nu_X(\mathrm{d}t). \tag{9}$$

The following theorem presents the solutions of problem (9); the proof is similar to that of Theorem 3.1 and is thus omitted.

Theorem 3.3 *Suppose that* $\gamma \geq 0$ *and the* m *types of distortion functions of the insurer are* $g_{ij}(t) = \mathbf{1}_{(1-\alpha_j,1]}(t)$, *for* $t \in [0,1]$ *and* $j = 1, \ldots, m$, *with* $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. *The solution of problem (9) is given as follows:*

- *Case 1:* $\gamma \geq \frac{1}{1-\alpha_1} - 1$. *The solution set to problem (9) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy* $\nu_X\{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$, *and* $\nu_X\{t \in [1-\alpha_1, 1) : g_r^*(t) \leq 1\} = 0$. *Moreover, for any optimal* g_r^*, *we have* $f^*_{\{g_r^*, g_{i1}\}}(X) = f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, *and* $W(g_r^*) = 0$.

- *Case 2:* $\frac{1}{1-\alpha_{j+1}} - 1 \leq \gamma < \frac{1}{1-\alpha_j} - 1$, *for some* $j = 1, \ldots, m-1$. *The solution set to problem (9) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy (i)* $g_r^*(t) = 1$ *for* $t \in (1-\alpha_j, \frac{1}{1+\gamma}]$, *and (ii)* $\nu_X\{t \in [0, 1-\alpha_j) : g_r^*(t) = 0\} = 0$ *and* $\nu_X\left\{t \in \left[\frac{1}{1+\gamma}, 1\right) : g_r^*(t) \leq 1\right\} = 0$. *Moreover, for any optimal* g_r^*, *we have* $f^*_{\{g_r^*, g_{ik}\}}(X) = 0$, *for* $k = j+1, \ldots, m$

$$f^*_{\{g_r^*, g_{ir}\}}(x) = \min\left\{ \left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_r) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right) \right\},$$

for $r = 1, \ldots, j$ *and* $x \in \mathbb{R}_+$, *and*

$$W(g_r^*) = \sum_{r=1}^{j} p_r \left[F_X^{-1}(\alpha_r) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right) - (1+\gamma) \int_{1-\alpha_r}^{\frac{1}{1+\gamma}} t\nu_X(\mathrm{d}t) \right].$$

- *Case 3:* $0 \le \gamma < \frac{1}{1-\alpha_m} - 1$. *The solution set to problem (9) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy (i)* $g_r^*(t) = 1$ *for* $t \in (1-\alpha_1, \frac{1}{1+\gamma}]$, *and (ii)* $\nu_X \{t \in [0, 1-\alpha_1) : g_r^*(t) = 0\} = 0$ *and* $\nu_X \left\{ t \in \left[\frac{1}{1+\gamma}, 1\right) : g_r^*(t) \le 1 \right\} = 0$. *Moreover, for any optimal* g_r^*, *we have*

$$f^*_{\{g_r^*, g_{im}\}}(x) = \min\left\{ \left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_m) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right) \right\}, x \in \mathbb{R}_+,$$

$$f^*_{\{g_r^*, g_{ir}\}}(x) = \min\left\{ \left(x - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right)\right)_+, F_X^{-1}(\alpha_r) - F_X^{-1}\left(1 - \frac{1}{1+\gamma}\right) \right\},$$

for $r = 1, \ldots, m-1$ *and* $x \in \mathbb{R}_+$, *and*

$$W(g_r^*) = \sum_{r=1}^{m-1} \int_{1-\alpha_r}^{1-\alpha_m} [1 - (1+\gamma)t]\, \nu_X(\mathrm{d}t) + \int_{1-\alpha_m}^{\frac{1}{1+\gamma}} [1 - (1+\gamma)t]\, \nu_X(\mathrm{d}t).$$

4 Solutions for the case when the insurer is endowed with a VaR risk measure or a convex distortion risk measure

In this section, we investigate problem (8) when one of the types of insurers adopts distortion function $g_{i1}(t) = \mathbf{1}_{(1-\alpha_1,1]}(t)$ (a VaR risk measure) while the other type of insurer adopts a strictly concave distortion function $g_{i2} \in \mathcal{G}_{cv}$.

Theorem 4.1 *Suppose that* $\gamma \ge 0$ *and the two types of distortion functions of the insurer are* $g_{i1}(t) = \mathbf{1}_{(1-\alpha_1,1]}(t)$ *and* $g_{i2} \in \mathcal{G}_{cv}$ *is strictly concave. The solution of problem (8) is given as follows:*

(i) If $g'_{i2}(0) \ge \frac{1}{1-\alpha_1}$, *then*

- *Case 1:* $\gamma \ge g'_{i2}(0) - 1$. *The solution set to problem (8) contains precisely those* $g_r^* \in \mathcal{G}$ *that satisfy*

$$\nu_X \{t \in [0,1) : g_r^*(t) \le \max\{g_{i1}(t), g_{i2}(t)\}\} = 0.$$

Furthermore, $f^*_{\{g_r^*,g_{i1}\}}(X)=0$, $f^*_{\{g_r^*,g_{i2}\}}(X)=0$, *and* $W(g_r^*)=0$.

– *Case 2:* $g'_{i2}(0)-1>\gamma\geq\frac{1}{1-\alpha_1}-1$. *Let* t_1 *be the solution of the equation*

$$g_{i2}(t)=(1+\gamma)t,\quad \text{for } t\in(0,1).$$

Then the solution set of (8) should be such that (i) $g_r^*(t)=g_{i2}(t)$, *for* $t\in[0,t_1]$; *and (ii)* $\nu_X\left\{t\in[t_1,1):g_r^*(t)\leq\max\{g_{i1}(t),g_{i2}(t)\}\right\}=0$. *Moreover,* $f^*_{\{g_r^*,g_{i1}\}}(x)=0$, $f^*_{\{g_r^*,g_{i2}\}}(x)=\left(x-F_X^{-1}(1-t_1)\right)_+$, *for* $x\in\mathbb{R}_+$, *and*

$$W(g_r^*)=(1-p)\int_0^{t_1}[g_{i2}(t)-(1+\gamma)t]\nu_X(\mathrm{d}t).$$

– *Case 3:* $\gamma<\frac{1}{1-\alpha_1}-1$. *Let* t_1 *and* t_2 *be the intersections of the straight line* $(1+\gamma)t$ *with* $g_{i2}(t)$ *and* $g_{i1}(t)$, *respectively. For points* $1-\alpha_1\leq t_3\leq t_4\leq t_1$, *let us denote*

$$\begin{aligned}V(t_3,t_4)=p\int_{1-\alpha_1}^{t_3}[g_{i2}(t_3)-(1+\gamma)t]\nu_X(\mathrm{d}t)+\int_{t_3}^{t_4}[g_{i2}(t)-(1+\gamma)t]\nu_X(\mathrm{d}t)\\+p\int_{t_4}^{t_1}[1-(1+\gamma)t]\nu_X(\mathrm{d}t).\end{aligned}\tag{10}$$

Let (t_3^*,t_4^*) *be a solution of the optimization problem* $\max_{1-\alpha_1\leq t_3\leq t_4\leq t_1}V(t_3,t_4)$. *Then, the solution of (8) is given as follows: (i)* $g_r^*(t)=g_{i2}(t)$, *for* $t\in[0,1-\alpha_1]$; *(ii)* $g_r^*(t)=g_{i2}(t_3^*)$, *for* $t\in(1-\alpha_1,t_3^*]$; *(iii)* $g_r^*(t)=g_{i2}(t)$, *for* $t\in(t_3^*,t_4^*]$; *(iv)* $g_r^*(t)=1$, *for* $t\in(t_4^*,t_2]$; *and (v)* $\nu_X\left\{t\in[t_2,1):g_r^*(t)\leq 1\right\}=0$. *Furthermore, we have*

$$f^*_{\{g_r^*,g_{i1}\}}(x)=\min\left\{\left(x-F_X^{-1}(1-t_2)\right)_+,F_X^{-1}(\alpha_1)-F_X^{-1}(1-t_2)\right\},\quad x\in\mathbb{R}_+,$$

and

$$\begin{aligned}f^*_{\{g_r^*,g_{i2}\}}(x)\;=\;&\min\left\{\left(x-F_X^{-1}(1-t_4^*)\right)_+,F_X^{-1}(1-t_3^*)-F_X^{-1}(1-t_4^*)\right\}\\&+\left(x-F_X^{-1}(\alpha_1)\right)_+,\quad x\in\mathbb{R}_+.\end{aligned}$$

The net profit for the reinsurer is

$$W(g_r^*)\;=\;(1-p)\int_0^{1-\alpha_1}[g_{i2}(t)-(1+\gamma)t]\nu_X(\mathrm{d}t)+p\int_{1-\alpha_1}^{t_3^*}[g_{i2}(t_3^*)-(1+\gamma)t]\nu_X(\mathrm{d}t)$$

$$+\int_{t_3^*}^{t_4^*}[g_{i2}(t)-(1+\gamma)t]\nu_X(\mathrm{d}t)+p\int_{t_4^*}^{t_2}[1-(1+\gamma)t]\nu_X(\mathrm{d}t).$$

(ii) If $g_{i2}'(0)<\frac{1}{1-\alpha_1}$, then

- *Case 1: $\gamma\geq\frac{1}{1-\alpha_1}-1$. The solution set to problem (8) contains precisely those $g_r^*\in\mathcal{G}$ that satisfy*

$$\nu_X\left\{t\in[0,1):g_r^*(t)\leq\max\{g_{i1}(t),g_{i2}(t)\}\right\}=0.$$

  Furthermore, $f^*_{\{g_r^*,g_{i1}\}}(X)=0$, $f^*_{\{g_r^*,g_{i2}\}}(X)=0$, and $W(g_r^*)=0$.

- *Case 2: $\frac{1}{1-\alpha_1}-1>\gamma\geq g_{i2}'(0)-1$. Let t_1 be the solution of the equation $(1+\gamma)t=g_{i1}(t)$ on $t\in(0,1)$, i.e., $t_1=1/(1+\gamma)$. Then, the solution set to problem (8) should be such that (i) $\nu_X\left\{t\in[0,1-\alpha_1):g_r^*(t)\leq g_{i2}(t)\right\}=0$; (ii) $g_r^*(t)=1$, for $t\in(1-\alpha_1,t_1]$; and (iii) $\nu_X\left\{t\in[t_1,1):g_r^*(t)\leq 1\right\}=0$. Moreover,*

$$f^*_{\{g_r^*,g_{i1}\}}(x)=\min\left\{\left(x-F_X^{-1}(1-t_1)\right)_+,F_X^{-1}(\alpha_1)-F_X^{-1}(1-t_1)\right\},$$

  $f^*_{\{g_r^*,g_{i2}\}}(x)=0$, for $x\in\mathbb{R}_+$, and

$$W(g_r^*)=p\int_{1-\alpha_1}^{t_1}[1-(1+\gamma)t]\nu_X(\mathrm{d}t).$$

- *Case 3: $\gamma<g_{i2}'(0)-1$. Then, the solution is the same as that given in Case 3 of (i).*

Proof. We only consider the case when $g_{i2}'(0)\geq\frac{1}{1-\alpha_1}$. The proof for the case when $g_{i2}'(0)<\frac{1}{1-\alpha_1}$ is similar and is omitted.

Case 1: $\gamma\geq g_{i2}'(0)-1$. For this case, we first observe that

$$(1+\gamma)t>g_{i1}(t)\quad\text{and}\quad(1+\gamma)t>g_{i2}(t),\quad\text{for } t\in(0,1),$$

which implies that the value of $W(g_r)$ in (8) is non-positive. Thus, the maximum value of $W(g_r)$ is 0 provided that $\nu_X\left\{t\in[0,1):g_r^*(t)\leq\max\{g_{i1}(t),g_{i2}(t)\}\right\}=0$. Hence, $f^*_{\{g_r^*,g_{i1}\}}(X)=0$, $f^*_{\{g_r^*,g_{i2}\}}(X)=0$, and $W(g_r^*)=0$.

Case 2: $g_{i2}'(0)-1>\gamma\geq\frac{1}{1-\alpha_1}-1$. For this case, it can be noted that (Figure 2)

$$(1+\gamma)t>g_{i1}(t)\quad\text{and}\quad(1+\gamma)t>g_{i2}(t),\quad\text{for } t\in(t_1,1),$$

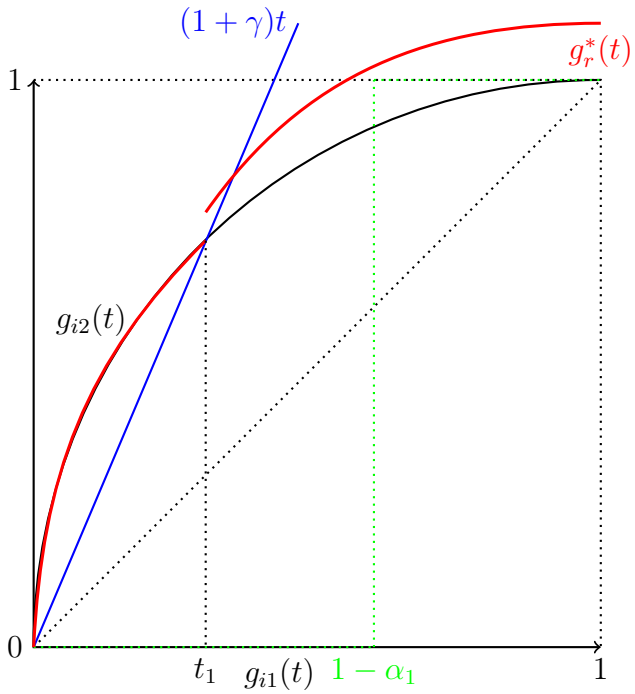

Figure 2: Illustrations of Case 2(i) in Theorem 4.1.

which indicate that $\nu_X\left\{t \in [t_1, 1) : g_r^*(t) \leq \max\{g_{i1}(t), g_{i2}(t)\}\right\} = 0$. Now, let us consider the form of $g_r^*(t)$ on $t \in [0, t_1)$. Clearly,

$$(1+\gamma)t > g_{i1}(t) \quad \text{and} \quad (1+\gamma)t < g_{i2}(t), \quad \text{for } t \in [0, t_1).$$

Based on the previous analysis, the optimal g_r^* must satisfy $\nu_X\left\{t \in [0, t_1) : g_r^*(t) \leq 0\right\} = 0$. Among this subclass, it is obvious that the optimal g_r^* must be following g_{i2} strictly on $[0, t_1)$ in order to maximize the value of $W(g_r)$. Then, by applying Lemma 2.1 we have $f^*_{\{g_r^*, g_{i1}\}}(x) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(x) = \left(x - F_X^{-1}(1-t_1)\right)_+$, for $x \in \mathbb{R}_+$, and

$$W(g_r^*) = (1-p)\int_0^{t_1} [g_{i2}(t) - (1+\gamma)t]\nu_X(\mathrm{d}t).$$

Case 3: $\gamma < \frac{1}{1-\alpha_1} - 1$. For this case, we know that the straight line $(1+\gamma)t$ intersects with $g_{i2}(t)$ and $g_{i1}(t)$ at the points $(t_1, g_{i2}(t_1))$ and $(t_2, 1)$, respectively. First, it is obvious that

$$(1+\gamma)t > g_{i1}(t) \quad \text{and} \quad (1+\gamma)t > g_{i2}(t), \quad \text{for } t \in (t_2, 1),$$

which implies that $\nu_X\left\{t \in [t_2, 1) : g_r^*(t) \leq 1\right\} = 0$.

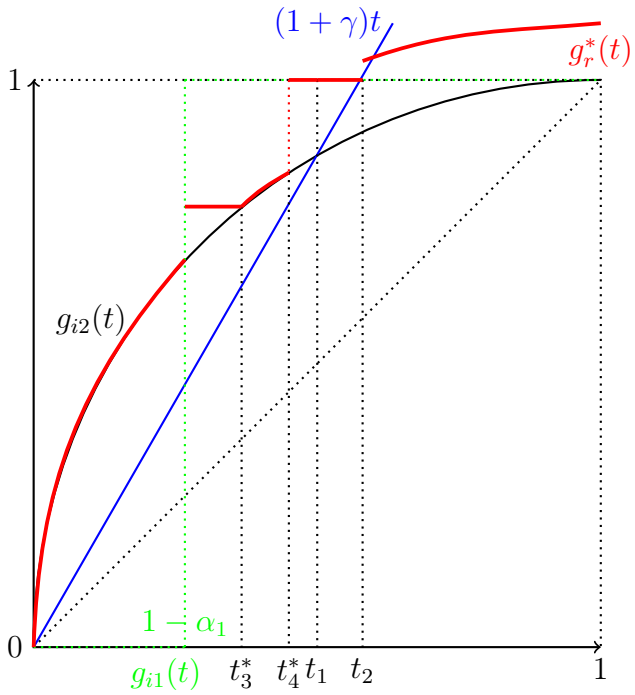

Figure 3: Illustrations of Case 3(i) in Theorem 4.1.

Furthermore, note that

$$(1+\gamma)t > g_{i1}(t) \quad \text{and} \quad (1+\gamma)t < g_{i2}(t), \quad \text{for } t \in (0, 1-\alpha_1],$$

from which we can conclude that $g_r^*(t)$ should be equal to $g_{i2}(t)$ for $t \in (0, 1-\alpha_1]$.

Next, we determine the form of $g_r^*(t)$ on the interval $t \in (1-\alpha_1, t_1)$. Since g_r^* is non-decreasing, it is sufficient to establish the following three results:

(a) if $g_r^*(t) > g_{i2}(t)$ for $t \in (a, b)$, then $g_r^*(t) = g_r^*(b)$ for all $t \in (a, b)$;

(b) if $g_r^*(t) = g_{i2}(b)$ for all $t \in (a, b]$, then $g_r^*(t) = g_{i2}(b)$ for all $t \in (1-\alpha_1, b]$;

(c) it is never optimal to have $g_r^*(t) < g_{i2}(t)$ for some $t \in [1-\alpha_1, t_1]$.

We start with (a). Suppose that it does not hold, i.e., there exists an interval (a, b) such that $g_r^*(t) > g_{i2}(t)$ for $t \in (a, b)$ and $g_r^*(t) < g_r^*(b)$ for some $t \in (a, b)$. According to the expression (8), it is easy to see that we can reset $g_r^*(t) = g_r^*(b)$ for $t \in (a, b)$ to increase the net profit of the reinsurer strictly, which means that we have our desired contradiction.

Next, we show (b). Suppose that $g_r^*(t) = g_{i2}(b)$ for all $t \in (a, b)$. Take a small $\varepsilon \in (0, a-(1-\alpha_1))$, and define the interval $A_\varepsilon = (a-\varepsilon, a]$. Clearly, if $g_r^*(t) > g_{i2}(t)$ for all $t \in A_\varepsilon$, then from the expression (8), it is easy to see that we can reset $g_r^*(t) = g_{i2}(b) > (1+\theta)t$

for all $t \in A_\varepsilon$ to increase the net profit of the reinsurer strictly. Moreover, if $g_r^*(t) < g_{i2}(t)$ for all $t \in A_\varepsilon$, then from the expression (8), we can reset $g_r^*(t) = g_{i2}(t) > (1+\theta)t$ to increase the net profit of the reinsurer strictly. Finally, we study $g_r^*(t) = g_{i2}(t)$ for all $t \in A_\varepsilon$. Define the function

$$\begin{aligned} h(t) &:= (g_{i2}(t) - (1+\theta)t) - p(g_{i2}(b) - (1+\theta)t) \\ &= g_{i2}(t) - p\, g_{i2}(b) - (1-p)(1+\theta)t, \end{aligned}$$

for $t \in (1-\alpha_1, b]$, which is concave since g_{i2} is assumed to be concave. Since $b < t_1$, it holds that $g_{i2}(b) \geq (1+\theta)b$ and thus $h(b) \geq 0$. Moreover, since $g_r^*(t) = g_{i2}(b)$ for all $t \in (a, b)$, it must hold that $\int_a^b h(t)\nu_X(dt) \leq 0$ due to the fact that the profit from the strategy $g_r^*(t) = g_{i2}(t)$ is no better than that from $g_r^*(t) = g_{i2}(b)$ on $t \in (a, b)$. Thus, since h is concave, it must hold that $h(a) \leq 0$ and $h(t) \leq 0$ for all $t \in A_\varepsilon$. From the expression (8), it is easy to see that we can reset $g_r^*(t) = g_{i2}(b) > (1+\theta)t$ for all $t \in A_\varepsilon$ to increase the net profit of the reinsurer strictly. Moreover, it follows by induction that it remains optimal to set $g_r^*(t) = g_{i2}(b) > (1+\theta)t$ for all $t \in (1-\alpha_1, b]$, and thus (b) is proven.

Result (c) follows trivially from expression (8) and $b < t_1$.

Now we show the main result. We distinguish four scenarios of $\lim_{t\downarrow 1-\alpha_1} g_r^*(t)$:

(i) $\lim_{t\downarrow 1-\alpha_1} g_r^*(t) < g_{i2}(1-\alpha_1)$: this is suboptimal by (c).

(ii) $\lim_{t\downarrow 1-\alpha_1} g_r^*(t) = g_{i2}(1-\alpha_1)$: by (c) the function g_r^* follows g_{i2}, and when it jumps up, by (a) and (b), it jumps to $g_r^*(t) = 1 = g_{i1}(t)$. Thus, $t_3^* = 1-\alpha_1$.

(iii) $1 > \lim_{t\downarrow 1-\alpha_1} g_r^*(t) > g_{i2}(1-\alpha_1)$: by (b), we have that there exists a $b \in [1-\alpha_1, t_1)$ such that $g_r^*(t) = g_{i2}(b)$ for all $t \in (1-\alpha_1, b)$. For $t \geq b$, we have from the same argument as (ii) that g_r^* firstly follows $g_{i2}(t)$ and then jumps to $g_{i1}(t) = 1$.

(iv) $\lim_{t\downarrow 1-\alpha_1} g_r^*(t) = 1$: Because the function is non-decreasing, it holds $g_r^*(t) = 1$ for all $t \in [1-\alpha_1, t_1)$. Thus, $t_3^* = t_4^* = 1-\alpha_1$.

The proof is finished. ■

In general, it may thus hold that $g_r^*(t) < g_{i1}(t)$ for some $t \in [0,1]$ and $g_r^*(t') < g_{i2}(t')$ for some other $t' \in [0,1]$. This implies that in the Bowley reinsurance solution it is possible for the insurer to strictly benefit as opposed to the status quo (no reinsurance). The insurer can benefit from the hidden information to the reinsurer to enjoy a cheaper premium. This is in sharp contrast to the Bowley solutions under symmetric and perfect information (Cheung et al., 2019).

A convex distortion risk measure ρ_g is called the Tail Value-at-Risk (TVaR) when $g(t) = \min\{1, \frac{t}{1-\alpha}\}$ for $t \in [0,1]$, where $\alpha \in [0,1)$. The distortion function g is concave, but not strictly concave. The following corollary can be obtained by a slight modification of the proof of Theorem 4.1.

Corollary 4.2 *Suppose that $\gamma \geq 0$ and the two types of distortion functions of the insurer are $g_{i1}(t) = \mathbf{1}_{(1-\alpha_1,1]}(t)$ and $g_{i2} = \min\{1, \frac{t}{1-\alpha_2}\}$ for $t \in [0,1]$, where $\alpha_1 \in (0,1)$ and $\alpha_2 \in [0,1)$. The solution of problem (8) is given as follows:*

(i) If $\alpha_1 < \alpha_2$, then

- *Case 1: $\gamma > \frac{\alpha_2}{1-\alpha_2}$. Then, $g_r^* \in \mathcal{G}$ should be such that*

$$\nu_X \left\{t \in [0,1) : g_r^*(t) \leq \max\{g_{i1}(t), g_{i2}(t)\}\right\} = 0.$$

  Furthermore, $f^*_{\{g_r^*, g_{i1}\}}(X) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, and $W(g_r^*) = 0$.

- *Case 2: $\frac{\alpha_2}{1-\alpha_2} \geq \gamma \geq \frac{\alpha_1}{1-\alpha_1}$. Let $t_1 = 1/(1+\gamma)$. Then, $g_r^* \in \mathcal{G}$ should be such that (i) $g_r^*(t) = \min\left\{1, \frac{t}{1-\alpha_2}\right\}$, for $t \in [0, t_1]$; and (ii) $\nu_X \left\{t \in [t_1, 1) : g_r^*(t) \leq 1\right\} = 0$. Moreover, $f^*_{\{g_r^*, g_{i1}\}}(x) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(x) = \left(x - F_X^{-1}(1-t_1)\right)_+$, for $x \in \mathbb{R}_+$, and*

$$W(g_r^*) = (1-p)\int_0^{t_1} \left[\min\left\{1, \frac{t}{1-\alpha_2}\right\} - (1+\gamma)t\right] \nu_X(\mathrm{d}t).$$

- *Case 3: $0 \leq \gamma < \frac{\alpha_1}{1-\alpha_1}$. Let $t_1 = 1/(1+\gamma)$. Then, the solution set of (8) is given as follows: (i) $g_r^*(t) = \min\left\{1, \frac{t}{1-\alpha_2}\right\}$, for $t \in [0, t_1]$; and (ii) $\nu_X \left\{t \in [t_1, 1) : g_r^*(t) \leq 1\right\} = 0$. Furthermore, we have*

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min\left\{\left(x - F_X^{-1}(1-t_1)\right)_+, F_X^{-1}(\alpha_1) - F_X^{-1}(1-t_1)\right\}, \quad x \in \mathbb{R}_+,$$

  and

$$f^*_{\{g_r^*, g_{i2}\}}(x) = \left(x - F_X^{-1}(1-t_1)\right)_+, \quad x \in \mathbb{R}_+.$$

  The net profit for the reinsurer is

$$W(g_r^*) = (1-p)\int_0^{1-\alpha_1} \left[\min\left\{1, \frac{t}{1-\alpha_2}\right\} - (1+\gamma)t\right] \nu_X(\mathrm{d}t) + \int_{1-\alpha_1}^{t_1} [1-(1+\gamma)t]\nu_X(\mathrm{d}t).$$

(ii) If $\alpha_1 \geq \alpha_2$, then

– *Case 1:* $\gamma \geq \frac{\alpha_1}{1-\alpha_1}$. *Then,* $g_r^* \in \mathcal{G}$ *should satisfy*

$$\nu_X \left\{ t \in [0,1) : g_r^*(t) \leq \max\{g_{i1}(t), g_{i2}(t)\} \right\} = 0.$$

Furthermore, $f^*_{\{g_r^*, g_{i1}\}}(X) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$, *and* $W(g_r^*) = 0$.

– *Case 2:* $\frac{\alpha_1}{1-\alpha_1} > \gamma \geq \frac{\alpha_2}{1-\alpha_2}$. *Let* $t_1 = 1/(1+\gamma)$. *Then, the solution set to (8) should be such that (i)* $\nu_X \left\{ t \in [0, 1-\alpha_1) : g_r^*(t) \leq t/(1-\alpha_2) \right\} = 0$; *(ii)* $g_r^*(t) = 1$, *for* $t \in (1-\alpha_1, t_1]$; *and (iii)* $\nu_X \left\{ t \in [t_1, 1) : g_r^*(t) \leq 1 \right\} = 0$. *Moreover,*

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min \left\{ \left(x - F_X^{-1}(1-t_1)\right)_+, F_X^{-1}(\alpha_1) - F_X^{-1}(1-t_1) \right\},$$

$f^*_{\{g_r^*, g_{i2}\}}(x) = 0$, *for* $x \in \mathbb{R}_+$, *and*

$$W(g_r^*) = p \int_{1-\alpha_1}^{t_1} [1 - (1+\gamma)t] \nu_X(\mathrm{d}t).$$

– *Case 3:* $0 \leq \gamma < \frac{\alpha_2}{1-\alpha_2}$. *Let* $t_1 = 1/(1+\gamma)$. *For any two points* $t_2, t_3 \in [1-\alpha_1, 1-\alpha_2]$ *such that* $t_2 \leq t_3$, *let us denote*

$$\begin{aligned} V(t_2, t_3) = p \int_{1-\alpha_1}^{t_2} \left[\frac{t_2}{1-\alpha_2} - (1+\gamma)t \right] \nu_X(\mathrm{d}t) \\ + \int_{t_2}^{t_3} \left[\frac{t}{1-\alpha_2} - (1+\gamma)t \right] \nu_X(\mathrm{d}t) + p \int_{t_3}^{1-\alpha_2} [1 - (1+\gamma)t] \nu_X(\mathrm{d}t). \end{aligned}$$

Let (t_2^*, t_3^*) *be a solution of the optimization problem*

$$\max_{t_2, t_3 \in [1-\alpha_1, 1-\alpha_2]} V(t_2, t_3),$$

that is, $(t_2^*, t_3^*) = \arg\max_{1-\alpha_1 \leq t_2 \leq t_3 \leq 1-\alpha_2} V(t_2, t_3)$. *Then, the solution of (8) is given as follows: (i)* $g_r^*(t) = \frac{t}{1-\alpha_2}$, *for* $t \in [0, 1-\alpha_1]$; *(ii)* $g_r^*(t) = \frac{t_2}{1-\alpha_2}$, *for* $t \in (1-\alpha_1, t_2^*]$; *(iii)* $g_r^*(t) = \frac{t}{1-\alpha_2}$, *for* $t \in [t_2^*, t_3^*]$; *(iv)* $g_r^*(t) = 1$, *for* $t \in (t_3^*, t_1]$; *and (v)* $\nu_X \left\{ t \in [t_1, 1) : g_r^*(t) \leq 1 \right\} = 0$. *Furthermore, we have*

$$f^*_{\{g_r^*, g_{i1}\}}(x) = \min \left\{ \left(x - F_X^{-1}(1-t_1)\right)_+, F_X^{-1}(\alpha_1) - F_X^{-1}(1-t_1) \right\}, \quad x \in \mathbb{R}_+,$$

and

$$\begin{aligned} f^*_{\{g_r^*, g_{i2}\}}(x) = \left(x - F_X^{-1}(1-t_1)\right)_+ - \left(x - F_X^{-1}(\alpha_2)\right)_+ + \left(x - F_X^{-1}(1-t_3^*)\right)_+ \\ - \left(x - F_X^{-1}(1-t_2^*)\right)_+ + \left(x - F_X^{-1}(\alpha_1)\right)_+, \quad x \in \mathbb{R}_+. \end{aligned}$$

The net profit for the reinsurer is

$$
\begin{aligned}
W(g_r^*) = (1-p)\int_0^{1-\alpha_1}\left[\frac{t}{1-\alpha_2}-(1+\gamma)t\right]\nu_X(\mathrm{d}t) \\
+p\int_{1-\alpha_1}^{t_2^*}\left[\frac{t_2^*}{1-\alpha_2}-(1+\gamma)t\right]\nu_X(\mathrm{d}t)+\int_{t_2^*}^{t_3^*}\left[\frac{t}{1-\alpha_2}-(1+\gamma)t\right]\nu_X(\mathrm{d}t) \\
+p\int_{t_3^*}^{1-\alpha_2}[1-(1+\gamma)t]\nu_X(\mathrm{d}t)+\int_{1-\alpha_2}^{t_1}[1-(1+\gamma)t]\nu_X(\mathrm{d}t).
\end{aligned}
$$

Next, we consider a numerical example where one type of insurer adopts the VaR risk measure while the other type adopts a proportional hazard (PH) distortion risk measure as proposed by Wang (1995).

Example 4.3 *Let $g_{i1}(t) = \mathbf{1}_{(1-\alpha_1,1]}(t)$ and $g_{i2}(t) = t^\beta$, for $\alpha_1, \beta \in (0,1)$. Clearly, $g_{i2} \in \mathcal{G}_{cv}$, and satisfies $g_{i2}'(0) = \infty > \frac{1}{1-\alpha_1}$ for any $\alpha_1 \in (0,1)$. Suppose that the claim X is exponentially distributed with mean 1. Let $\alpha_1 = 0.8$ and $p = 0.7$, which means that with probability 0.7 the insurer adopts the VaR risk measure with confidence level 0.8. Note that $\frac{1}{1-\alpha_1} = 5$.*

(i) Let $\gamma = 4$ and $\beta = 1/2$. For this case, we have $\gamma = \frac{1}{1-\alpha_1}$, which means that the solution of (8) corresponds to Case 2 in Theorem 4.1(i). It can be calculated that $t_1 = 0.04$, $f^*_{\{g_r^*,g_{i1}\}}(x) = 0$, $f^*_{\{g_r^*,g_{i2}\}}(x) = (x-3.2189)_+$, for $x \in \mathbb{R}_+$. This means that the reinsurer will not contract with the insurer if he adopts the VaR risk measure (i.e., a shut-down policy), while the reinsurance contract will be of the stop-loss treaty if the insurer adopts the PH distortion risk measure. Furthermore, the net profit can be computed as $W(g_r^*) = 0.06$.

(ii) Let $\gamma = 0.1$ and $\beta = 1/6$. Observe that $\gamma < \frac{1}{1-\alpha_1}$, which implies that the solution of (8) can be obtained from Case 3 in Theorem 4.1(i). It can be firstly computed that $t_1 = 0.8919$ and $t_2 = 0.9091$. The value of t_3^* and t_4^* can be derived through maximizing the function $V(t_3,t_4)$ on $t_3, t_4 \in [0.2, 0.8919]$ such that $t_3 \le t_4$. We find $t_3^* = 0.2$ and $t_4^* = 0.7961$, which means that $t_3^* = 1-\alpha_1$. Then, we have

$$
g_r^*(t) = \begin{cases} t^{0.5}, & t \in [0, 0.7961]; \\ 1, & t \in (0.7961, 0.9091]; \\ \text{any } \tilde{g}_r(t) \text{ such that} & \\ \nu_X\left\{t \in [0.9091, 1) : \tilde{g}_r(t) \le 1\right\} = 0, & t \in (0.9091, 1]. \end{cases}
$$

This means that g_r^* firstly follows the distortion function g_{i2} strictly up to the point

t_4^*, and then follows the expression of g_{i1} on $t \in (t_4^*, t_2]$. Beyond the point t_2 till the end point, there is no compensation from the reinsurer. By simple calculations, we have $f^*_{\{g_r^*, g_{i1}\}}(x) = \min\{(x - 0.0953)_+, 1.5141\}$ and $f^*_{\{g_r^*, g_{i2}\}}(x) = (x - 0.2280)_+$, for $x \in \mathbb{R}_+$, which implies that a layer reinsurance policy is contracted for the insurer with the VaR risk measure, while the traditional stop-loss compensation will be contracted for the insurer with the PH distortion risk measure. Furthermore, the net profit acquired by the reinsurer is $W(g_r^*) = 1.8464$.

5 Solutions for the case when the insurer is endowed with a convex distortion risk measure

Since the VaR risk measure is a special case of distortion risk measures but not belonging to the family of convex distortion risk measures, it is natural to seek for the Bowley reinsurance solution when both types of insurers adopt convex distortion risk measures.

In the next result, we discuss the Bowley reinsurance solution when both types of the distortion functions g_{i1} and g_{i2} are concave and have no intersection on $(0, 1)$.

Theorem 5.1 *Suppose that $\gamma \geq 0$ and $\{g_{i1}, g_{i2}\} \subset \mathcal{G}_{cv}$ such that $g_{i1}(t) > g_{i2}(t)$ for all $t \in (0, 1)$. The solution of problem (8) is given as follows:*

- *Case 1: $\gamma \geq g_{i1}'(0) - 1$. The solution set to problem (8) is any $g_r^* \in \mathcal{G}$ such that*

$$\nu_X \left\{ t \in [0, 1) : g_r^*(t) \leq g_{i1}(t) \right\} = 0.$$

  Besides, $f^*_{\{g_r^*, g_{i1}\}}(X) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$ and $W(g_r^*) = 0$.

- *Case 2: $g_{i2}'(0) - 1 \leq \gamma < g_{i1}'(0) - 1$. Let t_1 be the solution of the equation*

$$(1 + \gamma)t = g_{i1}(t), \quad t \in (0, 1).$$

  The optimal $g_r^* \in \mathcal{G}$ should satisfy (i) $g_r^*(t) = g_{i1}(t)$, $t \in [0, t_1]$, and (ii)

$$\nu_X \left\{ t \in [t_1, 1) : g_r^*(t) \leq g_{i1}(t) \right\} = 0.$$

  Moreover, $f^*_{\{g_r^*, g_{i2}\}}(x) = 0$, $f^*_{\{g_r^*, g_{i1}\}}(x) = \left(x - F_X^{-1}(1 - t_1)\right)_+$, $x \in \mathbb{R}_+$, and

$$W(g_r^*) = p \int_0^{t_1} \left[g_{i1}(t) - (1 + \gamma)t \right] \nu_X(\mathrm{d}t).$$

- *Case 3:* $0 \le \gamma < g'_{i2}(0) - 1$. *Let* t_2 *and* t_3 *be the respective solutions of the equations*

$$(1+\gamma)t = g_{i1}(t) \quad \text{and} \quad (1+\gamma)t = g_{i2}(t), \quad t \in (0,1).$$

The optimal $g_r^* \in \mathcal{G}$ *should be such that (i)* $g_r^*(t) = g_{i1}(t)$, $t \in (t_3, t_2]$, *and (ii)*

$$\nu_X \left\{ t \in [t_2, 1) : g_r^*(t) \le g_{i1}(t) \right\} = 0.$$

Moreover, for any $(a,b) \subset [0, t_3]$ *such that* $g_{i1}(t) > g_r^*(t) > g_{i2}(t)$ *for all* $t \in (a,b)$, *there exists a* $\hat{t} \in [b, t_3]$ *such that it holds that* $g_r^*(t) = g_r^*(b) = g_{i2}(\hat{t})$ *for all* $t \in (a, \hat{t}]$.

Proof. Case 1: $\gamma \ge g'_{i1}(0) - 1$. Note that $(1+\gamma)t > g_{i1}(t) > g_{i2}(t)$ for all $t \in (0,1)$. Thus, the integrands of $W(g_r, g_{i1})$ and $W(g_r, g_{i2})$ in (8) are non-positive for all admissible functions $g_r \in \mathcal{G}$ on $t \in [0,1]$. This further implies that the maximum value of $W(g_r)$ is zero and is attained at those $g_r \in \mathcal{G}$ such that

$$\nu_X \left\{ t \in [0,1) : g_r(t) \le g_{i1}(t) \right\} = 0.$$

The solution set to problem (8) is any $g_r^* \in \mathcal{G}$ such that

$$\nu_X \left\{ t \in [0,1) : g_r^*(t) \le g_{i1}(t) \right\} = 0.$$

Besides, $f^*_{\{g_r^*, g_{i1}\}}(X) = 0$, $f^*_{\{g_r^*, g_{i2}\}}(X) = 0$ and $W(g_r^*) = 0$.

Case 2: $g'_{i2}(0) - 1 \le \gamma < g'_{i1}(0) - 1$. For this case, we know that the straight line $(1+\gamma)t$ always lies above the curve $g_{i2}(t)$ for $t \in (0,1]$, and intersects with the curve $g_{i1}(t)$ at some point, say $t = t_1$, where t_1 is the solution of the equation

$$(1+\gamma)t = g_{i1}(t), \quad t \in (0,1).$$

Obviously, we have $g_{i1}(t) > (1+\gamma)t > g_{i2}(t)$ for $t \in (0, t_1)$ and $(1+\gamma)t > g_{i1}(t) > g_{i2}(t)$ for $t \in (t_1, 1)$. Thus, by applying the result of Case 1, we should ensure that the partial profit $W(g_r, g_{i2})$ should be 0, which implies that the optimal pricing function g_r^* should belong to the space

$$\mathcal{G}_{g_r, g_{i2}} := \left\{ g_r : g_r \in \mathcal{G},\ \nu_X \left\{ t \in [0,1) : g_r(t) \le g_{i2}(t) \right\} = 0 \right\}.$$

Within the class $\mathcal{G}_{g_r, g_{i2}}$, it holds that

$$\max_{g_r \in \mathcal{G}_{g_r, g_{i2}}} W(g_r) = p \max_{g_r \in \mathcal{G}_{g_r, g_{i2}}} W(g_r, g_{i1}).$$

Let us now consider the optimal g_r^* that maximizing $W(g_r, g_{i1})$. First, based on the analysis in Case 1, it can be derived that the optimal $g_r^* \in \mathcal{G}_{g_r,g_{i2}}$ should be such that

$$\nu_X \{t \in [t_1, 1) : g_r(t) \leq g_{i1}(t)\} = 0.$$

Then, it can be seen that

$$\max_{g_r \in \mathcal{G}_{g_r,g_{i2}}} W(g_r) = p \max_{g_r \in \mathcal{G}_{g_r,g_{i2}}} \int_0^{t_1} [g_r(t) - (1+\gamma)t] \, \mathbf{1}_{\{g_r(t) \leq g_{i1}(t)\}}(t) \nu_X(\mathrm{d}t). \tag{11}$$

By applying the observation that $g_{i1}(t) > (1+\gamma)t$ for $t \in (0, t_1)$, it immediately holds that the optimal $g_r \in \mathcal{G}$ should satisfy $g_r(t) = g_{i1}(t)$ for all $t \in [0, t_1]$ by using a similar argument in Case 2 of Theorem 4.1.

Thus, the optimal $g_r^* \in \mathcal{G}$ should satisfy (i) $g_r^*(t) = g_{i1}(t)$, $t \in [0, t_1]$, and (ii)

$$\nu_X \{t \in [t_1, 1) : g_r^*(t) \leq g_{i1}(t)\} = 0.$$

Moreover, $f^*_{\{g_r^*,g_{i2}\}}(x) = 0$, $f^*_{\{g_r^*,g_{i1}\}}(x) = \left(x - F_X^{-1}(1-t_1)\right)_+$, $x \in \mathbb{R}_+$ and

$$W(g_r^*) = p \int_0^{t_1} [g_{i1}(t) - (1+\gamma)t] \, \nu_X(\mathrm{d}t).$$

Case 3: $0 \leq \gamma < g_{i2}'(0) - 1$. We only give the proof for the general case $0 < \gamma < g_{i2}'(0) - 1$. Clearly, the straight line $(1+\gamma)t$ definitely intersects with the curves $g_{i1}(t)$ and $g_{i2}(t)$ at two points respectively, say $t = t_2$ and $t = t_3$ (see Figure 4). In other words, t_2 and t_3 are the respective solutions of the equations

$$(1+\gamma)t = g_{i1}(t) \quad \text{and} \quad (1+\gamma)t = g_{i2}(t), \quad t \in (0, 1).$$

First, by applying the argument in the proof of Case 2, it can be noted that the optimal g_r^* should be such that

$$\nu_X \{t \in [t_2, 1) : g_r(t) \leq g_{i1}(t)\} = 0 \quad \text{and} \quad \nu_X \{t \in [t_3, 1) : g_r(t) \leq g_{i2}(t)\} = 0.$$

Since $g_{i1}(t) > g_{i2}(t)$ for $t \in (0, 1)$, the pricing function $g_r^*(t)$ on the interval $t \in [t_2, 1)$ satisfy that $\nu_X \{t \in [t_2, 1) : g_r(t) \leq g_{i1}(t)\} = 0$.

Next, let us consider the pricing function $g_r^*(t)$ on $t \in (t_3, t_2)$. Note that $t_2 > t_3$ and $g_{i1}(t) > (1+\gamma)t > g_{i2}(t)$ for $t \in (t_3, t_2)$. According to the analysis of Case 2, it immediately follows that $g_r^*(t)$ must be equal to $g_{i1}(t)$ on $t \in (t_3, t_2)$.

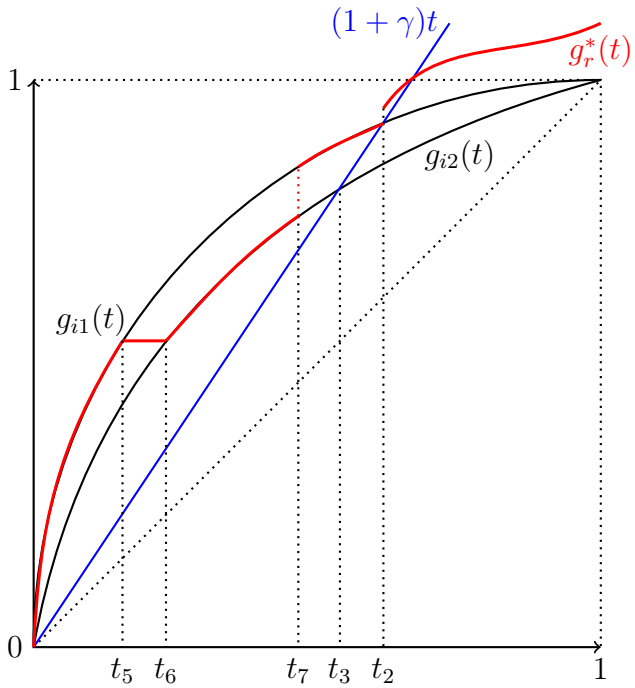

Figure 4: A possible solution of g_r^* for Case 3: $0 \leq \gamma < g_{i2}'(0) - 1$ in Theorem 5.1.

Finally, let us consider the pricing function $g_r^*(t)$ on $t \in (0, t_3)$. It is a direct result of

(a) if $g_{i1}(t) > g_r^*(t) > g_{i2}(t)$ for $t \in (a, b)$, then $g_r^*(t) = g_r^*(b)$ for all $t \in (a, b)$;

(b) if $g_{i1}(t) > g_r^*(t) > g_{i2}(t)$ for $t \in (a, b)$, then there exists a $\hat{t} \in [b, t_3]$ such that it holds that $g_r^*(t) = g_{i2}(\hat{t})$ for all $t \in (a, \hat{t}]$.

Here, (a) follows from the same arguments as in (a) in the proof of Theorem 4.1. Suppose that (b) does not hold. By (a), $g_r^*(t)$ remains flat until a point $\hat{t} \in [b, t_3]$. Suppose $g_r^*(t) < g_{i2}(\hat{t})$. This is a trivial violation of optimality as a result of expression (8). Suppose $g_r^*(t) > g_{i2}(\hat{t})$. By (a), it must then holds that $\lim_{t \downarrow \hat{t}} g_r^*(t) = g_{i1}(\hat{t})$. Then, we can reset $g_r^*(t) = g_{i1}(t)$ that yields a violation of optimality as a result of expression (8). This concludes the proof. ■

In light of Theorem 5.1, one can conclude that the reinsurer will not contract reinsurance with the insurer no matter what kind of risk attitude of the insurer is if the administration cost is too expensive to the reinsurer. If the cost is not that expensive, the reinsurer is only willing to contract with the insurer endowed with the larger distortion function. In this case, a shutdown policy is provided for the insurer adopting the smaller distortion function while a stop-loss policy is contracted with the insurer adopting the

larger distortion function. Finally, if the administration cost is low for the reinsurer, then the reinsurer will make a contract with the insurer regardless of the type.

Remark 5.2 *According to Case 3 of Theorem 5.1, the explicit expression of $g_r^*(t)$ can only be determined on the interval $[t_3, 1]$. Now, we focus our attention on the possible expressions of $g_r^* \in \mathcal{G}$ on the interval $[0, t_3]$. For any $t \in [0, t_3]$, we have $g_r^*(t) = g_{i1}(t)$, $g_r^*(t) = g_{i2}(t)$, or $g_r^*(t)$ is flat. If it is flat, it remains flat until it is equal to g_{i1}. Moreover, we have assumed that g_r^* is increasing. Here are several possibilities:*

1. *$g_r^*(t) = g_{i1}(t)$, for $t \in [0, t_3]$;*
2. *$g_r^*(t) = g_{i2}(t)$, for $t \in [0, t_3]$;*
3. *$g_r^*(t) = g_{i2}(t)$, for $t \in [0, t_4]$, and $g_r^*(t) = g_{i1}(t)$, for $t \in (t_4, t_3]$, where $t_4 \in (0, t_3)$;*
4. *$g_r^*(t) = g_{i1}(t)$, for $t \in [0, t_5]$, $g_r^*(t) = g_{i1}(t_5)$, for $t \in (t_5, t_6]$, $g_r^*(t) = g_{i2}(t)$, for $t \in [t_6, t_7]$, and $g_r^*(t) = g_{i1}(t)$, for $t \in (t_7, t_3]$.*

Note that other functional forms may be optimal.

Remark 5.3 *In Theorem 5.1, we find that if $\{g_{i1}, g_{i2}\} \subset \mathcal{G}_{cv}$ is such that $g_{i1}(t) > g_{i2}(t)$ for $t \in (0, 1)$, it may happen that $g_r^*(t) < g_{i1}(t)$ for some t. If the insurer is of type 1, the insurer may thus strictly benefit from reinsurance. This is in sharp contrast to the literature on Bowley solutions with symmetric information (Chan and Gerber, 1985; Cheung et al., 2019).*

Th next example demonstrates a situation where the pricing function g_r^* is flat somewhere.

Example 5.4 *Let X have realizations in $\{0, 1, 2\}$, each with probability $\frac{1}{3}$. Moreover, let $g_{i1}(\frac{1}{3}) = 1, g_{i1}(\frac{2}{3}) = 1, g_{i2}(\frac{1}{3}) = 0.5$, and $g_{i2}(\frac{2}{3}) = 0.95$. Note that by defining piecewise linear functions g_{i1}, g_{i2} in this way, it holds that $\{g_{i1}, g_{i2}\} \in \mathcal{G}_{cv}$. Moreover, $\gamma = 0$ and $p = 0.5$. Then, numerically we find $g_r^*(\frac{1}{3}) = g_r^*(\frac{2}{3}) = 0.95$. Thus, this is a case where g_r^* is flat on $[\frac{1}{3}, \frac{2}{3}]$. Moreover, if the insurer has the distortion function of the first type, then $f_{\{g_r^*, g_{i1}\}}(X) = X$, and otherwise $f_{\{g_r^*, g_{i2}\}}(X) = \mathbf{1}_{[1,2]}(X)$.*

6 Concluding Remarks

The optimal reinsurance decision problem plays a vital role in actuarial science. Most of the existing work deal with this interesting problem from the viewpoint of the insurer

under some constraints set by the reinsurer. In this paper, under the setting that the reinsurer is uncertain about the type of the distortion functions that the insurer adopts, we use a two-step optimization procedure to seek for the optimal ceded loss function for the insurer as well as the optimal pricing distortion function for the reinsurer by thoroughly analyzing three specific cases of the risk measures adopted by the two possible type of insurers: (i) two different VaR risk measures; (ii) a VaR risk measure and a convex distortion risk measure; and (iii) two different convex distortion risk measures. Our results show that the optimal ceded loss funtion has the form of layer reinsurance treaties depending on the cost coefficient and the explicit forms of the two types of distortion functions adopted by the insurer in the market.

In contrast to the literature on Bowley solutions with symmetric information, this paper shows that the insurer may also strictly benefit from buying reinsurance. As such, asymmetric information is a natural framework that allows for reciprocal contracts.

In most practical situations, in addition to maximizing his own net expected gain, the reinsurer would also like to control his risk exposure under some specified level at the same time. This leads naturally to the problems of identifying Bowley solutions under various kinds of risk constraints. Further study and investigations along this direction are expected in the future.

To conclude the paper, let us assume that there are two reinsurers in the market (a duopoly) and both of the reinsurers do not know the type of the single insurer. Under this new setting, extending the present analysis to two reinsurers yields a very different mechanism, and let us focus on competitive pricing as in Boonen et al. (2018). Then, the competition refers to the situation where there are (at least) two reinsurers that cannot cooperate with each other. The reinsurers compete by setting prices simultaneously and insurer selects the reinsurance contracts by minimizing a distortion risk measure of final risk. If two reinsurers charge the same prices, the insurers' demand is split evenly between them. If the cost-functions C are the same for both reinsurers, it follows from similar arguments as in Boonen et al. (2018) that the reinsurers are both making an expected profit of zero in the competitive reinsurance contracts. We wish to emphasize that an appealing characteristic of our monopolistic model is that both the insurer and the reinsurer may strictly benefit from the reinsurance transaction, and thus we leave the reinsurance problem with competition among multiple reinsurers for future research.

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