Peer-to-peer risk-sharing schemes with heterogeneity and infinite-mean losses

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Abstract

The increasing intensity of extreme catastrophic events in recent years highlights the critical need for insurance to protect against their potential disastrous impacts. However, such catastrophic losses are often regarded as uninsurable by insurers, and there is a lack of risk-management solutions for them. Contrary to basic intuition, the literature has shown that within a specific class of risk-sharing rules, diversifying infinite-mean Pareto losses is always harmful. Consequently, the optimal action is non-diversification, which effectively leads to a lack of protection against such catastrophic risks. In this paper, by considering a broader class of risk-sharing rules, we construct novel risk-sharing mechanisms as alternatives to non-diversification for managing catastrophic Pareto risks. To establish a foundation, we first study linear risk-sharing rules in a peer-to-peer risk-sharing setting with heterogeneity. Within this class, a Pareto optimal risk-sharing rule is obtained: uniform risk sharing among agents with finite-mean Pareto losses and no risk sharing among agents with infinite-mean Pareto losses. Next, by introducing non-linearity, we construct two novel risk-sharing rules for managing infinite-mean catastrophic Pareto risks. We then present theoretical results to justify the benefits of the proposed risk-sharing rules, supported by numerical illustrations.

Keywords: Peer-to-peer risk sharing, heterogeneous risks, infinite-mean losses, first-order stochastic dominance, convex order.

JEL classification: G22.

1 Introduction

Catastrophic and extreme losses have become more common and intense in recent years (Embrechts et al., 1999; Sheremet and Lucas, 2009; Cui et al., 2021). Natural disasters, like earthquakes, floods, and hurricanes, can lead to significant harm and losses to different countries. In view of this, insurance protection against such events is highly desirable.

Nevertheless, the traditional insurance and reinsurance markets for the protection of extreme losses have failed and been lacking worldwide (Ibragimov et al., 2009; Sheremet and Lucas, 2009; Wu, 2020; Cui et al., 2021), mainly because such heavy-tailed catastrophic losses are often deemed uninsurable. Reasons for the uninsurability include the difficulty in pricing and the loss of benefits from risk pooling. Pricing becomes difficult because actuarial pricing techniques often rely heavily on expectations; see, e.g., Kaas et al. (2008). Risk pooling loses its effectiveness due to the inapplicability of the *Law of Large Numbers*, a fundamental principle for managing the risk when offering insurance (Wu, 2020; Tavanaie Marvi and Linders, 2021). In fact, pooling infinite-mean losses could *increase* the likelihood of bankruptcy instead (Ibragimov et al., 2009; Wu, 2020).

In spite of this, there are well-established frameworks for modeling infinite-mean catastrophic losses. A common framework is the Extreme Value Theory (EVT), which allows us to model catastrophic losses resulting from extreme and tail events using *infinite-mean* and *heavy-tailed* models. Examples of utilizing

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infinite-mean models for modeling different kinds of risks in insurance and finance can be found in Chen and Wang (2025).

As we associate catastrophic losses with infinite-mean losses, some strange and unusual implications for catastrophic risk management emerge. In standard cases (e.g., mean-variance portfolio theory), *diversification* is shown to be beneficial, and so it is widely accepted and believed that diversification opportunities should be captured whenever they appear. Nonetheless, many such arguments are implicitly based on the assumption that the underlying random variables are of *finite mean*; when the losses are of *infinite mean*, many standard theories and results break down and are no longer applicable. In fact, it has been shown counter-intuitively that in certain cases, diversifying infinite-mean losses would only worsen the outcome (Ibragimov et al., 2009; Chen et al., 2025b), and thus the optimal action turns out to be *non-diversification* in such cases. This can also serve as an explanation to the failure of the market for insuring against catastrophic losses.

However, by having no diversification, the exposure to the extreme losses would just remain unchanged, thus the agents would still be prone to their great impact. In view of this intricacy, in this paper, we will demonstrate a novel approach for managing infinite-mean catastrophic Pareto losses by utilizing specially designed schemes of peer-to-peer (P2P) risk sharing. Our approach provides alternatives to having no risk sharing when facing catastrophic losses, based on a separation argument: Each infinite-mean loss will be separated into two parts: a finite-mean part to be shared with other agents, and the residual part to be retained.

Traditional insurance operates in a centralized manner: There is an insurer (central party) offering insurance to the policyholders, and they share their risks to the insurer. This contrasts with a decentralized or P2P design, where no central party exists, and agents instead participate in an agreement to share risks with each other. As the idea of decentralization has gained more attention, there has been increasing literature studying various aspects of P2P risk sharing recently; see, e.g., Denuit and Dhaene (2012), Abdikerimova and Feng (2022), Denuit and Robert (2023), Feng et al. (2023), and Chen et al. (2025a,b). P2P risk sharing is not only of theoretical interest, but also gets implemented in practice, e.g., P2P insurance is offered by the companies Friendsurance and Lemonade. P2P insurance carries numerous advantages, including cost reduction and higher flexibility in the design of the underlying risk-sharing mechanisms (Abdikerimova and Feng, 2022). This paper shows another advantage of the P2P design, namely the possibility of protecting against infinite-mean catastrophic losses, which is difficult to achieve in the traditional insurance setting as discussed previously.

The recent literature on P2P risk sharing utilizes two common approaches. In the first approach, individual losses are first pooled and subsequently redistributed among the participants. An example of this kind is *conditional mean risk sharing* (CMRS), which is an extensively studied risk-sharing rule; see, e.g., Denuit and Dhaene (2012) and Denuit and Robert (2023). In CMRS, the loss after risk sharing for each agent is simply the conditional mean of the initial loss, given the sum of all initial losses for the agents involved. If the initial losses are independent and identically distributed (iid), then CMRS simplifies to an equal share of the aggregate risk in the market, also known as *uniform risk sharing*. Various properties of P2P risk-sharing rules following the first approach are discussed in the literature, e.g., *actuarially fair*, *fully allocating*, and *Pareto optimal*, whose definitions are briefly reviewed in the following:

- A risk-sharing contract is called **actuarially fair** if it preserves the same expectation before and after risk sharing. This concept was introduced by Arrow (1963).
- A risk-sharing contract is called **fully allocating** if all risk will be allocated to agents, i.e., the economy is market clearing (also called self-financing), and no money is added or subtracted from the market.
- A risk-sharing contract is called **Pareto optimal** if there does not exist another contract that is an improvement for all agents and a strict improvement for at least one agent.

In the second approach, the transfer of risk between agents is explicitly derived. This approach is discussed by Abdikerimova and Feng (2022), Feng et al. (2023), and Abdikerimova et al. (2024), which impose a particular structure on the risk-sharing contracts. Their focuses are on (i) pro-rata (linear) risk allocations, making it explicit how much risk one agent shares with another, and (ii) a social planner with mean-variance preference. But because mean-variance preferences are used, the losses must have finite means and variances. This limitation is the key focus of the papers by Chen et al. (2025a,b), which investigate P2P risk sharing in a pool of identically distributed infinite-mean losses — *homogeneous* risk; a special case is that the pool contains iid infinite-mean Pareto losses. Chen et al. (2025b) have shown that in such a scenario, having no risk sharing is preferred to any *linear* risk allocation.

In this paper, we contribute to the literature on P2P risk sharing by proposing risk-sharing schemes that allow some infinite-mean Pareto losses to be shared without worsening the outcomes. We consider two main extensions of the case studied in Chen et al. (2025a,b), namely *non-linearity* in risk allocations and *heterogeneity* in loss distributions.[1] These two extensions are motivated by the following research questions:

1. What is the impact of including *finite-mean* losses in the pool, thereby introducing *heterogeneity*?
2. What is the impact of introducing *non-linearity* in P2P risk sharing? Is it possible that some non-linear risk sharing rules can lead to further improvements?

In general, P2P risk sharing starts with n random variables $X_1, \dots, X_n$, representing the losses before risk sharing for agents $1, \dots, n$, respectively. After risk sharing, the losses for agents $1, \dots, n$ are often denoted by $Y_1, \dots, Y_n$, respectively. The extensions discussed here fall within the realm of P2P risk sharing with two independent heterogeneous groups: one containing iid finite-mean losses and the other containing iid infinite-mean losses. As these conditions will be frequently imposed throughout the paper, we call them "two-group conditions" for convenience, which are formally defined below.

Definition 1.1 (Two-group conditions)**.** The **two-group conditions** refer to the following three conditions:

1. The losses $X_1, \dots, X_n$ are independent.
2. The first m losses $X_1, \dots, X_m$ are iid finite-mean random variables.
3. The remaining $n - m$ losses $X_{m+1}, \dots, X_n$ are iid infinite-mean random variables.

Practically, the $n-m$ agents having iid infinite-mean losses may be interpreted as (not too geographically close) countries exposing to different natural disasters. With non-linearity and heterogeneity, we show the possibility of designing P2P risk-sharing rules that mutually benefit agents with finite-mean and infinite-mean Pareto losses, though our argument for the agents with infinite-mean losses is more heuristic. As a result, we can incorporate infinite-mean catastrophic Pareto losses into P2P risk-sharing processes, thereby offering protection against them. This approach can control their exposures to catastrophic and tail events, a critical element in catastrophic risk management.

The risk-sharing schemes discussed in this paper are briefly described below:

- **Scheme [L]: linear risk-sharing scheme.** For every loss initially faced by the agents, a certain fixed proportion is distributed to each agent in the pool, such that the whole loss is covered, i.e., the proportions add up to 1. This is known as the fully allocating property in the P2P risk sharing literature.
- **Scheme [FR]: risk-sharing scheme with infinite-mean losses decomposed into finite and residual losses.** This scheme incorporates non-linearity, and allows for the exchange of risks between finite-mean and infinite-mean agents by separating each infinite-mean loss into two parts; one of which is a synthetic finite-mean loss. With the original and synthetic finite-mean losses, we are able to construct risk-sharing rules that allow parts of infinite-mean losses to be shared, while being beneficial to all the agents involved.
- **Scheme [LS]: risk-sharing scheme with infinite-mean losses decomposed into limited-loss and stop-loss variables.** This scheme is also non-linear and provides us another way to separate infinite-mean losses. Since the separation is based on limited-loss coverage for agents with infinite-mean losses, this scheme may be more natural and understandable than the scheme [FR]. From a theoretical perspective, this scheme also yields improvements over the scheme [FR] for agents with finite-mean losses.

[1] Recently, Chen et al. (2025) have also studied risk sharing with heterogeneous random variables. Their focus is on markets with only infinite-mean Pareto distributed losses. Extensions through generalizing the loss distribution being considered have also been made; see, e.g., Arab et al. (2025), Müller (2025), and Chen and Shneer (2026).

This paper is set out as follows. We will briefly introduce some notations and relevant concepts in Section 2. In Sections 3 to 5, we will discuss and compare different P2P risk-sharing schemes with theoretical results justifying the benefits of utilizing risk-sharing rules constructed therein for managing infinite-mean Pareto losses. Section 6 conducts numerical studies, and Section 7 concludes this paper.

2 Preliminaries

Let us first introduce some notations and review some relevant concepts for this paper.

Throughout this paper, random variables are real-valued and defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We then state some notational conventions and terms about the two-group conditions in Definition 1.1:

- We shall suppose that m is a positive integer less than n, so that both finite-mean and infinite-mean losses are present. The case where $m = 0$, i.e., only iid infinite-mean losses are present, is covered by Chen et al. (2025b) when the losses are Pareto distributed. The case where $m = n$, i.e., only iid finite-mean losses are present, is studied extensively in the literature; see, e.g., Denuit et al. (2005).
- Agents $1, \dots, m$ are called **finite-mean agents**, and agents $m+1, \dots, n$ are called **infinite-mean agents**.
- A **risk allocation** or a **rule** is any vector $(Y_1, \dots, Y_n)$ of losses after risk sharing for agents $1, \dots, n$. Recall the concept of *fully allocating* from Section 1; mathematically, a risk allocation $(Y_1, \dots, Y_n)$ is **fully allocating** when the aggregate loss stays the same before and after risk sharing, i.e., $\sum_{i=1}^n Y_i = \sum_{i=1}^n X_i$. We refer to a **scheme** as a particular subset of risk allocations that are fully allocating.

Next, we introduce some relevant concepts to be used later in this paper. Let X and Y be two random variables. Then, X is smaller than Y in **first-order stochastic dominance**, denoted by $X \le_{\text{st}} Y$, if $\mathbb{P}(X > t) \le \mathbb{P}(Y > t)$ for all $t \in \mathbb{R}$; it is in **strict first-order stochastic dominance** if also $\mathbb{P}(X > t) < \mathbb{P}(Y > t)$ for some $t \in \mathbb{R}$. Moreover, X is smaller than Y in **convex order**, denoted by $X \le_{\text{cx}} Y$, if $\mathbb{E}[\varphi(X)] \le \mathbb{E}[\varphi(Y)]$ for all convex functions φ such that both expectations are finite. Throughout this paper, the terms "convex" and "increasing" are both understood in the non-strict (weak) sense.

A random variable X follows a **Pareto distribution** with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$, denoted by $X \sim \text{Pareto}(\alpha, \theta)$, if its cumulative distribution function (CDF) is given by:

$$F(x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}, \quad x \ge \theta.$$

Without loss of generality, in this paper we shall focus on $\text{Pareto}(\alpha, 1)$ random variables where $\alpha > 0$, with CDF

$$F(x) = 1 - \left(\frac{1}{x}\right)^{\alpha}, \quad x \ge 1.$$

We will refer to this distribution using the shorthand notation $\text{Pareto}(\alpha)$.

A random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ is said to be **comonotonic** if there exists a random variable Z and increasing functions $t_1, \dots, t_n$ such that $\boldsymbol{X} \stackrel{\text{d}}{=} (t_1(Z), \dots, t_n(Z))$, where $\stackrel{\text{d}}{=}$ denotes equality in distribution. Components in a comonotonic random vector can be interpreted as being perfectly positively dependent.

Throughout this paper, we will also use the following shorthand notations:

- $\Delta_n = \{(\theta_1, \dots, \theta_n) \in [0,1]^n : \sum_{i=1}^n \theta_i = 1\}$,
- $x \wedge y = \min\{x, y\}$ for all $x, y \in \mathbb{R}$,
- $x_+ = \max\{x, 0\}$ for all $x \in \mathbb{R}$.

3 Scheme [L]

We start by introducing the set of all linear risk allocations.

Definition 3.1 (Scheme [L])**.** Let the two-group conditions in Definition 1.1 hold. The **scheme [L]** is the set of all risk allocations that take the form: $Y_i = \sum_{j=1}^{n} \theta_{ij} X_j$ for all $i = 1, \ldots, n$, where $(\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n$ for all $i = 1, \ldots, n$, and $(\theta_{1j}, \ldots, \theta_{nj}) \in \Delta_n$ for all $j = 1, \ldots, n$. Symbolically, we can express the set as

$$\mathcal{A} = \Bigg\{(Y_1, \ldots, Y_n) : Y_i = \sum_{j=1}^{n} \theta_{ij} X_j \text{ for all } i = 1, \ldots, n,$$
$$\text{with } (\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n\ \forall i = 1, \ldots, n, \text{ and } (\theta_{1j}, \ldots, \theta_{nj}) \in \Delta_n\ \forall j = 1, \ldots, n\Bigg\}.$$

Remark 3.2.

- A vector of losses $\boldsymbol{Y} = (Y_1, \ldots, Y_n) \in \mathcal{A}$ after risk sharing in the scheme [L] is called a **linear allocation**.
- For notational convenience, we often suppress the first index in the subscript of θ, interpreting it as fixed at a value in $\{1, \ldots, n\}$.
- Requiring $(\theta_{1j}, \ldots, \theta_{nj}) \in \Delta_n$ for all $j = 1, \ldots, n$ corresponds to enforcing the fully allocating property in the scheme.

3.1 Risk allocations for finite-mean agents under Pareto distribution

A major result for the scheme [L] is that it is always better off, in the first-order stochastic dominance sense, for finite-mean agents to not take any infinite-mean loss, when the losses are Pareto distributed. This result is shown using the following lemma.

Lemma 3.3. *Let the two-group conditions in Definition 1.1 hold with $m = 1$ and, moreover, assume that $X_1 \sim \text{Pareto}(\alpha)$ with $\alpha > 1$ and $X_2, \ldots, X_n \sim \text{Pareto}(\beta)$ with $\beta \leq 1$. Then,*

$$X_1 \leq_{\text{st}} \sum_{i=1}^{n} \theta_i X_i,$$

where $(\theta_1, \ldots, \theta_n) \in \Delta_n$.

Proof. We will prove this by induction. Consider the base case $n = 2$. The result holds trivially when $(\theta_1, \theta_2) = (1, 0)$. When $(\theta_1, \theta_2) = (0, 1)$, the result follows from the fact that $\mathbb{P}(X_1 > t) \leq \mathbb{P}(X_2 > t)$ for all $t \in \mathbb{R}$, where $X_1 \sim \text{Pareto}(\alpha)$ with $\alpha > 1$ and $X_2 \sim \text{Pareto}(\beta)$ with $\beta \leq 1$. So, henceforth assume that $\theta_1, \theta_2 \in (0, 1)$ with $\theta_1 + \theta_2 = 1$.

For all $t < 1$,

$$\mathbb{P}(\theta_1 X_1 + \theta_2 X_2 > t) = 1 = \mathbb{P}(X_1 > t),$$

so the result holds in this case. Now fix any $t \geq 1$ and let $\delta = (t - \theta_1)/\theta_2 = (t - 1 + \theta_2)/\theta_2 \geq 1$. Here, we will use the inequality

$$\mathbb{P}(\theta_1 X_1 + \theta_2 X_2 > t) \geq \mathbb{P}(X_2 > \delta) + \mathbb{P}(X_1 > t/\theta_1, X_2 \leq \delta)$$

for all $t \geq 1$, which is illustrated in Figure 1. Now consider:

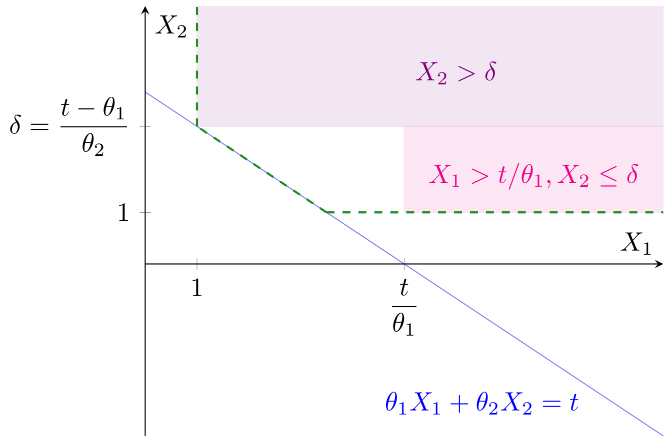

Figure 1: A geometrical illustration of the inequality $\mathbb{P}(\theta_1 X_1 + \theta_2 X_2 > t) \geq \mathbb{P}(X_2 > \delta) + \mathbb{P}(X_1 > t/\theta_1, X_2 \leq \delta)$ for all $t \geq 1$, where $\theta_1, \theta_2 \in (0,1)$ with $\theta_1 + \theta_2 = 1$.

$$\begin{aligned}
\mathbb{P}(\theta_1 X_1 + \theta_2 X_2 > t) - \mathbb{P}(X_1 > t) &\geq \mathbb{P}(X_2 > \delta) + \mathbb{P}(X_1 > t/\theta_1, X_2 \leq \delta) - \mathbb{P}(X_1 > t) \\
&= \mathbb{P}(X_2 > \delta) + \mathbb{P}(X_1 > t/\theta_1)\mathbb{P}(X_2 \leq \delta) - \mathbb{P}(X_1 > t) \\
&= (1/\delta)^\beta + (\theta_1/t)^\alpha \big(1 - (1/\delta)^\beta\big) - (1/t)^\alpha \\
&= (\theta_1/t)^\alpha + \underbrace{\big(1 - (\theta_1/t)^\alpha\big)}_{>0} \underbrace{(1/\delta)^\beta}_{\geq 1/\delta} - (1/t)^\alpha \\
&\geq \frac{1}{\delta} + \frac{\theta_1^\alpha}{t^\alpha}\left(1 - \frac{1}{\delta}\right) - \frac{1}{t^\alpha} \\
&= \frac{\theta_2}{t-\theta_1} + \frac{\theta_1^\alpha}{t^\alpha} \cdot \frac{t-1}{t-\theta_1} - \frac{1}{t^\alpha} \\
&= \frac{\overbrace{t^\alpha \theta_2 + \theta_1^\alpha (t-1) - t + \theta_1}^{(*)}}{\underbrace{t^\alpha (t-\theta_1)}_{>0}}.
\end{aligned}$$

Note that $(*) = 0$ when $t = 1$. Differentiating $(*)$ with respect to t, we have for all $t > 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(t^\alpha \theta_2 + \theta_1^\alpha (t-1) - t + \theta_1\big) &= \alpha \underbrace{t^{\alpha-1}}_{\geq 1} \theta_2 + \underbrace{\theta_1^\alpha}_{(1-\theta_2)^\alpha} - 1 \\
&\geq \alpha\theta_2 + 1 - \alpha\theta_2 - 1 \qquad \text{(Bernoulli's inequality)} \\
&= 0.
\end{aligned}$$

This implies that $(*)$ is increasing on $[1, \infty)$ as a function of t, hence $\mathbb{P}(\theta_1 X_1 + \theta_2 X_2 > t) - \mathbb{P}(X_1 > t) \geq 0$ for all $t \geq 1$. This proves the base case.

After that, assume for induction that the case $n = k$ holds, with $k \geq 2$. Then consider the case $n = k+1$. When $\theta_{k+1} = 1$, the result holds readily since $\mathbb{P}(X_1 > t) \leq \mathbb{P}(X_{k+1} > t)$ for all $t \in \mathbb{R}$. When $\theta_{k+1} = 0$, the result directly follows from the inductive hypothesis. So, from now on we assume that $\theta_{k+1} \in (0,1)$.

Let $X^* = \frac{1}{1-\theta_{k+1}}(\theta_1 X_1 + \cdots + \theta_k X_k)$. Then we can write $\theta_1 X_1 + \cdots + \theta_{k+1} X_{k+1} = (1-\theta_{k+1})X^* + \theta_{k+1} X_{k+1}$.

Since X^* and X_{k+1} are independent, after letting $\delta = (t-(1-\theta_{k+1}))/\theta_{k+1}$ like the base case, we have

$$\begin{aligned}
&\mathbb{P}(\theta_1 X_1 + \cdots + \theta_{k+1}X_{k+1} > t) - \mathbb{P}(X_1 > t)\\
&\quad = \mathbb{P}((1-\theta_{k+1})X^* + \theta_{k+1}X_{k+1} > t) - \mathbb{P}(X_1 > t)\\
&\quad \geq \mathbb{P}(X_{k+1} > \delta) + \mathbb{P}(X^* > t/(1-\theta_{k+1}))\mathbb{P}(X_{k+1} \leq \delta) - \mathbb{P}(X_1 > t)\\
&\quad \geq \mathbb{P}(X_{k+1} > \delta) + \mathbb{P}(X_1 > t/(1-\theta_{k+1}))\mathbb{P}(X_{k+1} \leq \delta) - \mathbb{P}(X_1 > t) \quad \text{(inductive hypothesis)}\\
&\quad \geq 0 \quad \text{(using the same argument as the base case).}
\end{aligned}$$

Thus the result holds for the case $n = k+1$, completing the proof by induction. □

Theorem 3.4. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that* $X_1, \dots, X_m \sim \text{Pareto}(\alpha)$ *with* $\alpha > 1$ *and* $X_{m+1}, \dots, X_n \sim \text{Pareto}(\beta)$ *with* $\beta \leq 1$. *Then,*

$$\theta_1 X_1 + \cdots + \theta_{m-1}X_{m-1} + (1-\theta_1 - \cdots - \theta_{m-1})X_m \leq_{\text{st}} \sum_{i=1}^n \theta_i X_i, \tag{1}$$

for all $(\theta_1, \dots, \theta_n) \in \Delta_n$.

Proof. When $m = 1$, the left-hand side of (1) reduces to X_1. Therefore, the case where $m = 1$ follows readily from Lemma 3.3. So, henceforth we assume $m \geq 2$. For $\theta_m = \cdots = \theta_n = 0$, the result holds trivially. Next, fix any $(\theta_1, \dots, \theta_n) \in \Delta_n$ such that $\theta_m + \cdots + \theta_n = 1 - \theta_1 - \cdots - \theta_{m-1} > 0$. Applying Lemma 3.3 with changes in labelling, we get

$$X_m \leq_{\text{st}} \sum_{j=m}^n \frac{\theta_j}{1-\theta_1 - \cdots - \theta_{m-1}} X_j,$$

which implies

$$(1-\theta_1 - \cdots - \theta_{m-1})X_m \leq_{\text{st}} \theta_m X_m + \theta_{m+1}X_{m+1} + \cdots + \theta_n X_n.$$

For all $i = 1, \dots, m-1$, we trivially have $\theta_i X_i \leq_{\text{st}} \theta_i X_i$. Hence, since $X_1, \dots, X_n$ are independent, by stability of $\leq_{\text{st}}$ under convolution[2] (Denuit et al., 2005, Proposition 3.3.17) we have

$$\theta_1 X_1 + \cdots + \theta_{m-1}X_{m-1} + (1-\theta_1 - \cdots - \theta_{m-1})X_m \leq_{\text{st}} \sum_{i=1}^n \theta_i X_i,$$

as desired. □

The left-hand side of (1) can be interpreted as the risk allocation after reassigning all original weights for infinite-mean losses to the loss X_m. If we required actuarial fairness for risk allocations of finite-mean agents in the scheme [L], i.e., after risk sharing, each of them must have a loss with the same mean as before, then it would be immediate that none of them can possibly take any infinite-mean losses in the scheme [L], as that would result in a loss with an infinite mean after risk sharing. Thinking in this way, Theorem 3.4 can be seen as a theoretical justification for imposing actuarial fairness for risk allocations of finite-mean agents in the scheme [L].

By Theorem 3.4, we know that under Pareto distributions, finite-mean agents are better off not taking any infinite-mean loss, meaning that the optimal risk-sharing rule in the scheme [L] must allocate finite-mean losses to finite-mean agents. In the following, we will show that it is indeed optimal to allocate those losses *uniformly*, i.e.,

$$Y_i = \frac{1}{m}\sum_{k=1}^m X_k$$

for all $i = 1, \dots, m$. The following proposition suggests that this uniform risk sharing is attractive for finite-mean agents.

[2] Stability of $\leq_{\text{st}}$ under convolution means that when $Y_1, \dots, Y_n$ are independent and $Z_1, \dots, Z_n$ are independent, if $Y_i \leq_{\text{st}} Z_i$ for all $i = 1, \dots, n$, then $\sum_{i=1}^n Y_i \leq_{\text{st}} \sum_{i=1}^n Z_i$.

Proposition 3.5 (Denuit et al. (2005, Corollary 3.4.24)). *Let $X_1, \dots, X_m$ be m iid random variables with finite mean. Then,*

$$\frac{1}{m}\sum_{k=1}^{m} X_k \leq_{\mathrm{cx}} \sum_{k=1}^{m} \theta_k X_k,$$

for all $(\theta_1, \dots, \theta_m) \in \Delta_m$.

Remark 3.6.

- The two-group conditions in Definition 1.1 imply that the finite-mean losses $X_1, \dots, X_m$ are iid. Thus, Proposition 3.5 is readily applicable under the two-group conditions.
- CMRS is equivalent to uniform risk sharing for iid finite-mean losses (Denuit and Dhaene, 2012), that is:

$$\mathbb{E}\left[X_i \middle| \sum_{k=1}^{m} X_k\right] = \frac{1}{m}\sum_{k=1}^{m} X_k,$$

for all $i = 1, \dots, m$.

3.2 Pareto optimal risk-sharing rule [L*] under Pareto distribution

Section 3.1 shows that it is desirable to allocate finite-mean Pareto losses uniformly to finite-mean agents. In such case, the only losses remaining to be allocated to infinite-mean agents are the iid infinite-mean Pareto losses. In the case where only iid infinite-mean Pareto losses are to be shared, Chen et al. (2025b) show that it is optimal for infinite-mean agents to retain their own losses, in the sense of first-order stochastic dominance. In light of this, it is quite appealing to adopt a risk-sharing rule in which finite-mean agents share their losses uniformly while infinite-mean agents retain their own losses, allowing finite-mean agents to benefit in convex order without affecting infinite-mean agents. In this section, we are going to show that this appealing risk-sharing rule is indeed *Pareto optimal* in the scheme [L]. For ease of referencing, we shall name this risk-sharing rule as [L*]:

Definition 3.7 (Rule [L*]). The **rule [L*]** is the risk allocation with (i) uniform risk sharing among finite-mean losses, and (ii) no risk sharing for infinite-mean agents in scheme [L], i.e., $Y_i = \frac{1}{m}\sum_{k=1}^{m} X_k$ for all $i = 1, \dots, m$, and $Y_j = X_j$ for all $j = m+1, \dots, n$.

The concept of Pareto optimality is defined with respect to a particular preference relation. Since we have shown some theoretical results about stochastic orders in Section 3.1, it is natural to consider a preference related to stochastic orders, namely the first-order stochastic dominance $\leq_{\mathrm{st}}$ and the convex order $\leq_{\mathrm{cx}}$. It is clear by definition that the convex order is incompatible with infinite-mean losses. Moreover, (strict) first-order stochastic dominance leads to a (strict) ordering in expectations, which makes it hard to unify with the actuarial fairness property for finite-mean losses in risk sharing. Therefore, we will consider a preference relation $\preceq_{\mathrm{sc}}$ that is defined through $\leq_{\mathrm{st}}$ or $\leq_{\mathrm{cx}}$, depending on whether the random variables in comparison have finite mean or not.

Definition 3.8 (Preference $\preceq_{\mathrm{sc}}$). Let $\mathcal{A}$ denote the set of all possible allocations in the scheme [L] (see Definition 3.1). For all allocations $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ and $\boldsymbol{Z} = (Z_1, \dots, Z_n)$ in $\mathcal{A}$, we have $Y_i \preceq_{\mathrm{sc}} Z_i$, i.e., Z_i is weakly preferred to Y_i, if:

$$\begin{cases} Z_i \leq_{\mathrm{cx}} Y_i & \text{when } Y_i \text{ and } Z_i \text{ both have finite mean,} \\ Z_i \leq_{\mathrm{st}} Y_i & \text{otherwise.} \end{cases}$$

Remark 3.9. We define the corresponding strict preference $\prec_{\mathrm{sc}}$ in a standard manner: For all $i = 1, \dots, n$, $Y_i \prec_{\mathrm{sc}} Z_i$ if $Y_i \preceq_{\mathrm{sc}} Z_i$ and $Z_i \not\preceq_{\mathrm{sc}} Y_i$.

Using this preference relation, we can then define the notion of Pareto optimality and Pareto improvement under $\preceq_{\mathrm{sc}}$ as follows.

Definition 3.10 (Pareto improvement and Pareto optimality). A linear allocation $\boldsymbol{Z} \in \mathcal{A}$ is a **Pareto improvement** over another linear allocation $\boldsymbol{Y} \in \mathcal{A}$ if

1. $Y_i \preceq_{\text{sc}} Z_i$ for all $i = 1, \dots, n$, and
2. $Y_i \prec_{\text{sc}} Z_i$ for some $i = 1, \dots, n$

where $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ and $\boldsymbol{Z} = (Z_1, \dots, Z_n)$. A linear allocation $\boldsymbol{Y} \in \mathcal{A}$ is **Pareto optimal** in the scheme [L] if there is no linear allocation $\boldsymbol{Z} \in \mathcal{A}$ that is a Pareto improvement over $\boldsymbol{Y}$.

With these definitions, we are now ready to prove that the rule [L*] is a Pareto optimal allocation in the scheme [L] under $\preceq_{\text{sc}}$.

Theorem 3.11. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that* $X_1, \dots, X_m \sim \text{Pareto}(\alpha)$ *with* $\alpha > 1$*, and* $X_{m+1}, \dots, X_n \sim \text{Pareto}(\beta)$ *with* $\beta \le 1$*. Then, the rule [L*] is Pareto optimal in the scheme [L] under* $\preceq_{\text{sc}}$*.*

Proof. Denote the risk allocation for the rule [L*] by $\boldsymbol{Y} = (Y_1, \dots, Y_n)$. Assume, for contradiction, that there exists a Pareto improvement $\boldsymbol{Z} = (Z_1, \dots, Z_n) \in \mathcal{A}$ over $\boldsymbol{Y}$. For a contradiction, it suffices to show that that Y_i is weakly preferred to Z_i, i.e., $Z_i \preceq_{\text{sc}} Y_i$, for *all* $i = 1, \dots, n$.

By the assumption that $\boldsymbol{Z}$ is a Pareto improvement over $\boldsymbol{Y}$, we have $Y_i \preceq_{\text{sc}} Z_i$ for all $i = 1, \dots, m$. Since $Y_1, \dots, Y_m$ all have a finite mean, this forces $Z_1, \dots, Z_m$ to all have a finite mean as well. To see this, suppose on the contrary that Z_i had an infinite mean for some $i = 1, \dots, m$. Then, by the definition of $\preceq_{\text{sc}}$, we would have $Z_i \le_{\text{st}} Y_i$ for some $i = 1, \dots, m$, which would then imply

$$\mathbb{E}[Z_i] = \int_0^\infty S_{Z_i}(t)\,\mathrm{d}t \le \int_0^\infty S_{Y_i}(t)\,\mathrm{d}t = \mathbb{E}[Y_i] < \infty,$$

where S_{Y_i} and S_{Z_i} are survival functions of Y_i and Z_i, respectively. This contradicts with the assumption that Z_i has infinite mean.

Consequently, according to the scheme [L], the θ_{ik}'s for the infinite-mean losses $X_{m+1}, \dots, X_n$ must be zero for $Z_1, \dots, Z_m$. Hence, we can express them simply as

$$Z_i = \sum_{k=1}^{m} \theta_{ik} X_k,$$

for all $i = 1, \dots, m$.

By the fully-allocating property from scheme [L], all finite-mean losses $X_1, \dots, X_m$ would then be exhausted by finite-mean agents. So, in the allocation $\boldsymbol{Z}$, only the infinite-mean losses $X_{m+1}, \dots, X_n$ remain to be shared among infinite-mean agents. In other words, we must have

$$Z_j = \sum_{k=m+1}^{n} \theta_{jk} X_k,$$

for all $j = m+1, \dots, n$.

So far, we have established that (i) finite-mean losses must go to finite-mean agents and (ii) infinite-mean losses must go to infinite-mean agents, in the allocation $\boldsymbol{Z}$. With this separation, we can show our desired result easily.

By Proposition 3.5, we have

$$Y_i = \frac{1}{m} \sum_{k=1}^{m} X_k \le_{\text{cx}} \sum_{k=1}^{m} \theta_{ik} X_k = Z_i,$$

for all $i = 1, \dots, m$. By definition of $\preceq_{\text{sc}}$, this implies that $Z_i \preceq_{\text{sc}} Y_i$ for all $i = 1, \dots, m$. It then remains to show that $Z_j \preceq_{\text{sc}} Y_j$ for all $j = m+1, \dots, n$ as well. From Chen et al. (2025b, Theorem 1), we have

$$Y_j = X_j \le_{\text{st}} \sum_{k=m+1}^{n} \theta_{jk} X_k = Z_j,$$

for all $j = m+1, \dots, n$. This implies that $Z_j \preceq_{\text{sc}} Y_j$ for all $j = m+1, \dots, n$, completing the proof. □

4 Scheme [FR]

The key idea in the scheme [FR] is to artificially "create" more iid finite-mean losses for constructing a risk-sharing rule that is better than the rule [L*]. To do this, we will utilize two classical results from probability, namely *probability integral transform* and *inverse transform sampling*. Let F^{-1} denote the *generalized inverse* of a CDF F, which is given by $F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$. The scheme [FR] is defined formally below.

Definition 4.1 (Scheme [FR]). Let the two-group conditions in Definition 1.1 hold, F_i denote the CDF of X_i for all $i = 1, \ldots, n$, $X_j^{\text{finite}} := F_1^{-1}(F_j(X_j))$ for all $j = m+1, \ldots, n$, and F_j be continuous for all $j = m+1, \ldots, n$. The **scheme [FR]** is the set of all risk allocations with the following form:

$$Y_i = \sum_{k=1}^{m} \theta_{ik} X_k + \sum_{k=m+1}^{n} \theta_{ik} X_k^{\text{finite}} \qquad \text{for all } i = 1, \ldots, m,$$

and

$$Y_j = \sum_{k=1}^{m} \theta_{jk} X_k + \sum_{k=m+1}^{n} \theta_{jk} X_k^{\text{finite}} + \left(X_j - X_j^{\text{finite}}\right) \qquad \text{for all } j = m+1, \ldots, n,$$

where $(\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n$ for all $i = 1, \ldots, n$, and $(\theta_{1k}, \ldots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \ldots, n$.

Remark 4.2.

- *(Interpretation of X_j^{finite})* Under the two-group conditions, applying probability integral transform and then inverse transform sampling, we know $X_j^{\text{finite}} \overset{\text{d}}{=} X_1$ for all $j = m+1, \ldots, n$, which implies that the random variables $X_1, \ldots, X_m, X_{m+1}^{\text{finite}}, \ldots, X_n^{\text{finite}}$ are iid. Here we have separated each infinite-mean loss X_j into a finite-mean portion X_j^{finite} and the residual portion $X_j - X_j^{\text{finite}}$. Through this separation, we have increased the number of iid finite-mean losses from m in the risk-sharing rule [L*] to n here. This gives us some insights on why the scheme [FR] can potentially allow us to construct a risk-sharing rule that is even better than the rule [L*].
- For every $j = m+1, \ldots, n$, the finite-mean portion X_j^{finite} and the original infinite-mean loss X_j are comonotonic, because the composition $F_1^{-1} \circ F_j$ is increasing.
- The requirement $(\theta_{1k}, \ldots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \ldots, n$ again corresponds to enforcing the fully allocating property in the scheme. The requirement $(\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n$ for all $i = 1, \ldots, n$ can also be found in the definition for the scheme [L] (Definition 3.1). It is again imposed here to allow for more natural and fair comparisons between the schemes [FR] and [L]. A consequence of this requirement is that the risk allocations for finite-mean agents always satisfy actuarial fairness, as $\theta_{i1} + \cdots + \theta_{in} = 1$ for all $i = 1, \ldots, m$.
- We have excluded the sharing of the residual portion $X_j - X_j^{\text{finite}}$ in the structure of the scheme [FR], as sharing this infinite-mean portion may not be helpful for diversification, and our argument will primarily focus on the finite-mean portion X_j^{finite}.

4.1 Risk allocations for finite-mean agents

Like what was done in Section 3, we first investigate risk allocations for finite-mean agents. It turns out that uniform risk sharing among the finite-mean losses is again optimal. But "uniform risk sharing" here carries a different meaning from the one for the scheme [L], due to the availability of additional synthetic finite-mean losses. Instead of just allocating the m original finite-mean losses uniformly, we will do that on all the n iid finite-mean losses in the scheme [FR]. To be more precise, the uniform risk sharing for finite-mean agents is given by

$$Y_i = \frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right),$$

for all $i = 1, \ldots, m$. The optimality of this uniform risk sharing is then established by the following proposition.

Proposition 4.3. *Let the two-group conditions in Definition 1.1 hold. Then, for all $(\theta_1, \ldots, \theta_n) \in \Delta_n$, we have*

$$\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) \leq_{\text{cx}} \sum_{k=1}^{m} \theta_k X_k + \sum_{k=m+1}^{n} \theta_k X_k^{\text{finite}},$$

where X_k^{finite} is defined in Definition 4.1.

Proof. It follows from Denuit et al. (2005, Corollary 3.4.24) since the losses $X_1, \ldots, X_m, X_{m+1}^{\text{finite}}, \ldots, X_n^{\text{finite}}$ are iid. □

By the definition of convex order, this result suggests that uniform risk sharing is the most preferred allocation for every finite-mean, risk-averse agent, that is, an agent with a concave utility function. Specifically, their expected utility is maximized under this allocation.

Particularly, setting $\theta_1 = \cdots = \theta_m = 1/m$ and $\theta_{m+1} = \cdots = \theta_n = 0$ in Proposition 4.3 yields

$$\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) \leq_{\text{cx}} \frac{1}{m}\sum_{k=1}^{m} X_k,$$

which shows that, for finite-mean agents, the uniform risk sharing here is better than the rule [L*]. This prepares us to construct a risk-sharing rule in the scheme [FR], referred to as the rule [FR*], which is better than the rule [L*].

4.2 Risk-sharing rule [FR*] under Pareto distribution

Definition 4.4 (Rule [FR*])**.** The **rule [FR*]** is the risk allocation given by

$$Y_i = \frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) \qquad \text{for all } i = 1, \ldots, m,$$

and

$$Y_j = \frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) + \left(X_j - X_j^{\text{finite}}\right) \qquad \text{for all } j = m+1, \ldots, n.$$

Remark 4.5. After risk sharing, the only difference between the loss of each infinite-mean agent j and that of a finite-mean agent is the residual portion $X_j - X_j^{\text{finite}}$. Under the two-group conditions, if $X_1 \leq_{\text{st}} X_{m+1}$, then $F_1^{-1} \leq F_j^{-1}$ and thus the residual portion $X_j - X_j^{\text{finite}}$ is always nonnegative. This condition $X_1 \leq_{\text{st}} X_{m+1}$ is particularly satisfied by Pareto losses: specifically, $X_1 \sim \text{Pareto}(\alpha)$ and $X_{m+1} \sim \text{Pareto}(\beta)$ with $\alpha > 1$ and $\beta \leq 1$. This implies that every infinite-mean agent needs to bear a loss at least as large as that of a finite-mean agent after risk sharing based on the rule [FR*], as one may expect.

To demonstrate that the rule [FR*] is attractive compared to the rule [L*], we again need to assume the losses to be Pareto distributed, which leads to significant simplifications of the scheme [FR], mainly due to the following result:

Lemma 4.6. *For $W \sim \text{Pareto}(\alpha)$ and $X \sim \text{Pareto}(\beta)$, it holds that*

$$F_W^{-1}(F_X(X)) = X^{\beta/\alpha},$$

where F_W and F_X are the CDF's of W and X, respectively.

Proof. The CDF of W is given by $F_W(w) = 1 - w^{-\alpha}$, $w \geq 1$, and its inverse is given by $F_W^{-1}(p) = (1-p)^{-1/\alpha}$, $0 < p < 1$. Furthermore, we have $F_X(X) = 1 - X^{-\beta}$. Hence,

$$F_W^{-1}(F_X(X)) = \left[1 - (1 - X^{-\beta})\right]^{-1/\alpha} = X^{\beta/\alpha}.$$

□

Thus, when we assume $X_1, \dots, X_m \sim \text{Pareto}(\alpha)$ with $\alpha > 1$ and $X_{m+1}, \dots, X_n \sim \text{Pareto}(\beta)$ with $\beta \le 1$, we can express the risk-sharing rule in the scheme [FR] as:

$$Y_i = \sum_{k=1}^{m} \theta_{ik} X_k + \sum_{k=m+1}^{n} \theta_{ik} X_k^{\beta/\alpha} \qquad \text{for all } i = 1, \dots, m,$$

and

$$Y_j = \sum_{k=1}^{m} \theta_{jk} X_k + \sum_{k=m+1}^{n} \theta_{jk} X_k^{\beta/\alpha} + \left(X_j - X_j^{\beta/\alpha}\right) \qquad \text{for all } j = m+1, \dots, n,$$

where $(\theta_{i1}, \dots, \theta_{in}) \in \Delta_n$ for all $i = 1, \dots, n$, and $(\theta_{1k}, \dots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \dots, n$.

In the scheme [L], we have examined the benefits of the rule [L*] by considering Pareto optimality with respect to the preference $\preceq_{\text{sc}}$, which is related to stochastic orders. Here, we will also examine the benefits of the rule [FR*] through stochastic orders.

In Section 4.1, we have already shown that the rule [FR*] is better than [L*] for finite-mean agents. To investigate whether the same holds for infinite-mean agents, a natural approach is to compare the risk allocations for infinite-mean agents in these two rules via stochastic orders. Due to their infinite-mean nature, the convex order $\le_{\text{cx}}$ becomes obsolete. So, it then appears that we should consider the first-order stochastic dominance $\le_{\text{st}}$. However, due to the strong requirement of $\le_{\text{st}}$ and the complexity of the expressions here (still rather complex even under Pareto distribution), it is quite difficult to analyze. With the inapplicability of both $\le_{\text{st}}$ and $\le_{\text{cx}}$, here we will instead utilize a heuristic argument under the two-group conditions in Definition 1.1.

In the rule [L*], for each infinite-mean agent j, we can write $Y_j = X_j = X_j^{\text{finite}} + (X_j - X_j^{\text{finite}})$, hence the post-risk-sharing "finite-mean portion" and "residual portion" in the rule [L*] are X_j^{finite} and $X_j - X_j^{\text{finite}}$, respectively. The argument then compares the behaviours of finite-mean and residual portions in the rules [L*] and [FR*], by considering improvements in two aspects: (i) improvement for finite-mean portion and (ii) improvement from diversification benefits.

Improvement for finite-mean portion. For this part, the Pareto distribution assumption is not yet needed, so we would not impose this assumption for the moment. We will show that the finite-mean portion of each infinite-mean loss improves in convex order after risk sharing. In other words, we will show that the post-risk-sharing finite-mean portion for the rule [FR*], $\frac{1}{n}\left(\sum_{k=1}^m X_k + \sum_{k=m+1}^n X_k^{\text{finite}}\right)$, is smaller than that for the rule [L*], X_j^{finite}, in convex order. This improvement indeed follows readily from Proposition 4.3, as the following corollary suggests.

Corollary 4.7. *Let the two-group conditions in Definition 1.1 hold. Then,*

$$\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) \le_{\text{cx}} X_j^{\text{finite}},$$

for all $j = m+1, \dots, n$.

Proof. Setting the jth entry of $(\theta_1, \dots, \theta_n)$ in Proposition 4.3 as 1 and other entries as 0 yields the result. □

Improvement from diversification benefits. We need to assume the Pareto distribution in this part, so that we have the simplification $X_j^{\text{finite}} = X_j^{\beta/\alpha}$. We will show that, before risk sharing, the finite-mean portion $X_j^{\beta/\alpha}$ and the residual portion $X_j - X_j^{\beta/\alpha}$ are comonotonic, which is considered the least favorable in terms of diversification. Then, according to the rule [FR*], some losses that are independent from the residual portion $X_j - X_j^{\beta/\alpha}$ would be mixed into the finite-mean portion after risk sharing, yielding diversification benefits. The comonotonicity of $X_j^{\beta/\alpha}$ and $X_j - X_j^{\beta/\alpha}$ is established by the following result.

Proposition 4.8. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that* $X_1, \dots, X_m \sim \text{Pareto}(\alpha)$ *with* $\alpha > 1$ *and* $X_{m+1}, \dots, X_n \sim \text{Pareto}(\beta)$ *with* $\beta \le 1$. *Then,* $(X_j^{\beta/\alpha}, X_j - X_j^{\beta/\alpha})$ *is comonotonic for all* $j = m+1, \dots, n$.

Proof. Define functions t_1 and t_2 on $[1,\infty)$ by $t_1(x) = x^{\beta/\alpha}$, and $t_2(x) = x - x^{\beta/\alpha}$. Note that the functions t_1 and t_2 are increasing and

$$(X_j^{\beta/\alpha}, X_j - X_j^{\beta/\alpha}) \stackrel{\mathrm{d}}{=} (t_1(X_j), t_2(X_j)).$$

Hence, $(X_j^{\beta/\alpha}, X_j - X_j^{\beta/\alpha})$ is comonotonic. □

With these two improvements, the rule [FR*] is considered better than the rule [L*] for infinite-mean agents.

Thus, in general, the rule [FR*] yields improvements over the rule [L*], as it is preferred by both finite-mean and infinite-mean agents. Furthermore, this rule allows parts of infinite-mean losses to be shared, unlike the rule [L*]. Therefore, we have constructed a risk-sharing rule [FR*] that permits management of catastrophic infinite-mean Pareto losses, while still being beneficial to all the agents. In Section 5, we will investigate another possible construction of such a risk-sharing rule.

5 Scheme [LS]

In this section, we study an alternative scheme to the scheme [FR]. Similar to the scheme [FR], the scheme [LS] involves the separation of infinite-mean losses, but they are instead decomposed into *limited-loss variables* and *stop-loss variables*, which may be more natural and familiar to practitioners.

Definition 5.1 (Scheme [LS])**.** Let the two-group conditions in Definition 1.1 hold, and $d \in \mathbb{R}$ be a value such that $\mathbb{E}[X_{m+1} \wedge d] = \mathbb{E}[X_1]$, which exists by the continuity of the mapping $d \mapsto \mathbb{E}[X_{m+1} \wedge d]$ and the Intermediate Value Theorem. Then the **scheme [LS]** is the set of all risk allocations which take the following form:

$$Y_i = \sum_{k=1}^{m} \theta_{ik} X_k + \sum_{k=m+1}^{n} \theta_{ik}(X_k \wedge d) \qquad \text{for all } i = 1, \dots, m,$$

and

$$Y_j = \sum_{k=1}^{m} \theta_{jk} X_k + \sum_{k=m+1}^{n} \theta_{jk}(X_k \wedge d) + (X_j - d)_+ \qquad \text{for all } j = m+1, \dots, n,$$

where $(\theta_{i1}, \dots, \theta_{in}) \in \Delta_n$, for all $i = 1, \dots, n$, and $(\theta_{1k}, \dots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \dots, n$.

Remark 5.2.

- The value d is set such that each limited loss variable $X_j \wedge d$ has the same mean as the original finite-mean loss X_i. The intuitive idea is that, while this way of separation cannot make them equal in distribution like the scheme [FR], such choice of d can at least make them comparable by requiring equality in *mean*.
- The requirements $(\theta_{1k}, \dots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \dots, n$ and $(\theta_{i1}, \dots, \theta_{in}) \in \Delta_n$ for all $i = 1, \dots, n$ are familiar and have appeared in the definitions of both schemes [L] and [FR]. The former is again enforcing the fully allocating property in the scheme [LS]. Due to how the value of d is chosen, the latter requirement implies that the risk allocations for finite-mean agents always satisfy actuarial fairness, like the scheme [FR].

A general closed-form formula for d in Definition 5.1 is not available because the situation described therein is quite general. However, under the special case where the losses are Pareto distributed, a formula for d can be derived easily. Such formula will be helpful for numerical computations related to the scheme [LS]; see, e.g., Section 6.2.

Proposition 5.3. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that* $X_1, \dots, X_m \sim \mathrm{Pareto}(\alpha)$ *with* $\alpha > 1$ *and* $X_{m+1}, \dots, X_n \sim \mathrm{Pareto}(\beta)$ *with* $\beta \le 1$*. Then, the value of* d *in the scheme [LS] is given by*

$$d = \begin{cases} \left[1 + \left(\dfrac{\alpha}{\alpha - 1} - 1\right)(-\beta + 1)\right]^{1/(-\beta+1)} & \text{if } \beta \in (0,1), \\ \exp\left(\dfrac{\alpha}{\alpha - 1} - 1\right) & \text{if } \beta = 1. \end{cases}$$

Proof. Note that

$$\mathbb{E}[X_{m+1} \wedge d] = \int_0^d \mathbb{P}(X_{m+1} > t)\,\mathrm{d}t = \int_0^1 \mathbb{P}(X_{m+1} > t)\,\mathrm{d}t + \int_1^d \left(\frac{1}{t}\right)^{\beta} \mathrm{d}t = \begin{cases} 1 + \dfrac{d^{-\beta+1} - 1}{-\beta + 1} & \text{if } \beta \in (0,1), \\ 1 + \ln d & \text{if } \beta = 1. \end{cases}$$

Solving $\mathbb{E}[X_{m+1} \wedge d] = \mathbb{E}[X_1] = \dfrac{\alpha}{\alpha - 1}$ for d yields the desired result. ☐

5.1 Risk allocation for finite-mean agents

Apart from having a more intuitively appealing separation, another major motivation for studying the scheme [LS] is that uniform risk sharing for finite-mean agents is better in the scheme [LS] than in the scheme [FR] if $X_1 \leq_{\mathrm{st}} X_{m+1}$, which is satisfied when the losses are Pareto distributed. Here, "uniform risk sharing" again carries a different meaning from the other schemes; it refers to the following risk allocation:

$$Y_i = \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right),$$

for all $i = 1, \ldots, m$.

The proof of our claim about the improvement of uniform risk sharing in the scheme [LS] over that in the scheme [FR] is based on the following lemma.

Lemma 5.4. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that $X_1 \leq_{\mathrm{st}} X_{m+1}$[3]. Then, it holds that*

$$X_j \wedge d \leq_{\mathrm{cx}} X_j^{\mathrm{finite}} \stackrel{\mathrm{d}}{=} X_1,$$

for all $j = m+1, \ldots, n$, where X_j^{finite} is defined in Definition 4.1.

Proof. Let $F_{X_j \wedge d}$ and F_1 be the CDF's of $X_j \wedge d$ and X_1, respectively. Since $X_1 \leq_{\mathrm{st}} X_{m+1}$, we have $F_{X_j \wedge d}(t) \leq F_1(t)$ for all $t < d$. Also, for all $t > d$, it is clear that $F_1(t) \leq 1 = F_{X_j \wedge d}(t)$. Then the result follows by Ohlin (1969, Lemma 2), since $\mathbb{E}[X_j \wedge d] = \mathbb{E}[X_1]$ by construction of d. ☐

Using this lemma, we can prove our claim straightforwardly.

Proposition 5.5. *Let the two-group conditions in Definition 1.1 hold and, moreover, assume that $X_1 \leq_{\mathrm{st}} X_{m+1}$. Then,*

$$\frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) \leq_{\mathrm{cx}} \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j^{\mathrm{finite}}\right),$$

where X_j^{finite} is defined in Definition 4.1.

Proof. By Lemma 5.4, we have $X_j \wedge d \leq_{\mathrm{cx}} X_j^{\mathrm{finite}}$ for all $j = m+1, \ldots, n$. Then, by stability of $\leq_{\mathrm{cx}}$ under convolution, we have

$$\frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) \leq_{\mathrm{cx}} \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j^{\mathrm{finite}}\right),$$

as desired. ☐

[3] The condition $X_1 \leq_{\mathrm{st}} X_{m+1}$ is satisfied if $X_1 \sim \mathrm{Pareto}(\alpha)$ and $X_{m+1} \sim \mathrm{Pareto}(\beta)$ with $\alpha > 1$ and $\beta < 1$.

5.2 Risk-sharing rule [LS*]

Similar to what we did for the scheme [FR], in the scheme [LS] we will again construct a risk-sharing rule, called rule [LS*], that yields an improvement over the rule [L*] and also allows parts of infinite-mean losses to be shared. The rule [LS*] has a similar structure to that of the rule [FR*]:

Definition 5.6 (Rule [LS*])**.** The **rule [LS*]** is the risk allocation given by

$$Y_i = \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) \qquad \text{for all } i = 1, \dots, m,$$

and

$$Y_j = \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) + (X_j - d)_+ \qquad \text{for all } j = m+1, \dots, n.$$

For finite-mean agents, we know from Proposition 5.5 that the rule [LS*] is better than the rule [FR*] in convex order, and from Section 4.1 that the rule [FR*] is in turn better than the rule [L*] in convex order. The transitivity of convex order implies that the rule [LS*] yields an improvement over the rule [L*] for finite-mean agents. It remains to argue that the same holds true for *infinite-mean* agents as well.

Here, as in the discussion on the rule [FR*], we will show the benefits by considering the two improvements investigated previously in Section 4.2: (i) improvement for finite-mean portion and (ii) improvement from diversification benefits, under the two-group conditions in Definition 1.1. By writing $Y_j = X_j = X_j \wedge d + (X_j - d)_+$ for each infinite-mean agent j in the rule [L*], we can treat the post-risk-sharing "finite-mean portion" and "residual portion" for the rule [L*] as $X_j \wedge d$ and $(X_j - d)_+$, respectively.

Improvement for finite-mean portion. Variance will be used as the criterion for showing this improvement. More specifically, we will show that the variance of the post-risk-sharing finite-mean portion for the rule [LS*], $\mathrm{Var}\big(\frac{1}{n}\big(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d\big)\big)$, is smaller than that for the rule [L*], $\mathrm{Var}(X_j \wedge d)$. The following result gives a lower bound on n for having an improvement in variance, given a fixed m.

Proposition 5.7. *Let the two-group conditions in Definition 1.1 hold with* $\mathrm{Var}(X_1)$ *and* $\mathrm{Var}(X_{m+1} \wedge d)$ *being positive, and fix a positive integer* m *that is less than* n*. Moreover, assume that* $X_1 \leq_{\mathrm{st}} X_{m+1}$*. Then,* $\mathrm{Var}(X_{m+1} \wedge d) \leq \mathrm{Var}(X_1)$*. Also, for all* $j = m+1, \dots, n$,

$$\mathrm{Var}\left(\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d\right)\right) \leq \mathrm{Var}(X_j \wedge d)$$

if and only if the positive integer n *satisfies* $n \geq \frac{1}{2}\left(1 + \sqrt{1 + 4m\left(\frac{\mathrm{Var}(X_1)}{\mathrm{Var}(X_{m+1} \wedge d)} - 1\right)}\right)$.

Proof. Let σ_{fin}^2 and σ_{lim}^2 denote $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_{m+1} \wedge d)$, respectively. First, Lemma 5.4 implies that $\sigma_{\text{fin}}^2 \geq \sigma_{\text{lim}}^2$, which follows from a standard property of the convex order $\leq_{\mathrm{cx}}$. Next, for all $j = m+1, \dots, n$, we have

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d\right)\right) \leq \mathrm{Var}(X_j \wedge d)\\
&\iff \frac{1}{n^2}\left[m\sigma_{\text{fin}}^2 + (n-m)\sigma_{\text{lim}}^2\right] \leq \sigma_{\text{lim}}^2\\
&\iff \frac{m}{n^2}\sigma_{\text{fin}}^2 \leq \frac{n^2 - n + m}{n^2}\sigma_{\text{lim}}^2\\
&\iff \frac{m}{n^2 - n + m} \leq \frac{\sigma_{\text{lim}}^2}{\sigma_{\text{fin}}^2}\\
&\iff n^2 - n + m \geq \frac{m\sigma_{\text{fin}}^2}{\sigma_{\text{lim}}^2}\\
&\iff n^2 - n - m\left(\frac{\sigma_{\text{fin}}^2}{\sigma_{\text{lim}}^2} - 1\right) \geq 0.
\end{aligned}$$

Since $\sigma^2_{\text{fin}} \geq \sigma^2_{\text{lim}}$, the discriminant of the quadratic polynomial $n^2 - n - m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right)$ is

$$\Delta = (-1)^2 - 4(1)\left[-m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right)\right] = 1 + 4m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right) \geq 1 + 4m(1-1) \geq 1 > 0.$$

Then, we consider the quadratic equation

$$n^2 - n - m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right) = 0.$$

Using the quadratic formula, this yields

$$n = \frac{1 \pm \sqrt{\Delta}}{2}.$$

Since $\frac{1-\sqrt{\Delta}}{2} \leq \frac{1-1}{2} = 0$ and the coefficient of n^2 in the quadratic equation is positive, we have

$$n^2 - n - m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right) \geq 0 \iff n \geq \frac{1+\sqrt{\Delta}}{2} = \frac{1}{2}\left[1 + \sqrt{1 + 4m\left(\frac{\sigma^2_{\text{fin}}}{\sigma^2_{\text{lim}}} - 1\right)}\right],$$

as n is always positive, completing the proof. □

Remark 5.8. When m is small and $\text{Var}(X_1)$ is not much greater than $\text{Var}(X_{m+1} \wedge d)$ (which is typically the case), the lower bound on n would not be too large, and thus this is not an asymptotic result. For example, suppose that $m = 3$ and $\text{Var}(X_1) = 8\,\text{Var}(X_{m+1} \wedge d)$. Then, the lower bound on n is just

$$\frac{1}{2}\left(1 + \sqrt{1 + 4m\left(\frac{\text{Var}(X_1)}{\text{Var}(X_{m+1} \wedge d)} - 1\right)}\right) = \frac{1}{2}\left(1 + \sqrt{1 + 4(3)(8-1)}\right) = \frac{1}{2}\left(1 + \sqrt{85}\right) \approx 5.1098,$$

so this inequality on the variances holds whenever the positive integer n is at least 6. That is, as long as there are at least 3 more infinite-mean agents joining the pool, there is an improvement in variance for the finite-mean portion.

Improvement from diversification benefits. In the rule [LS*], note first that the finite-mean and residual portions before risk sharing, $X_j \wedge d$ and $(X_j - d)_+$, are comonotonic for all $j = m+1, \ldots, n$, because both are increasing functions of X_j. Like the rule [FR*], the rule [LS*] adds some losses that are independent from the residual portion $(X_j - d)_+$ into the finite-mean portion after risk sharing, which gives us some diversification benefits by deviating from the least favorable case (comonotonicity).

With these two improvements, the rule [LS*] is considered better than the rule [L*] for infinite-mean agents. Hence, besides the rule [FR*], we have constructed another risk-sharing rule [LS*] that can serve as an alternative to having no risk sharing in the context of catastrophic risk management.

In addition to mutually benefiting both finite-mean and infinite-mean agents and handling infinite-mean losses, the rule [LS*] has some additional benefits over the rule [FR*], from both the theoretical and practical aspects. Theoretically, the rule [LS*] benefits finite-mean agents more than the rule [FR*]. From a practical perspective, the rule [LS*] should be more understandable to practitioners.

6 Numerical studies

6.1 Illustration for first-order stochastic dominance in the scheme [L]

The first-order stochastic dominance from Theorem 3.4 suggests that, with $X_1, \ldots, X_m \sim \text{Pareto}(\alpha)$ for $\alpha > 1$ and $X_{m+1}, \ldots, X_n \sim \text{Pareto}(\beta)$ for $\beta \leq 1$, we have

$$\mathbb{P}(\theta_1 X_1 + \cdots + \theta_{m-1} X_{m-1} + (1 - \theta_1 - \cdots - \theta_{m-1}) X_m > t) \leq \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i > t\right),$$

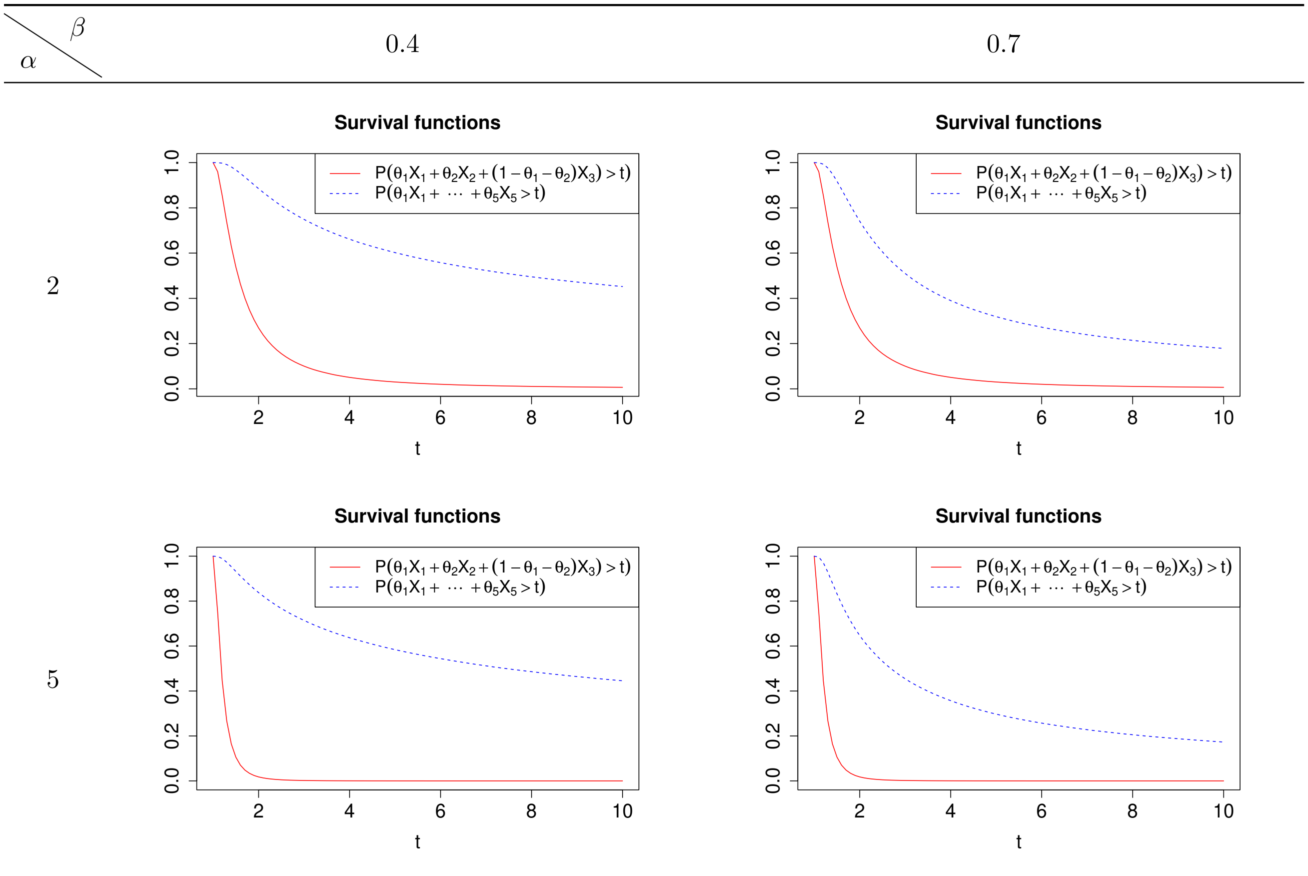

Figure 2: Plots of survival functions for the expressions $\mathbb{P}(\theta_1 X_1 + \cdots + \theta_{m-1} X_{m-1} + (1 - \theta_1 - \cdots - \theta_{m-1}) X_m > t)$ and $\mathbb{P}(\sum_{i=1}^{n} \theta_i X_i > t)$ from Theorem 3.4, with $m = 3$, $n = 5$, $\theta_1 = 0.1$, $\theta_2 = 0.15$, $\theta_3 = 0.2$, $\theta_4 = 0.15$, $\theta_5 = 0.4$, and different combinations of α (vertical) and β (horizontal). In each plot, the solid curve is always below the dashed curve, which is consistent with the first-order stochastic dominance suggested in Theorem 3.4. The plots are obtained by simulation.

for all $t \in \mathbb{R}$. Hence, when we plot these survival functions, the left-hand one should always lie below the right-hand one. To illustrate this numerically, we first fix the parameters at specific values and then plot the corresponding survival functions, obtained from a large number of simulations. The plots illustrating the survival functions for different combinations of α and β can be found in Figure 2.

We find that all plots in Figure 2 are consistent with the first-order stochastic dominance in Theorem 3.4. Furthermore, when we vary the parameters α and β, the magnitude of the differences between the survival functions varies. This indicates that the benefits from redistributing all original weights for infinite-mean losses to the loss X_m for finite-mean agents depend on the parameters α and β of the underlying Pareto distributions.

By comparing the plots in Figure 2, we observe that the differences between the survival functions shrink when the shape parameter α for the finite-mean Pareto loss is closer to the shape parameter β for the infinite-mean Pareto loss. In each row of Figure 2, the plot on the right, where α and β are closer, shows a smaller difference than the plot on the left. In each column of Figure 2, the upper plot, where α and β are closer, again shows a smaller difference than the plot below. An intuitive reason for the appearance of this pattern is that the distributions Pareto(α) and Pareto(β) become more similar as α and β get closer, thereby reducing the impact of redistributing the weights.

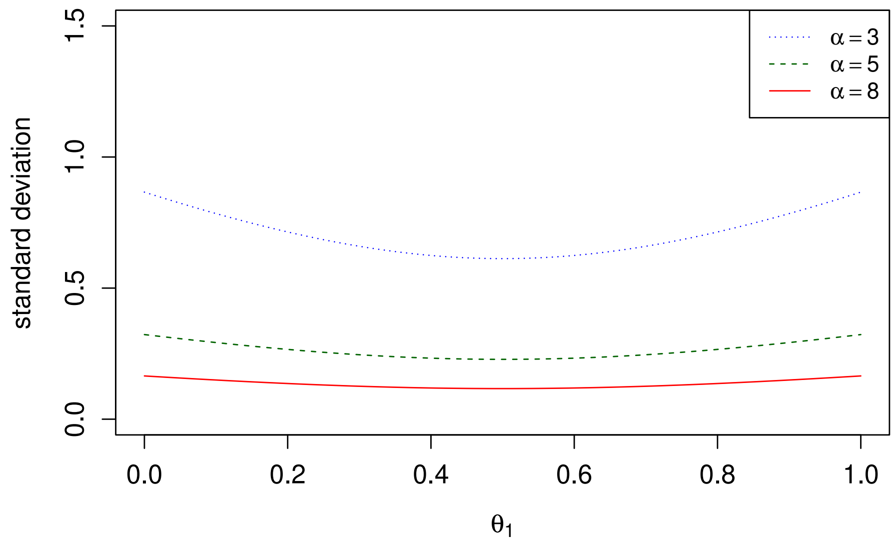

Figure 3: Plot of standard deviations of Y_i for a finite-mean agent i, with $n = 2$ and different values of α, under the Pareto distribution and the setting in the scheme [FR]. The standard deviations are independent of the choice of the infinite-mean Pareto loss parameter β.

6.2 Illustration for convex orders for the rules [FR*] and [LS*]

For the rule [FR*], it follows from Proposition 4.3 that the following convex-order relationship holds for a finite-mean agent i:

$$\frac{1}{n}\left(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\right) \leq_{\text{cx}} \sum_{k=1}^{m} \theta_k X_k + \sum_{k=m+1}^{n} \theta_k X_k^{\text{finite}},$$

which implies that the standard deviation of the loss after *uniform* risk allocation should be minimal. To illustrate this feature numerically, we use Pareto losses with $n = 2$ as an example: $X_1 \sim \text{Pareto}(\alpha)$ with $\alpha > 2$ and $X_2 \sim \text{Pareto}(\beta)$ with $\beta \leq 1$. Under Pareto distributions, closed-form formulas for the standard deviations of these expressions are available if $\alpha > 2$; see, e.g., Klugman et al. (2019). The plot for the standard deviations with different values of α can be found in Figure 3.

As expected, the plot is consistent with the fact that the standard deviation of the risk allocation is always minimized at $\theta_1 = 0.5$, corresponding to the uniform risk allocation. Also, we can see that the graphs corresponding to higher α are "flatter", which indicates that the variations in standard deviations as θ_1 changes becomes smaller when α increases. This occurs because as α increases, the standard deviation of a Pareto(α) loss decreases, thereby reducing the absolute magnitude of the variations in standard deviation.

Next, we investigate the following convex-order relationship between the risk allocation for a finite-mean agent in the rules [LS*] and [FR*] from Proposition 5.5:

$$\frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) \leq_{\text{cx}} \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j^{\text{finite}}\right).$$

This implies that the standard deviation of the risk allocation for the rule [LS*] is smaller than that for the rule [FR*]. To illustrate this difference, again we will take Pareto losses with $\alpha > 2$ as an example and compare the standard deviations for these expressions with different values of parameters. Closed-form

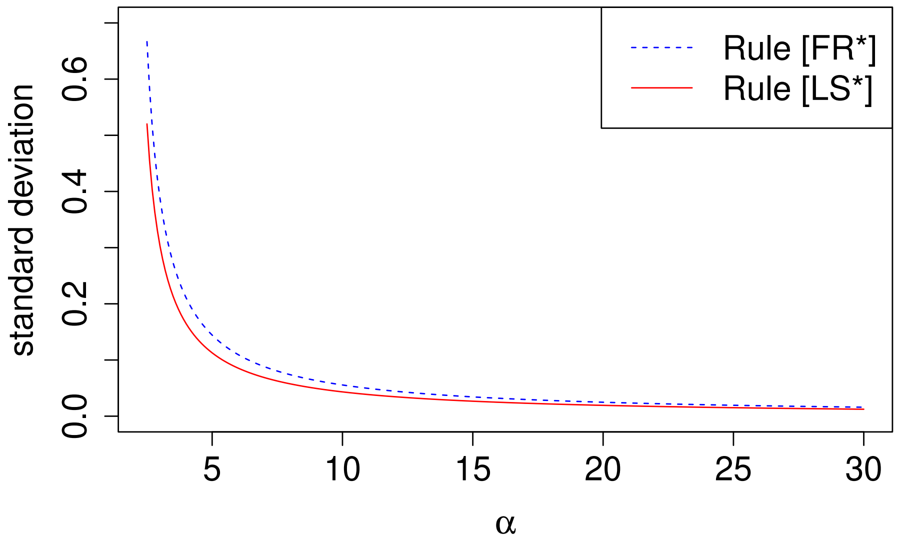

Figure 4: Comparison between standard deviations of the risk allocations for a finite-mean agent, for the rules [FR*] and [LS*], with $m = 3$, $n = 5$, $\beta = 0.5$, and different α. The formula in Proposition 5.3 is utilized for computing the value of d for every α. While the standard deviation of the risk allocation for the rule [LS*] depends on β, the changes in the standard deviation from varying β are tiny and hardly observable visually, thus we choose not to include this factor in the plot.

formulas for these expressions are again available under Pareto distribution; see, e.g., Klugman et al. (2019). The plot for comparing the standard deviations can be found in Figure 4.

As expected, the standard deviation of the expression for the rule [LS*] is always smaller than that for the rule [FR*]. Furthermore, as α increases, the standard deviations of both expressions drop, and the absolute difference between the standard deviations reduces. However, the standard deviations actually have similar *relative* difference when α changes, as seen in Figure 5.

6.3 Quantile comparisons for the rules [L*], [FR*], and [LS*]

In previous sections we have used stochastic orders to compare the rules [L*], [FR*], and [LS*]. Here we will use a different tool for the comparison, namely the quantile function. Since analytical treatment of the quantile function is quite challenging, we resort to numerical investigations. Instead of the probabilistic functions that are more commonly used in numerical illustrations, like CDF and density function, we consider the quantile function here due to its foundational role for the scheme [FR], in which synthetic finite-mean losses are generated via quantile functions. For all numerical illustrations in Section 6.3, we always work under the Pareto distribution, with the shape parameters $\alpha = 2$ for finite-mean Pareto losses, and $\beta = 0.7$ for infinite-mean Pareto losses.

First, we provide a preliminary numerical illustration to gain a basic understanding of how the two types of "synthetic" finite-mean losses X_j^{finite} (from the scheme [FR]) and $X_j \wedge d$ (from the scheme [LS]) differ in nature based on their quantile functions. The quantile functions of $X_j \wedge d$ and X_j^{finite}, for an infinite-mean agent j, are plotted in Figure 6. We observe that the two quantile functions intersect at a single point, apart from the initial intersection at $p = 0$. Furthermore, by construction, the limited loss variable $X_j \wedge d$ has a maximum value of d, so its quantile function is capped at d. In contrast, the quantile function for X_j^{finite} is unbounded. In this regard, the loss $X_j \wedge d$ may be seen as having a less heavy tail than X_j^{finite} when we

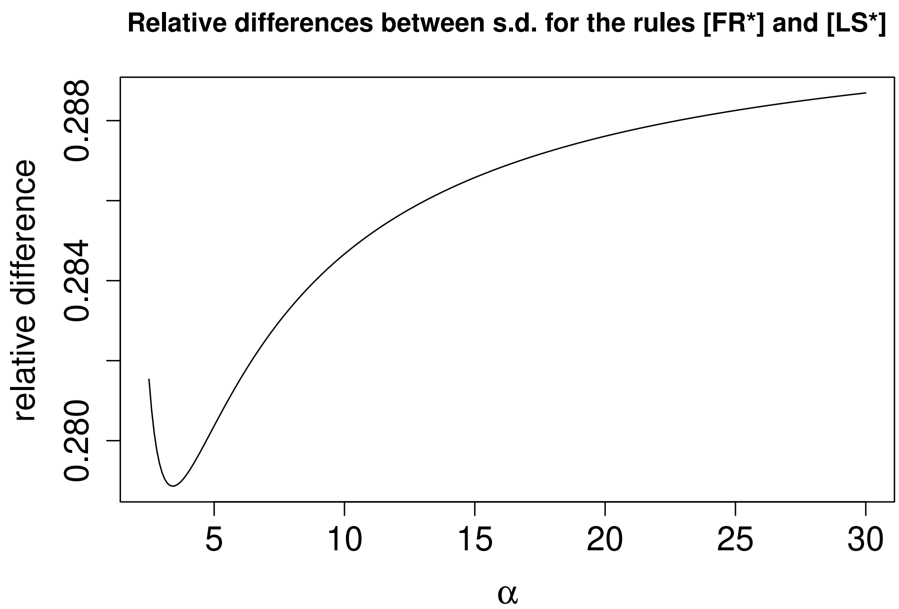

Figure 5: Plot of relative difference between standard deviations (s.d.) of the risk allocation for a finite-mean agent, for the rules [FR*] and [LS*], with $m = 3$, $n = 5$, $\beta = 0.5$, and different α. The formula in Proposition 5.3 is utilized for computing the value of d for every α. Here, the relative difference is computed by the formula $|\sigma_{\mathrm{FR}} - \sigma_{\mathrm{LS}}|/\sigma_{\mathrm{LS}}$ where σ_{FR} and σ_{LS} denote the s.d. for the rules [FR*] and [LS*] respectively.

associate the tail heaviness with the values taken by the quantile function for input probabilities approaching 1.

Second, we compare the risk allocation for an infinite-mean agent j across the three rules [L*], [FR*], and [LS*]. Here, we set $m = 3$ and $n = 5$, and then compare the quantile functions of the following three allocations from the three rules for an infinite-mean agent j (which is agent 4 or 5). For notational convenience, below we add extra superscripts for the Y_j's to identify the rule in consideration.

- *Rule [L*]:* $Y_j^{\mathrm{L}} = X_j$,
- *Rule [FR*]:* $Y_j^{\mathrm{FR}} = \frac{1}{5}(X_1 + X_2 + X_3 + X_4^{\mathrm{finite}} + X_5^{\mathrm{finite}}) + (X_j - X_j^{\mathrm{finite}})$,
- *Rule [LS*]:* $Y_j^{\mathrm{LS}} = \frac{1}{5}(X_1 + X_2 + X_3 + X_4 \wedge d + X_5 \wedge d) + (X_j - d)_+$.

In Figure 7, we plot the ratios between two pairs of quantiles, namely the ratio between the quantiles of Y_j^{FR} and Y_j^{LS}, and that between the quantiles of Y_j^{LS} and Y_j^{L}. These ratios provide an impression of the potential benefits offered by the rules [FR*] and [LS*]. A common characteristic shared in the two ratio plots is that the ratio approaches 1 as the input probability gets close to 1, demonstrating the similarity of the tail behaviours of the three quantiles. Besides, we can observe that both ratio plots cross the value 1 at a certain input probability p, with the ratio being higher (lower) than 1 when the input probability is below (above) p. Hence, when assessing the three rules based on quantiles, none of them is the "universally best" one in the sense of having the smallest quantile for every input probability p. Indeed, since tail behaviours often receive the most attention for risk management purposes, the similarity in their tail behaviours may lead to similar preferences for all three risk-sharing rules.

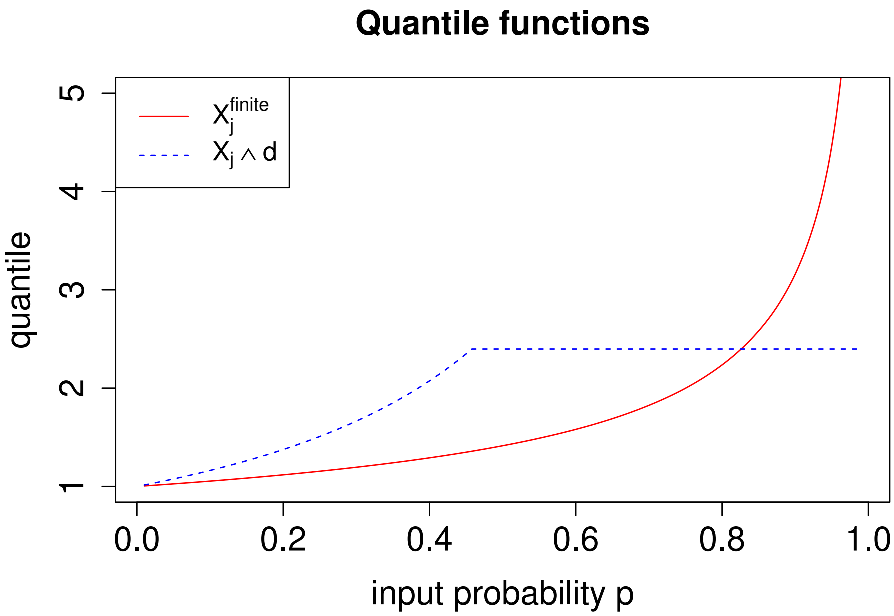

Figure 6: Comparison between quantile functions of $X_j \wedge d$ and X_j^{finite}, where $X_j \sim \text{Pareto}(\beta = 0.7)$ is an infinite-mean Pareto loss, and $\alpha = 2$. The value of d is calculated by the formula in Proposition 5.3.

7 Conclusion

This paper explores three P2P risk-sharing schemes: [L], [FR], and [LS]. The latter two schemes are non-linear in nature, which allows us to construct two risk-sharing rules, [FR*] and [LS*], that yield improvements over the rule [L*], which is Pareto optimal in the scheme [L]. To summarize and illustrate the findings in this paper, we present our key results graphically in Figure 8.

The common theme across these three risk-sharing rules is the application of uniform risk sharing, but applied to different groups of losses for these three rules. While the rule [L*] only involves the original finite-mean losses, the rules [FR*] and [LS*] also involve additional finite-mean losses, which are generated in two distinct ways: The additional finite-mean losses for the rule [FR*] are $X_{m+1}^{\text{finite}}, \ldots, X_n^{\text{finite}}$, obtained via probability integral transform and inverse transform sampling, whereas the additional losses for the rule [LS*] are $X_{m+1} \wedge d, \ldots, X_n \wedge d$, which are capped infinite-mean losses.

The rule [LS*] outperforms the rule [FR*] for finite-mean agents in convex order, as demonstrated in Proposition 5.5. Together with the more understandable separation of infinite-mean losses in the scheme [LS], the rule [LS*] is considered the most appealing choice among the three rules. After applying the rule [LS*] to a P2P risk-sharing agreement, participation from both finite-mean and infinite-mean agents becomes appealing.

Finally, we discuss some limitations of this paper. First, many results in this paper are based on the assumption that the losses are Pareto distributed. Therefore, when applying the approaches discussed in this paper to catastrophic risk management, one needs to assess whether the Pareto loss assumption is reasonable by carefully analyzing the specific context. Extending these results beyond the Pareto distribution is an interesting direction for future research. Second, heuristic arguments are used to demonstrate the improvements of the rules [FR*] or [LS*] over the rule [L*] for infinite-mean agents. Such arguments are not always conclusive, and future work could focus on providing more rigorous analytical justifications.

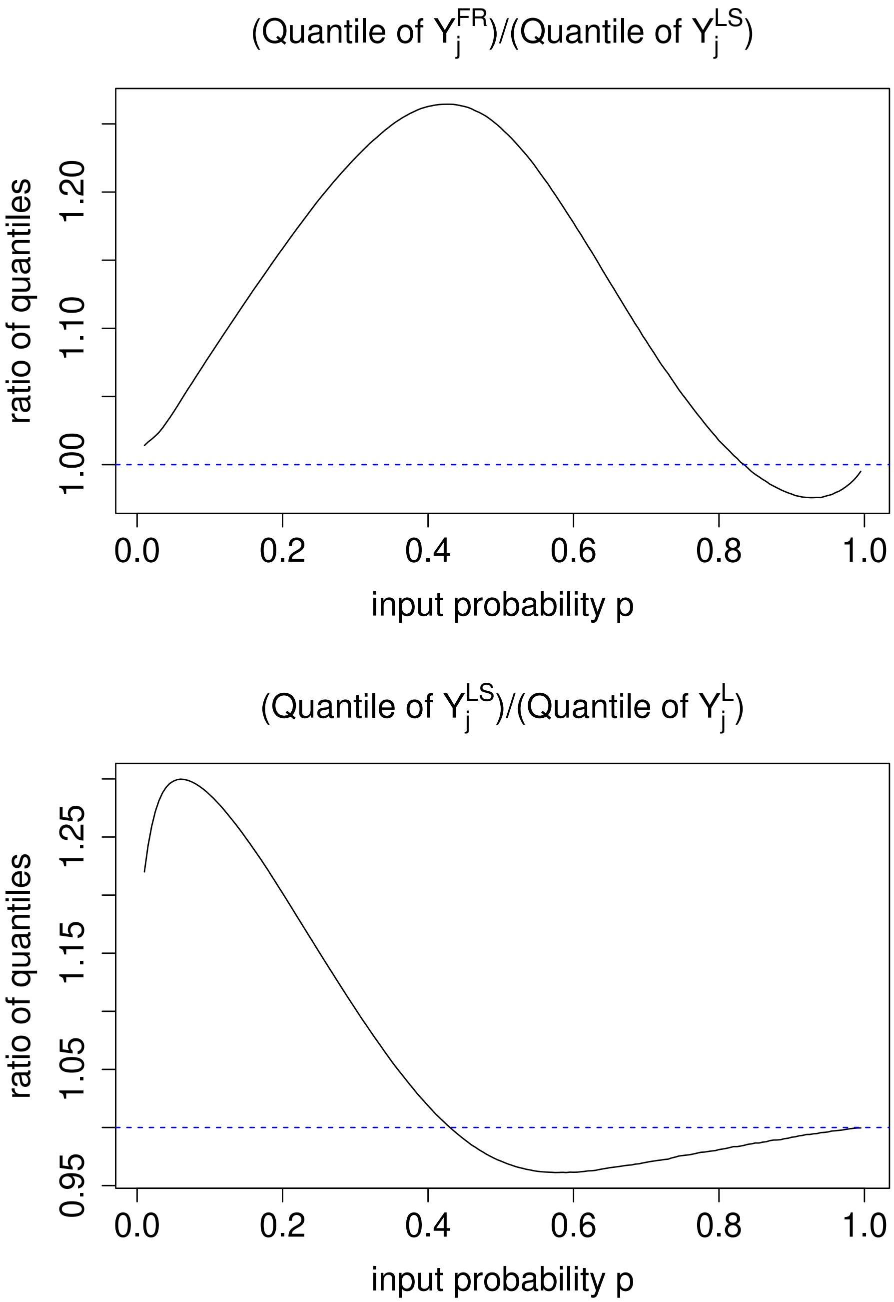

Figure 7: Comparison between quantile functions of the risk allocations for an infinite-mean agent j for the three rules [L*], [FR*], and [LS*], namely Y_j^{L}, Y_j^{FR}, and Y_j^{LS} respectively, with $\alpha = 2$, $\beta = 0.7$, $m = 3$, and $n = 5$, through two plots of quantile ratios. The value of d is calculated by the formula in Proposition 5.3. The plots are obtained by simulation.

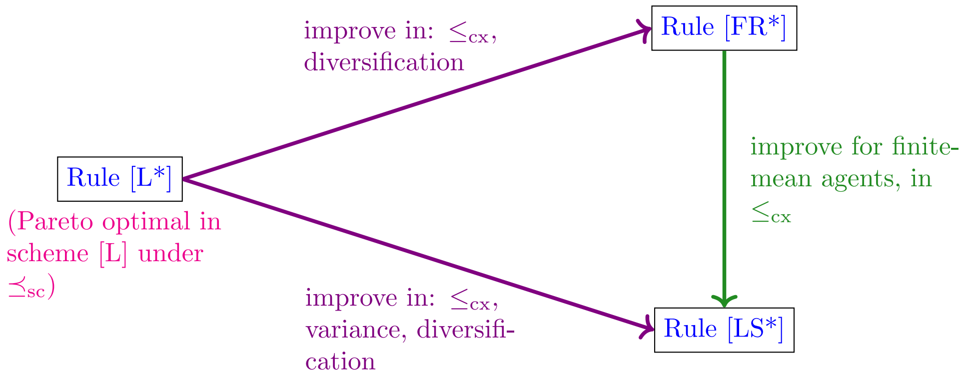

Figure 8: Illustration of the key results of this paper.

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Disclosure statement

No potential competing interests are reported by the authors.

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