(NO-)BETTING PARETO OPTIMA UNDER RANK-DEPENDENT UTILITY

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Abstract. In a pure-exchange economy with no aggregate uncertainty, we characterize in closed-form and in full generality Pareto-optimal allocations between two agents who maximize (non-concave) rank-dependent utilities (RDU). We then derive a necessary and sufficient condition for Pareto optima to be no-betting allocations (i.e., deterministic allocations - or full insurance allocations). This condition depends only on the probability weighting functions of the two agents, and not on their (concave) utility of wealth. Hence with RDU preferences, it is the difference in probabilistic risk attitudes given common beliefs, rather than heterogeneity or ambiguity in beliefs, that is a driver of betting behavior. As by-product of our analysis, we answer the question of when sunspots matter in this economy.

1. Introduction

Starting from an economic environment with no aggregate uncertainty, when will two economic agents want to bet (engage in speculative trade), thereby introducing uncertainty into the economy? One immediate and well-known case is when the agents are risk-averse Expected-Utility (EU) maximizers who disagree about probability assessments. If the agents

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have heterogeneous beliefs, then they find it Pareto-improving to engage in uncertainty-generating trade through betting. Conversely, heterogeneity in probabilistic beliefs between two risk-averse EU-maximizers is the only way in which betting can be Pareto-improving when starting from a no-betting allocation.

This, as Billot et al. [4, 5] argue, is hard to reconcile with the reality that significantly less betting than the prediction of the classical theory happens on all possible sources of uncertainty. Indeed, if heterogeneity in (subjective) beliefs arises from differences in preferences, as in the subjective EU (SEU) theory of De Finetti and Savage [13, 35], then the above discussion would imply that as long as individuals have different preferences over acts, they will engage in betting. Hence, if less betting takes place than predicted, then agents agree on preferences over acts more than hypothesized. Consequently, either agents are more homogeneous in their preferences than assumed, and therefore agree on beliefs much more than the predictions of SEU, or SEU itself is not a plausible descriptive model of behaviour. Following Billot et al. [4, 5], we opt for the latter explanation.

In this paper, similarly to Billot et al. [4], we contend that it is the inadequacy of SEU rather than the similarity in preferences that could explain why less betting takes place in reality than what is predicted by the classical model. However, unlike Billot et al. [4], we suggest that this might not be due to Ellsberg-type behavior, but rather to Allais-type behavior. In other words, we propose that it is not necessarily vagueness and heterogeneity in beliefs that could explain the aforementioned disconnect, but rather differential risk attitudes given a common objective belief, where risk attitude is not solely captured by marginal utility of wealth. The chief decision-theoretic model that captures such a second layer of risk attitude is the Rank-Dependent Utility (RDU) model of Quiggin [29, 30], which is based on a probability weighting function.

Specifically, we consider a standard pure-exchange economy with no aggregate uncertainty. The economy is populated by two RDU-maximizing agents, characterized by generic probability weighting (distortion) functions and concave utility functions. Our setting allows for non-convexity in preferences, as no assumption on the concavity or convexity of the distortion functions is imposed *a priori*. Our first main result (Theorem 3.1) provides a closed-form characterization of Pareto-optimal allocations in full generality. This allows us, in particular, to examine a sufficient condition for Pareto-optimal allocations to be no-betting allocations[1] (Corollary 3.5). This condition is satisfied for instance when the two agents are strongly risk-averse, that is, averse to mean-preserving spreads, and preferences hence display convexities. Our second main result (Theorem 4.1) provides a more general condition (Condition (4.1), or equivalently Condition (4.2)) that is both necessary and sufficient for Pareto-optimal allocations to be no-betting allocations. This suggests, among other things, that Pareto-optimal allocations can be no-betting allocations even in the presence of non-convexities in preferences. At the extreme, preferences that display risk seeking behavior lead to Pareto-optimal allocations that unsurprisingly involve betting (Corollary 3.4).

[1]It is important to note that what we call a *no-betting Pareto optimum* is a Pareto-optimal allocation that is constant across states of the world, and hence does not involve any betting. This is sometimes referred to as a *full insurance allocation* (e.g., Billot et al. [4, 5]), and it is not to be understood as an allocation that is optimal in the sense of the *no-betting Pareto dominance* criterion of Gilboa et al. [19] or Gilboa and Samuelson [18].

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As an illustration, we consider an example involving inverse-S shaped probability weighting functions. Such distortion functions have been shown to be plausible and descriptively relevant, as they are able to accommodate typical behavior involving the overweighting of extreme (good and bad) events (e.g., Quiggin [30] or Tversky and Kahneman [38]). We illustrate the effect of inverse S-shaped probability weighting functions on the shape of Pareto-optimal allocations, and we examine the effect of the parameters of such distortion functions on Condition (4.1), and hence on the betting vs. no-betting property of Pareto optima.

In a setting with two Choquet Expected Utility (CEU) agents (Schmeidler [36]) with constant initial endowments (and hence constant aggregate endowment), De Castro and Chateauneuf [12, Theorem 4.2] show that a condition similar to our Condition (4.2) is sufficient for the absence of any trade that leads to a Pareto-improvement over the initial constant allocation. However, it is not clear whether this condition is necessary in their setting. Additionally, in a finite state space setting and with two Choquet Expected Utility (CEU) agents with constant initial endowments (and hence constant aggregate endowment), Dominiak et al. [14, Theorem 4.1] also show that a similar condition to our Condition (4.2) is both necessary and sufficient for the absence of a Pareto-improving trade over the constant initial allocation. What our Theorem 4.1 contributes to this literature is the following. First, we only assume that the aggregate endowment is constant. Second, not only do we show that Condition (4.2) is both necessary and sufficient for the absence of a Pareto-improving trade over the initial allocation of endowments, but we also show that an allocation is Pareto-optimal if and only it is a no-betting allocation (that is, constant across states), which is also equivalent to Condition (4.2). Third, in the context of CEU, it is not possible to disentangle beliefs from risk attitudes that are not captured by marginal utility of wealth, while RDU provides this flexibility. In fact, we also show how risk aversion of the agents (any notion of risk aversion in RDU), or the weaker notion of pessimism, is sufficient for our Condition (4.2) to hold, and therefore for Pareto optima to be no-betting allocations (Proposition 4.8). In this sense, the driver of betting in RDU is probabilistic risk aversion, since beliefs are objective and common. We elaborate on this in Remark 4.2.

Our results also contribute to the literature on sunspot equilibria. Cass and Shell [6] propose the concept of a "sunspot" as an extrinsic source of uncertainty; and a sunspot equilibrium occurs when there exists a Pareto-optimal allocation that is contingent on this extrinsic source of uncertainty – hence *not* constant across states of the world, even when starting from a situation of deterministic initial endowments. In other words, if sunspots do not occur, then every Pareto-optimal allocation is a no-betting allocation. We do not attempt to argue that sunspots are realistic or empirically observed. Indeed, on the one hand, if sunspots do not matter, then betting or gambling cannot happen in *any* exchange market. On the other hand, if sunspots matter, then we would expect that many agents bet on events, as long as their probability weighting functions satisfy a certain condition. In other words, we would expect a significant amount of betting to occur on extrinsic events. As an immediate implication of our Theorem 4.1, we show that sunspots do not matter at a Pareto optimum, as long as our Condition (4.1) is satisfied. This happens, for instance, if preferences are convex. On the other hand, when preference are non-convex, sunspots do in fact matter at a Pareto optimum. This echoes the results of Tallon [37], who shows that in a standard sunspot pure-exchange economy with no aggregate uncertainty, but with CEU agents, sunspots matter at equilibrium if the agents are ambiguity-loving, that is their non-additive probability measures are

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submodular (concave); while sunspots do not matter if the non-additive probability measures are supermodular (convex).

Related Literature: We refer to Quiggin [31], Wakker [39], Chew et al. [10], Chateauneuf and Cohen [7], Chateauneuf et al. [8, 9], Cohen [11], Ryan [34], and Ghossoub and He [17] for more about the disentanglement of risk aversion from marginal utility in RDU, and the role played by the probability weighting in determining risk attitudes.

Billot et al. [4] argue that the disconnect between the predictions of SEU and the reality of markets might be due to vagueness in beliefs, or ambiguity. They extend the classical analysis to the case of agents who maximize Maxmin Expected Utility (MEU) preferences, as proposed by Gilboa and Schmeidler [20], and they show that, if agents exhibit decreasing marginal utility of wealth, then Pareto-optimal allocations are no-betting (i.e., full insurance) allocations if and only if the agents share at least one common prior on the state space. This result includes, as a special case, the CEU preferences of Schmeidler [36], with convex capacities. A significant extension of these results is given by Rigotti et al. [33], who consider a general class of ambiguity-sensitive convex preferences that includes MEU, CEU with convex capacities, the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji [25], and the variational preferences model of Maccheroni, Marinacci, and Rustichini [26], for instance. They introduce a notion of subjective beliefs as supporting hyperplanes (similar to Yaari [41]) and show that no-betting allocations are Pareto optimal if and only if these subjective beliefs have a non-empty intersection. A related analysis for the case of convex MEU-preferences is given in Kajii and Ui [24].

While the literature mentioned above focuses on convex ambiguity-sensitive preferences, similar results have been obtained in the case of non-convex ambiguity-sensitive preferences. For instance, Billot et al. [5] consider the case of CEU-maximizers with decreasing marginal utility of wealth but capacities that are not necessarily convex. They also show that Pareto-optimal allocations are no-betting allocations if and only if the agents' capacities have cores with a non-empty intersection, that is, they share at least one prior on the state space. Dominiak et al. [14] provide extensions of the results of Kajii and Ui [24] to the case of non-convex Choquet preferences, and Ghirardato and Siniscalchi [16] extend some of the results of Rigotti et al. [33] to a class of non-convex preferences.

The rest of this paper is set out as follows. Section 2 describes the economy with RDU-maximizing agents that we consider in this paper. In Section 3, we present our main result on the general characterization of Pareto-optimal allocations, examine two special cases, and look at a numerical example that illustrates the case of inverse S-shaped probability weighting functions. The general result given in Section 3 is then used in Section 4 to provide a necessary and sufficient condition for Pareto-optimal allocations to be no-betting allocations. We also provide an illustrating example. Finally, Section 5 concludes. All proofs are given in the Appendices.

2. The Economy

2.1. **Preferences.** Let $(S, \Sigma, \mathbb{P})$ be a non-atomic probability space, in which S is the set of states, and let $\mathcal{X}$ be a given collection of random variables on $(S, \Sigma, \mathbb{P})$ containing all *payoffs* available to decision makers. For any $Z \in \mathcal{X} \subset L^{\infty}(S, \Sigma, \mathbb{P})$, let F_Z and F_Z^{-1} denote the

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cumulative distribution function on $\mathbb{R}$ and quantile function on $[0,1]$, respectively, defined by:

$$F_Z(t) := \mathbb{P}\left(s \in S : Z(s) \leqslant t\right) \quad \text{and} \quad F_Z^{-1}(t) := \inf\left\{z \in \mathbb{R} \;\middle|\; F_Z(z) \geqslant t\right\}.$$

We consider a standard pure-exchange economy under uncertainty. The economy consists of two RDU-agents, with utilities $U_i : \mathcal{X} \to \mathbb{R}$ of the form

$$U_i(Z) := \int_S u_i(Z)\, \mathrm{d}T_i \circ \mathbb{P}, \qquad \text{for all } Z \in \mathcal{X}, \tag{2.1}$$

where u_i is the utility function and T_i is the probability weighting function of agent $i \in \{1,2\}$. We make the following assumption throughout.

Assumption 2.1. *The utility functions u_i are increasing, strictly concave, twice continuously differentiable, and satisfy the Inada conditions $\lim_{x\to-\infty} u_i'(x) = +\infty$ and $\lim_{x\to+\infty} u_i'(x) = 0$.*[2]

The probability weighting functions $T_i : [0,1] \to [0,1]$ are such that $T_i(0) = 0$, $T_i(1) = 1$, and the functions T_i are absolutely continuous and increasing.

The first part of Assumption 2.1 states standard requirements on the agents' utility indices. In particular, strict concavity is required in Theorem 3.1, where the inverse of u_i' enters in the explicit representation of the optimal sharing rule. The second part of Assumption 2.1 relates to the agents' weighting functions. These conditions are minimal requirements that are typically used in the literature.

Let $\widetilde{T}_i$ be the conjugate function of T_i given by $\widetilde{T}_i(t) := 1 - T_i(1-t)$, for all $t \in [0,1]$. The following simple and standard, but important, result allows us to focus on quantile functions rather than payoffs in $\mathcal{X}$.

Lemma 2.2. *For any $Z \in \mathcal{X}$ and $i \in \{1,2\}$, we have $U_i(Z) = \int_0^1 u_i\left(F_Z^{-1}(t)\right) \widetilde{T}_i'(t)\, \mathrm{d}t$.*

RDU is a special case of CEU, in which the agent's non-additive measure (capacity) ν is a distortion of a probability measure ($\nu = T \circ \mathbb{P}$). Convexity (resp. concavity) of the probability weighting function T yields convexity (resp. concavity) of the capacity ν. In CEU, a convex capacity reflects ambiguity-aversion, while a concave capacity reflects ambiguity-seeking behavior. In RDU, the interpretation is different, and it relates to risk attitudes rather than ambiguity attitudes. Specifically, if the probability weighting function T_i is convex (resp. concave) and u_i is concave, Chew et al. [10] show that agent i is strongly risk averse (resp. seeking), that is, averse to mean-preserving spreads. Various other notions of risk aversion in RDU have been proposed in the literature. See Section 4.2 for an overview thereof.

2.2. **Allocations and Optimality.** Let $W \in \mathbb{R}$ be the aggregate endowment, assumed to be constant across states. An allocation $(X_1, X_2) \in \mathcal{X} \times \mathcal{X}$ is feasible if $X_1 + X_2 = W$. An allocation $(X_1, X_2) \in \mathcal{X} \times \mathcal{X}$ is a *no-betting allocation* if it is feasible and $X_i = c$, $\mathbb{P}$-a.s., for some $i \in \{1,2\}$ and some constant $c \in \mathbb{R}$. A feasible allocation (X_1^*, X_2^*) is Pareto Optimal (PO) if there is no other feasible allocation (X_1, X_2) such that $U_i(X_i) \geqslant U_i(X_i^*)$ for $i \in \{1,2\}$, with at least one strict inequality.

[2] Throughout this paper, we mean "increasing" in the strictly increasing sense. We use the terminology "non-decreasing" to mean weakly increasing. We use the same convention for decreasing functions.

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3. A Closed-Form Characterization of Pareto-Optimal Allocations

Our first main result characterizes PO allocations in full generality. We then provide a sketch of the proof, and examine an illustrating example. Let $\mathtt{U}$ be a random variable on the non-atomic space $(S, \Sigma, \mathbb{P})$, with a uniform distribution on $(0,1)$.

Theorem 3.1. *Suppose that Assumption 2.1 holds. A feasible allocation (X_1^*, X_2^*) is PO if there exists some $\lambda^* > 0$ such that*

$$X_2^* \;=\; m^{-1}\left(\lambda^* \delta'\big(T_1\left(\mathtt{U}\right)\big)\right), \tag{3.1}$$

where $m(x) := \frac{u_1'(W-x)}{u_2'(x)}$, $\forall x \in \mathbb{R}$, and δ is the convex envelope[3] of the function $\Psi : [0,1] \to [0,1]$ given by

$$\Psi(t) \;:=\; \widetilde{T}_2\left(T_1^{-1}(t)\right), \quad \forall t \in [0,1]. \tag{3.2}$$

Moreover, for every PO allocation (X_1^*, X_2^*), there exists some $\lambda^* > 0$ such that

$$X_2^* \overset{\mathbb{P}}{\sim} m^{-1}\left(\lambda^* \delta'\left(T_1\left(\mathtt{U}\right)\right)\right).$$

We provide a short sketch of the proof in Section 3.1, while the full proof is given in Appendix A. Note that by Assumption 2.1, it follows that $m : \mathbb{R} \to \mathbb{R}_+$ is strictly increasing, with strictly increasing inverse $m^{-1} : \mathbb{R}_+ \to \mathbb{R}$. Moreover, Ψ is a distortion function, and we interpret it as a composed weighting function. When assigned to Agent 2, this composed weighting function allows to assign to Agent 1 a linear weighting function. In other words, the pair of weighting functions (T_1, T_2) and (id, Ψ) yield the same PO allocations, where id denotes the identity function.

Importantly, Theorem 3.1 implies that, unlike in SEU, common probabilistic beliefs (the same underlying probability measure $\mathbb{P}$) might still lead to a situation in which betting is Pareto improving, depending on the shape of the probability weighting functions. Indeed, eq. (3.1) implies that whether or not a PO allocation (X_1^*, X_2^*) is a no-betting allocation depends entirely on the linearity of δ (that is, δ' being a constant). This, in turn is solely a consequence of the convexity of the composed weighting function $\Psi = \widetilde{T}_2 \circ T_1^{-1}$. Hence, the shape of X_2^* is a function of the marginal utilities of wealth (through the function m). However, whether or not (X_1^*, X_2^*) is a no-betting allocation only depends on the probability weighting functions T_1 and T_2.

Additionally, eq. (3.1) shows precisely how potential uncertainty in PO allocations is generated through the random variable $\mathtt{U}$. Specifically, to obtain a PO allocation $(X_1^*, X_2^*) = (W - X_2^*, X_2^*)$, X_2^* is constructed by applying a non-decreasing transformation to $\mathtt{U}$ as in eq. (3.1). In Section 3.3 we discuss two special cases in which this non-decreasing transformation takes simpler forms.

[3] The convex envelope of a function is the greatest convex function that is pointwise dominated by that function: For a real-valued function f on $[0,1]$, the convex envelope of f on $[0,1]$ is defined as the greatest convex function g on $[0,1]$ such that $g(x) \leqslant f(x)$, for each $x \in [0,1]$. See He et al. [21] for details.

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3.1. **A Sketch of the Proof of Theorem 3.1.** The proof of the main result relies on a four-step procedure that is based on a convex envelope relaxation for the composition of arbitrary distortion functions. All details of the procedure described below are given in Appendix A.

Step 1 starts with determining PO allocations through the Pareto frontier of the set of attainable utilities:

$$(\mathcal{P}_{\bar{u}}) \qquad \sup_{Y\in\mathcal{X}} U_1(W-Y) \quad \text{s.t.} \quad U_2(Y) \geqslant \bar{u}.$$

PO allocations are characterized by solving $(\mathcal{P}_{\bar{u}})$ and varying $\bar{u} \in \mathbb{R}$. Due to the generality of the probability weighting functions, the optimization problem is non-concave.

Step 2 relies on a quantile-based reformulation of $(\mathcal{P}_{\bar{u}})$ in which the variable is a quantile function, rather than an element of $\mathcal{X}$. Since the aggregate endowment W is constant, the functionals $Y \mapsto U_1(W-Y)$ and $Y \mapsto U_2(Y)$ are both law-invariant with respect to $\mathbb{P}$ on $\mathcal{X}$. Letting $\widetilde{Y} := F_Y^{-1}(\mathtt{U}) \overset{\mathbb{P}}{\sim} Y$ gives $U_1(W-\widetilde{Y}) = U_1(W-Y)$ and $U_2(\widetilde{Y}) = U_2(Y)$. Consequently, solutions to Problem $(\mathcal{P}_{\bar{u}})$ are of the form $f^*(\mathtt{U})$, where f^* is a quantile function. As shown in Lemma A.2, we can focus on the following reformulation

$$\sup_{f\in\mathcal{Q}} \int u_1(W-f_t)\, T_1'(t)\,\mathrm{d}t \quad \text{s.t.} \quad \int u_2(f_t)\, T_2'(1-t)\,\mathrm{d}t \geqslant \bar{u}, \tag{3.3}$$

where $\mathcal{Q}$ is the collection of all quantile functions, i.e.,

$$\mathcal{Q} := \Big\{ f : (0,1) \to \mathbb{R} \;\Big|\; f \text{ is non-decreasing and left-continuous} \Big\}.$$

Letting $\Psi := \widetilde{T}_2 \circ T_1^{-1}$, where $\widetilde{T}_2$ is the conjugate of T_2, Problem (3.3) in turn can be solved via a change of variable:

$$\sup_{q\in\mathcal{Q}} \int u_1(W-q_t)\,\mathrm{d}t \quad \text{s.t.} \quad \int u_2(q_t)\,\Psi_t'\,\mathrm{d}t \geqslant \bar{u}, \tag{3.4}$$

with q^* being optimal for Problem (3.4) if and only if $f^* := q^* \circ T_1$ is optimal for Problem (3.3).

Step 3 solves Problem (3.4) via a pointwise optimization of the Lagrangian

$$\mathcal{L}(q,\lambda) := \int_0^1 u_1(W-q_t) + \lambda u_2(q_t)\,\Psi_t'\,\mathrm{d}t.$$

That is, we determine

$$\bar{q}_t := \arg\max_y \big\{ u_1(W-y) + \lambda u_2(y)\,\Psi_t' \big\}.$$

Since Ψ' might fail to be monotone, $\bar{q}$ might fail to be monotone and hence $\bar{q} \notin \mathcal{Q}$ may result. To overcome this difficulty (arising from the non-convexity of T_1 and T_2), we consider the following relaxation of Problem (3.4):

$$\sup_{q\in\mathcal{Q}} \int u_1(W-q_t)\,\mathrm{d}t \quad \text{s.t.} \quad \int u_2(q_t)\,\delta_t'\,\mathrm{d}t \geqslant \bar{u}, \tag{3.5}$$

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where the function δ is the (smooth) convex envelope of Ψ.[4] The convexity of δ yields monotonicity of δ', which guarantees that the pointwise optimizer q^* of the Lagrangian, given by

$$q_t^* := \arg\max_y \left\{u_1\left(W-y\right)+\lambda u_2\left(y\right)\delta_t'\right\},$$

is indeed a quantile function. Finally, we verify that q^*, defined for the relaxed problem, is also an optimizer for Problem (3.4) and obtain the first part of the main result.

Step 4 shows the uniqueness (in distribution) of (X_1^*, X_2^*), as stated in the second part of Theorem 3.1, by using Lemma A.8 in Appendix A.

3.2. **Marginal Rates of Substitution.** By identifying $\mathcal{Q} = B_+$ from Step 3 as the positive cone of the bounded-variation function space $B = \mathrm{BV}^l(0,1)$ – see [2] for details – utilities can be identified by $\widetilde{U}_i : \mathcal{Q} \to \mathbb{R}$ with $\widetilde{U}_i(f) := \int_0^1 u_i\left(f_t\right) T_i'\left(t\right) \mathrm{d}t$, as per Lemma 2.2.

Lemma 3.2. *Each functional $\widetilde{U}_i$ is concave and smooth: For any feasible direction $h \in F(f) := \{h \in B : f + \epsilon h \in \mathcal{Q},\ \text{for some } \epsilon > 0\}$, we have*

$$\lim_{\epsilon\to 0} \frac{\widetilde{U}_i(f+\epsilon h) - \widetilde{U}_i(f)}{\epsilon} = \langle \nabla \widetilde{U}_i(f), h\rangle = \int_0^1 h(t)\,\mathrm{d}\mu_i(t),$$

where the Gateaux gradient $\nabla\widetilde{U}_i(f)$ of $\widetilde{U}_i$ at $f \in \mathcal{Q}$ can be identified with some countably additive measure μ_i on $((0,1), \mathcal{B}(0,1))$, being absolutely continuous with respect to the Lebesgue measure on $(0,1)$ with a bounded utility gradient density given by

$$\frac{\mathrm{d}\mu_i(t)}{\mathrm{d}t} = \psi_i(t) = u_i'(f_t)T_i'(t).$$

With Lemma 3.2 at hand, we obtain the following necessary and sufficient first-order conditions of Pareto optimality. This follows from the concavity of $\widetilde{U}_i$, after a change of variable.

Corollary 3.3. *A Pareto-optimal allocation $(W - f^*, f^*)$ on the consumption set $\mathcal{Q}$ is characterized by the following first-order condition:*

$$\frac{u_1'(W-f_t^*)}{u_2'(f_t^*)} = \lambda^* \frac{T_2'(1-t)}{T_1'(t)}, \quad \text{for some } \lambda^* > 0. \tag{3.6}$$

Note that the corollary implicitly assumes that $f^* \in \mathcal{Q}$. From the perspective of the convex envelope in Problem (3.5), this aspect is captured when moving to the pointwise construction of PO allocations (see again Step 3 in Subsection 3.1). The first-order condition (3.6) is of limited use for the identification of an optimal f^*, since the ratio $\frac{T_2'(1-t)}{T_1'(t)}$ can be non-monotone and thus the inferred f^* may not be an element of $\mathcal{Q}$. For this reason, we introduce in Step 3 of the four-step procedure described in Subsection 3.1 the convex envelope of $\widetilde{T}_2 \circ T_1^{-1}$

[4]Note that $\delta = (\Psi^*)^*$ is the biconjugate of Ψ from convex analysis (see [22, Section E.1.3] for details). In a quite unrelated context of optimal auction design, Myerson [27] applies this technique (convexification of the epigraph; also known as ironing) to determine a seller's optimal auction design, when the buyer's value estimate of the object to be sold is unknown and given by a cumulative distribution function.

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(with monotone derivative) to account for the "missing" concavity within Problem ($\mathcal{P}_{\bar{u}}$) that characterizes PO allocations.

Of course, when T_i is the identity function for $i \in \{1,2\}$, we directly recover the well-known relation between marginal rates of substitution in an economy with EU agents.

3.3. **Two Special Cases.** Next, we show that Theorem 3.1 can be simplified under an Arrow-Pratt type condition for probability weighting functions. Recall first that $\widetilde{T}_i$ is the conjugate of T_i, defined by $\widetilde{T}_i(t) := 1 - T_i(1-t)$, for all $t \in [0,1]$.

Corollary 3.4. *Suppose that Assumption 2.1 holds and that for all $t \in (0,1)$,*

$$\frac{\widetilde{T}_i''(t)}{\widetilde{T}_i'(t)} > \frac{T_j''(t)}{T_j'(t)}, \quad \text{for } i,j \in \{1,2\},\ i \neq j. \tag{3.7}$$

A feasible allocation (X_1^*, X_2^*) is PO if there exists some $\lambda^* > 0$ such that $X_2^* = m^{-1}\left(\lambda^* \frac{T_2'(1-\mathtt{U})}{T_1'(\mathtt{U})}\right)$. Moreover, for every PO allocation (X_1', X_2'), there exists some $\lambda^* > 0$ such that

$$X_2' \overset{\mathbb{P}}{\sim} m^{-1}\left(\lambda^* \frac{T_2'(1-\mathtt{U})}{T_1'(\mathtt{U})}\right).$$

Condition (3.7) is satisfied, for instance, whenever T_1 and T_2 are concave, leading to non-convex preferences and displaying risk seeking attitudes. In that case, Corollary 3.4 states that the PO allocations involve betting; and the result shows precisely how these betting PO allocations depend on the primitives. Under Condition (3.7), Corollary 3.4 shows that unlike in SEU, common probabilistic beliefs (the same underlying $\mathbb{P}$) lead to a situation in which betting is Pareto improving. Condition (3.7) does not depend on the agents' marginal utility of wealth, but only on their probabilistic risk aversion.

Corollary 3.5. *Suppose that Assumption 2.1 holds and that for all $t \in (0,1)$,*

$$\frac{\widetilde{T}_i''(t)}{\widetilde{T}_i'(t)} \leqslant \frac{T_j''(t)}{T_j'(t)}, \quad \text{for } i,j \in \{1,2\},\ i \neq j. \tag{3.8}$$

A feasible allocation (X_1^*, X_2^*) is PO if and only if it is a no-betting allocation.

Condition (3.8) is satisfied, for instance, when both T_1 and T_2 are convex, implying that the two agents are strongly risk-averse, that is, averse to mean-preserving spreads (e.g., Chew et al. [10]). In that case, the agents have convex preferences, and Corollary 3.5 states that the PO allocations are risk-free. Corollary 3.5 is a consequence of Theorem 3.1, but it is also a special case of a more general result (Theorem 4.1 below), in which we characterize no-betting PO allocations by means of Ψ.

3.4. **A Numerical Example.** We now turn our attention to the empirically important case of inverse S-shaped probability weighting functions of Tversky and Kahneman [38].

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Example 3.6 (Inverse S-Shaped Probability Weighting Functions). We consider the Tversky and Kahneman [38] inverse S-shaped weighting functions. Specifically, we suppose that for $i \in \{1,2\}$, the probability weighting function T_i on $[0,1]$ is given by

$$T_i(t) := \frac{t^{\gamma_i}}{\left(t^{\gamma_i} + (1-t)^{\gamma_i}\right)^{1/\gamma_i}}, \tag{3.9}$$

for some $\gamma_i \in (0.279, 1]$. Rieger and Wang [32] show that T_i in eq. (3.9) is increasing and inverse-S shaped if $\gamma_i \in (0.279, 1)$, and T_i is the identity function for $\gamma_i = 1$. It then follows that the function $\Psi : [0,1] \to [0,1]$ defined in eq. (3.2) is given by

$$\Psi(t) = 1 - \frac{\left(1 - T_1^{-1}(t)\right)^{\gamma_2}}{\left(\left(T_1^{-1}(t)\right)^{\gamma_2} + \left(1 - T_1^{-1}(t)\right)^{\gamma_2}\right)^{\frac{1}{\gamma_2}}}.$$

For values $\gamma_1 = 0.5$ and $\gamma_2 = 0.9$, it is easily verified from Figure 1 that there exists $t_0 \in [0,1]$ such that Ψ is convex on $[0, t_0]$ and concave on $[t_0, 1]$. Recall that δ is the convex envelope of Ψ on $[0,1]$. Then $\Psi(0) = \delta(0) = 0$ and $\Psi(1) = \delta(1) = 1$. Moreover, since δ is affine on the set $\left\{t \in [0,1] : \delta_t < \Psi_t\right\}$, there exists some $z_0 \in (0, t_0)$ such that δ is given by

$$\delta(t) = \begin{cases} \Psi(t) & \text{if } t \leqslant z_0; \\ \Psi(z_0) + \left(\frac{1-\Psi(z_0)}{1-z_0}\right)(t - z_0) & \text{if } t \geqslant z_0. \end{cases}$$

Note that δ is continuously differentiable by continuity of Ψ, and moreover it holds that $\delta'(z) = \frac{1-\Psi(z_0)}{1-z_0}$ for all $z \geqslant z_0$. Numerical computation gives $z_0 \approx 0.16$. Figure 1 plots the graph of the functions T_1, T_2, Ψ, and δ.

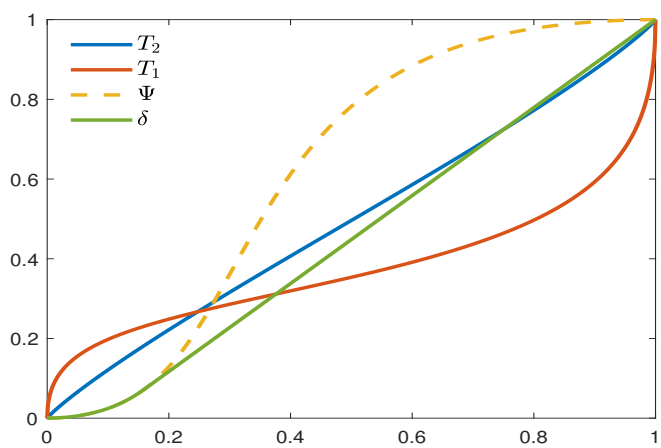

FIGURE 1. Plots of the functions T_1, T_2, Ψ, and the convex envelope δ of Ψ.

Consider now a more specific setup with $W = 0$ and $u_i(x) = \frac{-\exp(-\beta_i x)}{\beta_i}$, with $\beta_i > 0$ for $i \in \{1,2\}$. We then have $m(x) = \exp\left((\beta_1 + \beta_2)\, x\right)$ for $x \in \mathbb{R}$, and so $m^{-1}(y) = \ln(y)/(\beta_1 + \beta_2)$, for $y > 0$. In general, m^{-1} is increasing with $\lim_{y\to 0} m^{-1}(y) = -\infty$ and $\lim_{y\to\infty} m^{-1}(y) = \infty$. An increase in the aggregate risk aversion $\beta_1 + \beta_2$ moves $m^{-1}(\cdot)$ closer to the x-axis, and thus

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PO allocations are in turn closer to zero. To see this, let $\beta_1 = \beta_2 = 0.5$, so that $m^{-1}(y) = \ln(y)$. Theorem 3.1 then allows to identify all PO allocation (X_1^*, X_2^*) via

$$X_2^* = m^{-1}\left(\lambda^* \delta'\Big(T_1\left(\mathtt{U}\right)\Big)\right) = \frac{1}{\beta_1 + \beta_2} \ln\left(\lambda^* \delta'\Big(T_1\left(\mathtt{U}\right)\Big)\right) = \ln\left(\lambda^*\right) + I^*\left(\mathtt{U}\right), \tag{3.10}$$

where $I^*\left(x\right) := \ln\big(\delta'(T_1\left(x\right))\big)$ and for some $\lambda^* > 0$. By varying side-payments $\ln\left(\lambda^*\right)$, we span the set of PO allocations. Based on (3.10) and the point z_0, we can now identify betting and no betting events. Since $\delta'(z)$ is constant on $[z_0, 1]$, it follows from eq. (3.10) that on the event $T_1(\mathtt{U}) \in [z_0, 1]$, no-betting is prevalent. Specifically, on the interval $[T_1^{-1}(z_0), 1] \approx [0.05, 1]$, PO allocations are constant across states.

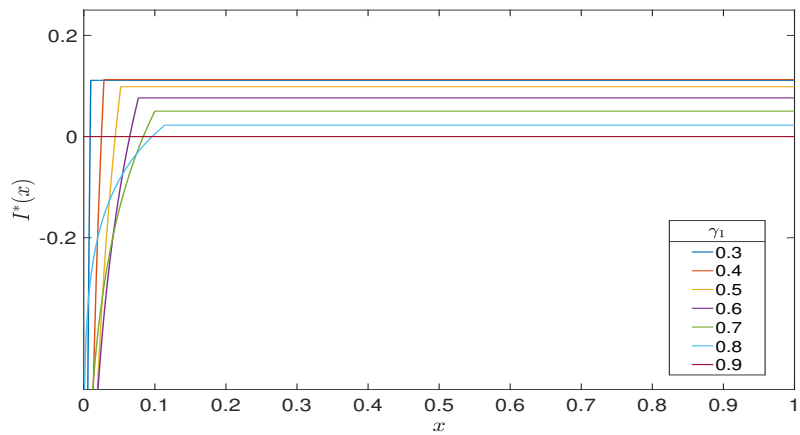

FIGURE 2. Graph of $I^*\left(x\right) = \ln\big(\delta'(T_1\left(x\right))\big)$ with fixed $\gamma_2 = 0.9$ and varying γ_1.

On the other hand, for realizations of $\mathtt{U}$ on $[0, T_1^{-1}(z_0)] = [0, 0.05]$, X_2^* is strictly increasing in $\mathtt{U}$. For $\lambda^* = 1$, $\ln(\lambda^*) = 0$ and thus the side-payment is equal to zero, implying that $X_2^* = I^*(\mathtt{U})$. We display in Figure 2 the function I^* with varying γ_1. Recall that a positive realization of X_2^* means a wealth transfer from Agent 1 to Agent 2, while a negative realization means a wealth transfer from Agent 2 to Agent 1. Since I^* is increasing, a small value of $\mathtt{U}$ yields a "good" realization for Agent 1 and a "bad" realization for Agent 2.

Figure 2 plots the function $x \mapsto I^*\left(x\right)$, by keeping $\gamma_2 = 0.9$ fixed and varying γ_1. The plot highlights a similar functional form for various choices of γ_1: it is flat over a large region. Moreover, the probability of a "betting event" decreases in γ_1 (i.e., it increases in the S-shaped curvature of T_1). Betting events are contained in the interval $[0, T_1^{-1}(z_0)]$, where $I^*\left(x\right)$ is strictly increasing and thus non-flat.

We now look at certainty equivalents for the case $\lambda^* = 1$ with zero side payment. The certainty equivalents at the corresponding PO allocation for Agent 1 and Agent 2 are defined, respectively, as

$$\mathbb{C}_1 := u_1^{-1}\left(U_1\left(W - X_2^*\right)\right) \quad \text{and} \quad \mathbb{C}_2 := u_2^{-1}\left(U_2\left(X_2^*\right)\right).$$

For $(\gamma_1, \gamma_2) = (0.5, 0.9)$, numerical computation then yields $\mathbb{C}_1 \approx 3.53\%$ and $\mathbb{C}_2 \approx 3.24\%$. Since both are positive, the function I^* displayed in Figure 2 yields a higher utility for both agents than in the absence of risk sharing (that is, when $X_2^* \equiv 0$). By varying $\lambda^* > 0$, one alters the risk allocation with zero-sum deterministic side-payments, and thus the aggregate

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certainty equivalents remain the same, i.e., $\mathbb{C}_1 + \mathbb{C}_2 = 6.77\%$. Any allocation $(\mathbb{C}_1, \mathbb{C}_2)$ such that $\mathbb{C}_1 + \mathbb{C}_2 = 6.77\%$ can be obtained by varying the deterministic side-payments.

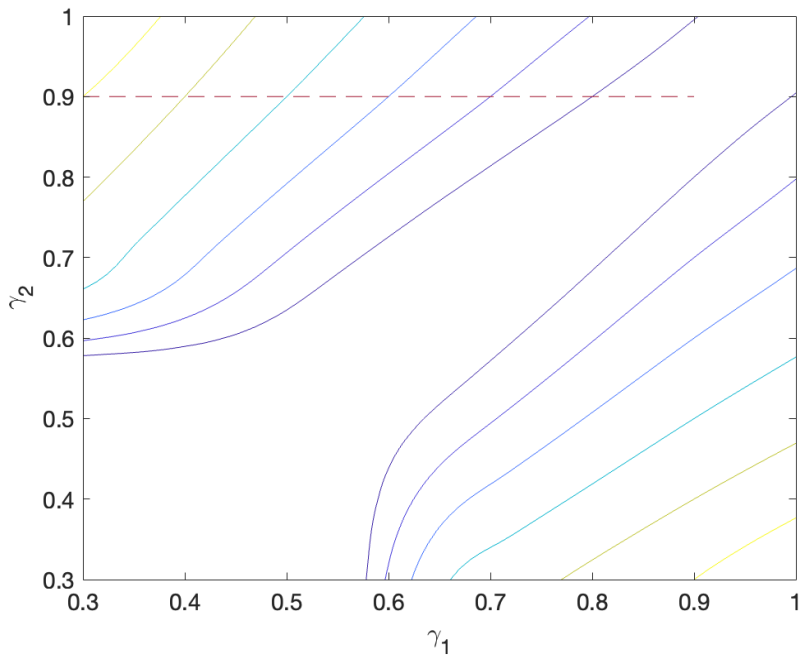

FIGURE 3. The lines represent the parameters (γ_1, γ_2) that yield the same outcome of $\mathbb{C}_1 + \mathbb{C}_2$. The lines are chosen to include the outcomes of $\gamma_2 = 0.9$ and $\gamma_1 = 0.3, \ldots, 0.9$, as shown in Table 1.

γ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\mathbb{C}_1$	7.66%	5.69%	3.53%	1.83%	0.72%	0.14%	0%
$\mathbb{C}_2$	7.63%	5.72%	3.24%	1.81%	0.71%	0.14%	0%
$\mathbb{C}_1 + \mathbb{C}_2$	15.29%	11.41%	6.77%	3.64%	1.43%	0.28%	0%

TABLE 1. The certainty equivalents $\mathbb{C}_1$ and $\mathbb{C}_2$ of $I^*(x)$ corresponding to Example 3.6. Here, we fix $\gamma_2 = 0.9$, and we vary γ_1.

In Table 1, we display the corresponding certainty equivalents $\mathbb{C}_1$ and $\mathbb{C}_2$. Recall here that $\mathbb{C}_1 + \mathbb{C}_2 = 0$ if the parameters are the same: $\gamma_1 = \gamma_2$. In Example 4.3 below, we identify all (γ_1, γ_2) pairs around the 45-degree line that yield no-betting Pareto optimality (see Figure 4(A)). In contrast to the probability of a "betting event", welfare gains from risk sharing increase with a decreasing γ_1 with $\gamma_1 \leqslant \gamma_2$. Figure 3 displays the set of parameters (γ_1, γ_2) that yield the same outcome of $\mathbb{C}_1 + \mathbb{C}_2$ as in Table 1. Due to symmetry in the roles of Agents 1 and 2, these lines are symmetric in the 45-degree line. Furthermore, we find that if the parameters γ_1 and γ_2 are further apart, then $\mathbb{C}_1 + \mathbb{C}_2$ becomes larger. This effect is even more pronounced for larger values of (γ_1, γ_2).

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Example 3.6 above examine situations where the PO allocations are constant with a high probability. In the next section, we look at necessary and sufficient conditions for PO allocations to be no-betting allocations, i.e., constant across states.

4. No-Betting Allocations

This section presents a new criterion that characterizes risk-free PO allocations, even when preferences are no longer convex.

4.1. A Characterization of No-Betting Allocations.

The following result provides necessary and sufficient conditions under which there exist no-betting PO allocations.

Theorem 4.1. *Let the function $\Psi : [0,1] \to [0,1]$ be defined as in (3.2). If Assumption 2.1 holds, then the following are equivalent:*

(1) *For all $t \in [0,1]$,*
$$\Psi(t) \geqslant t. \tag{4.1}$$

(2) *There exists a PO no-betting allocation.*

(3) *Any PO allocation is a no-betting allocation.*

(4) *Every no-betting allocation is PO.*

Note that Theorem 4.1 also characterizes situations in which sunspots exist, i.e., when PO allocations depend on an extrinsic random variable. In contrast to MEU preferences, sunspots can exist with RDU preferences when probabilistic beliefs are common and given exogenously (equal to $\mathbb{P}$), and when the source of trading is solely differential risk attitudes. In fact, it may be Pareto optimal to bet when the probability weighting functions are not convex, as shown in Theorem 3.1.

As long as the probability weighting functions satisfy Condition (4.1) in Theorem 4.1, PO allocations are no-betting allocations (and conversely). Otherwise, betting is Pareto optimal. Condition (4.1) is equivalent to $T_1^{-1}(t) \geqslant \widetilde{T}_2^{-1}(t)$, for all $t \in [0,1]$. Letting $t = T_1(z)$, this rewrites as $T_1(z) \leqslant \widetilde{T}_2(z)$, or equivalently:
$$T_1(z) + T_2(1-z) \leqslant 1, \text{ for all } z \in [0,1]. \tag{4.2}$$
We interpret this as an elevation property of the probability weighting functions of the two agents jointly, since this writes as
$$T_1(z) - z + T_2(1-z) - (1-z) \leqslant 0, \text{ for all } z \in [0,1]. \tag{4.3}$$
For instance, if for a small $z \in (0,1)$, Agent 1 overweights good outcomes ($T_1(z) > z$) and Agent 2 underweights bad outcomes ($T_2(1-z) > 1-z$), then there is a desire to shift losses from Agent 1 to Agent 2, and thus random, i.e., non-constant PO allocations appear. Moreover, this also holds vice versa for large $z \in (0,1)$.

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Remark 4.2. Consider a pure-exchange economy with two Choquet-Expected Utility (CEU) maximizers, with respective capacities ν_1 and ν_2, and with constant initial endowments $W_1, W_2 \in \mathbb{R}$. Let $W = W_1 + W_2$ be the constant aggregate endowment. De Castro and Chateauneuf [12, Theorem 4.2] show that if

$$\nu_1(A) \leqslant 1 - \nu_2(A^c), \quad \forall A \in \Sigma, \tag{4.4}$$

where A^c denotes the complement of A, then there exists no Pareto improvement over the initial constant allocation (W_1, W_2), and hence betting cannot be Pareto-improving. For De Castro and Chateauneuf [12], Condition (4.4) is sufficient for betting not to be Pareto-improving over the initial constant allocation, but it was not shown to be necessary. Additionally, if the state space is finite, Dominiak et al. [14, Theorem 4.1] show that Condition (4.4) is both necessary and sufficient for there not to exist any Pareto improvement over the status quo (that is, the initial constant, hence no-betting allocation is Pareto-optimal).

Note if we take $\nu_i = T_i \circ \mathbb{P}$, for $i = 1, 2$, then (4.4) becomes

$$T_1(\mathbb{P}(A)) + T_2(1 - \mathbb{P}(A)) \leqslant 1, \ \forall A \in \Sigma,$$

which corresponds to our Condition (4.2) above. In our setting with RDU preferences, we do not assume that initial endowments are constant, but only that the aggregate endowment is constant. Furthermore, not only do we show that Condition (4.2) is both necessary and sufficient for the absence of a Pareto-improving trade over the initial allocation of endowments, but we also show that an allocation is Pareto-optimal if and only it is a no-betting allocation (that is, constant across states), and that this is also equivalent to Condition (4.2).

Finally, unlike CEU, RDU provides the ability to disantagle beliefs from any residual risk aversion that is not captured through marginal utility of wealth (i.e., probabilistic risk aversion). What Condition (4.2) illustrates is that the driver of betting in RDU is probabilistic risk aversion, since beliefs are objective and common. Proposition 4.8 below shows how risk aversion of the agents (any notion of risk aversion in RDU), or the weaker notion of pessimism, is sufficient for our Condition (4.2) to hold, and therefore for Pareto optima to be no-betting allocations.

When both agents use the same probability weighting function, the elevation property in Condition (4.3) is associated with *pessimism* or *source preference* (e.g., Abdellaoui et al. [1]), and it is a basic property imposed in Prospect Theory (Kahneman and Tversky [23]). For instance, Condition (4.3) holds when $T_1(t) = T_2(t) = t^\gamma/((t^\gamma + (1-t)^\gamma)^{1/\gamma})$ for $\gamma \in (0.279, 1]$, which is inverse-S shaped (Tversky and Kahneman [38]). Based on Example 3.6, the next example considers three parameterized economies.

Example 4.3. [No-Betting Allocations] (A) First, let T_1 and T_2 have the functional form of (3.9), that is, inverse S-shaped probability weighting functions as in Tversky and Kahneman [38]. Recall from Theorem 4.1 that the choice of utility functions is irrelevant for the existence of PO no-betting allocations. Then, for all $\gamma_1, \gamma_2 \in [0.3, 1]$, we display in Figure 4(A) the combinations of parameters that yield no risk sharing, i.e., the parameter combinations for which Condition (4.3) holds. We find that if γ_1 and γ_2 are identical or "close" ($\gamma_1 \approx \gamma_2$), then there exist PO no-betting allocations (see Theorem 4.1). On the other hand, if γ_1 and γ_2 are not "too close", hence displaying enough heterogeneity in probabilistic risk attitudes, then

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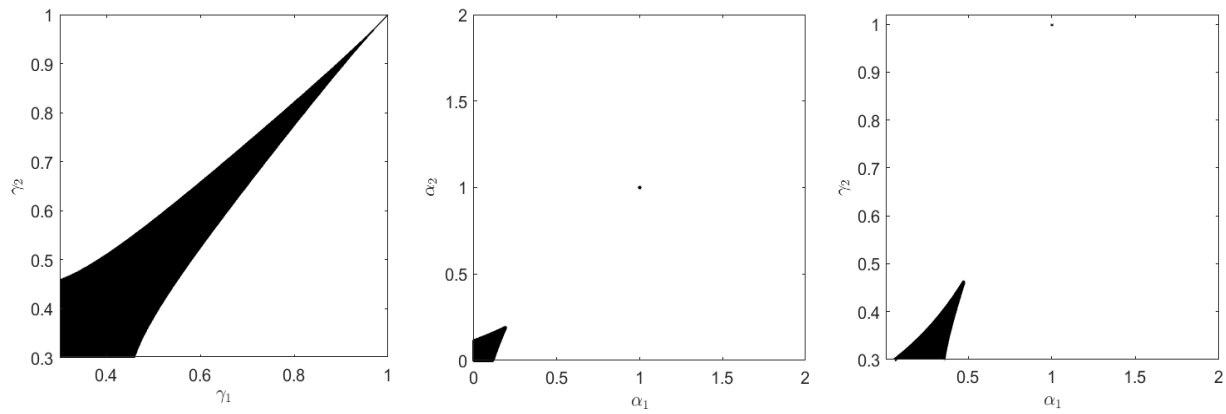
no-betting PO allocations do not exists, and therefore betting may be Pareto optimal. For most parameter combinations, we find that there exists a bet that is Pareto improving over a no-betting allocation.



(A) inverse S (B) Prelec-1 (C) mixed

FIGURE 4. Set of parameters under which there exist PO no-betting allocations (black area). (A) considers probability weighting functions as in (3.9). (B) considers probability weighting functions as in (4.5). (C) The first (resp. second) agent is endowed with a probability weighting function as in (4.5) (resp. (3.9)).

(B) Next, we consider the case where T_1 and T_2 belong to the class of Prelec-1 probability weighting functions, as introduced and characterized axiomatically by Prelec [28]. Specifically, T_i is given by

$$T_i(t) = \exp(-(-\ln(t))^{\alpha_i}), \ \forall t \in [0,1], \ \forall i \in \{1,2\}, \tag{4.5}$$

for some parameter $\alpha_i > 0$. Such distortion functions are inverse-S shaped when $\alpha_i \in (0,1)$, linear when $\alpha_i = 1$, and S-shaped when $\alpha_i > 1$. Then for all $\alpha_1, \alpha_2 \in (0,2)$, we display in Figure 4(B) the parameter combinations under which there is no risk sharing. We find that only when α_1 and α_2 are "close" and smaller than a value of approximately 0.19, or when $\alpha_1 = \alpha_2 = 1$ (implying no probability weighting), there exist PO no-betting allocations (cf. Theorem 4.1). Recall that the case $\alpha_1 = \alpha_2 = 1$ coincides with the well-known case where both agents are EU-maximizers (e.g., Billot et al. [4]).

(C) Finally, if the first agent uses a distortion function as in (4.5) and the second agent uses a distortion function as in (3.9), then the parameter combinations under which there is no risk sharing exhibit a similar pattern to when both agents use a distortion function as in (4.5). Specifically, there exist PO no-betting allocations only if α_1 and γ_2 are "close" and "small" (α_1 smaller than 0.47 and γ_2 between 0.3 and 0.46), or when $\alpha_1 = \gamma_2 = 1$ (no probability weighting).

The next result shows that Condition (3.8) of Corollary 3.5 is stronger than Condition (4.1) in Theorem 4.1, and thus sufficient for the Pareto optimality of no-betting allocations.

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Proposition 4.4. *Suppose that Assumption 2.1 holds. If there exists $i \in \{1,2\}$ such that for all $z \in (0,1)$,*

$$\frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} \leqslant \frac{T_j''(z)}{T_j'(z)}, \qquad i,j \in \{1,2\},\ i \neq j, \tag{4.6}$$

then $\Psi(t) \geqslant t$ for all $t \in [0,1]$.

As previously mentioned, Condition (4.6) holds, for instance, when T_1 and T_2 are convex. This implies that both agents are averse to mean-preserving spreads. Also, (4.6) holds when both T_1 and T_2 are linear, and thus when both agents are EU-maximizers. More generally, (4.6) can be seen as a requirement on the the degree of relative probabilistic risk aversion of one agent compared to the other.

4.2. **Risk Aversion and Pessimism.** Various notions of risk aversion exist for RDU preferences. We recall some below, and we refer to Chateauneuf et al. [8, 9], Chateauneuf and Cohen [7], Cohen [11], Ryan [34], and Ghossoub and He [17] for more. Following Quiggin [31], we define the following risk premia associated with a given RDU representation.

Definition 4.5. Consider a RDU representation U with utility function u and probability weighting function T. For $Z \in \mathcal{X}$, we define the following.

(i) The *certainty equivalent* of Z is given by $\mathrm{CE}(Z) := u^{-1}\big(U(Z)\big)$.

(ii) The *risk premium* of Z is given by $\Delta(Z) := \int Z dP - \mathrm{CE}(Z)$.

(iii) The *pessimism premium* of Z is given by $\Delta_T(Z) := \int Z dP - \int Z \mathrm{d}T \circ \mathbb{P}$.

(iv) The *outcome premium* of Z is given by $\Delta_{u,T}(Z) := \int Z \mathrm{d}T \circ \mathbb{P} - \mathrm{CE}(Z)$.

It follows from the above definitions that $\Delta(Z) = \Delta_T(Z) + \Delta_{u,T}(Z)$, for each $Z \in \mathcal{X}$. That is, in RDU, the risk premium is a sum of an outcome premium that depends on the marginal utility of wealth, and a pessimism premium that depends on the probability weighting function. Risk aversion is hence more than decreasing marginal utility of wealth, and risk attitudes are equally driven by the probability weighting function.

Again, following Quiggin [31], the above risk premia lead to the following notions of risk aversion. These notions of risk aversion are based only on risk premia, and not on spreads of distributions, as in strong risk aversion, monotone risk aversion, or location-independent risk aversion (see, e.g., Chateauneuf et al. [8, 9] or Ryan [34]).

Definition 4.6. For a RDU representation U with utility function u and probability weighting function T, we say that:

(i) U is *weakly risk averse* if $\Delta(Z) \geqslant 0$, for all $Z \in \mathcal{X}$.

(ii) U is *pessimistic* if $\Delta_T(Z) \geqslant 0$, for all $Z \in \mathcal{X}$.

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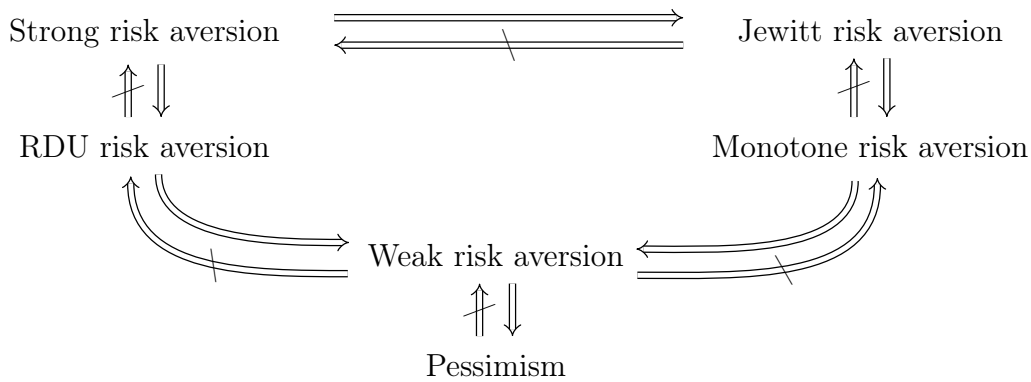

FIGURE 5. Relation between different notions of risk aversion under RDU.

(iii) U is *RDU risk averse* if $\Delta_T(Z) \geqslant 0$ and $\Delta_{u,T}(Z) \geqslant 0$, for all $Z \in \mathcal{X}$.

(iv) U is *Jewitt risk averse* if $U(X) \geqslant U(Y)$, whenever X and Y have the same mean but X is less location-independent risky than Y, i.e.,

$$\int_{-\infty}^{F_X^{-1}(q)} F_X(t)\mathrm{d}t \leqslant \int_{-\infty}^{F_Y^{-1}(q)} F_Y(t)\mathrm{d}t, \quad \forall\, q \in (0,1).$$

(v) U is *monotone risk averse* if $U(X) \geqslant U(Y)$, whenever X and Y have the same mean but X is less Bickel-Lehmann dispersed than Y, i.e., the function $q \mapsto F_X^{-1}(q) - F_Y^{-1}(q)$ is non-increasing on $(0,1)$.

The following proposition gathers some results from Quiggin [31], Chateauneuf and Cohen [7], Ryan [34], and Ghossoub and He [17], providing a characterization of the above notions of risk aversion.

Proposition 4.7. *Let U be a RDU with utility function u and probability weighting T.*

(1) *Strong risk aversion $\Rightarrow$ Jewitt risk aversion $\Rightarrow$ Monotone risk aversion $\Rightarrow$ Weak risk aversion.*
(2) *Strong risk aversion $\Rightarrow$ RDU risk aversion $\Rightarrow$ Weak risk aversion $\Rightarrow$ Pessimism.*
(3) *U is pessimistic if and only if $T(t) \leqslant t$, for all $t \in [0,1]$.*
(4) *If u is concave and U is pessimistic, then U is RDU risk averse.*
(5) *If u is concave, then U is weakly risk averse if and only if U is pessimistic.*

Hence, pessimism is the weakest notion of (probabilistic) risk aversion in RDU. The next result shows that pessimism is in fact sufficient for PO allocations to be no-betting allocations. In other words, if both agents display any notion of risk aversion, then PO allocations are no-betting allocations.

Proposition 4.8. *If both agents are pessimistic, then $\Psi(t) \geqslant t$ for all $t \in [0,1]$.*

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Proof. Proof. Suppose that both agents are pessimistic. Then, by Proposition 4.7-(iii), $T_i(t) \leqslant t$, for all $t \in [0,1]$ and $i \in \{1,2\}$. Then, since T_1 is increasing, we have $T_1^{-1}(t) \geqslant t$ and $\widetilde{T}_2(t) \geqslant t$, for all $t \in [0,1]$. Since in addition $\widetilde{T}_2$ is increasing, it then follows that

$$\Psi(t) = \widetilde{T}_2\left(T_1^{-1}(t)\right) \geqslant \widetilde{T}_2(t) \geqslant t, \ \forall t \in [0,1].$$

Hence, Condition (4.1) holds. □ □

An immediate implication of the above result is that if both agents are pessimistic, then PO allocations are no-betting allocations. Note also that since $T_i(0) = 1 - T_i(1) = 0$ for $i \in \{1,2\}$, if T_i is convex then it is also pessimistic. Hence, a special case of the above is when the two agents have convex distortion functions (e.g., strongly risk averse). In particular, Proposition 4.7-(i-ii) implies that a sufficient condition for Condition (4.1) to hold is that each one of the two agents is either strongly risk averse, RDU risk averse, Jewitt risk averse, monotone risk averse, weakly risk averse, or pessimistic.

5. Conclusion

This paper studies Pareto-optimal (PO) allocations in two-agent economy with no aggregate uncertainty and rank-dependent utilities. Our two main results allow for nonconvex preferences, as minimal assumptions on the distortion functions are imposed.

Our first main result (Theorem 3.1) provides a closed-form characterization of PO allocations in full generality. This allows us, in particular, to examine a sufficient condition for PO allocations to be no-betting allocations (Corollary 3.5). This condition only depends on the probability weighting functions of the agents, and not on their (concave) utility functions. It is satisfied for instance when the two agents are strongly risk-averse, that is, averse to mean-preserving spreads, and preferences hence display convexities.

Our second main result (Theorem 4.1) characterizes no-betting PO allocations via Condition (4.1) on the composed weighting function Ψ. This condition only depends on the probability weighting functions of the agents, and not on their (concave) utility functions. Hence with RDU preferences, it is the difference in probabilistic risk attitudes given common beliefs, rather than heterogeneity or ambiguity in beliefs, that drives betting. As by-product of our analysis, we answer the question of when sunspots matter in this economy. Finally, Condition (4.1) suggests that PO allocation are risk-free under nonconvex preferences. At the extreme, preferences displaying risk seeking lead to PO allocations that involve betting (Corollary 3.4).

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Appendix A. Proof of Theorem 3.1

Proposition A.1. *For any $\bar{u} \in \mathbb{R}$, the payoff $Y^* := m^{-1}\left(\lambda^* \delta'(T_1(\mathtt{U}))\right)$ is optimal for $(\mathcal{P}_{\bar{u}})$.*

The proof of the proposition is the content of the next three subsections (Step 1-3).

A.1. **Step 1.** First note that, since the space $(S, \Sigma, \mathbb{P})$ is non-atomic by assumption, there exists a random variable $\mathtt{U}$ on $(S, \Sigma, \mathbb{P})$ with a uniform distribution on $(0,1)$ (e.g., [15, Proposition A.31]). Recall Problem (3.3) on p. 7. We obtain the following result.

Lemma A.2. *If f^* solves Problem (3.3), then $Y^* := f^*(\mathtt{U})$ solves Problem $(\mathcal{P}_{\bar{u}})$.*

Proof. Let f^* be optimal for Problem (3.3) and $Y^* = f^*(\mathtt{U})$. Then, since $f^* \in \mathcal{Q}$, it follows that $F^{-1}_{Y^*} = f^*$. Moreover, using the variable $s = F_{u_2(Y^*)}(t)$ yields

$$
\begin{aligned}
U_2(Y^*) &= \int u_2(Y^*)\, \mathrm{d}T_2 \circ \mathbb{P} = \int_0^{+\infty} T_2\left(1 - F_{u_2(Y^*)}(t)\right)\, \mathrm{d}t + \int_{-\infty}^0 T_2\left(1 - F_{u_2(Y^*)}(t)\right) - 1\, \mathrm{d}t \\
&= \int_{F_{u_2(Y^*)}(0)}^1 T_2(1-s)\; dF^{-1}_{u_2(Y^*)}(s) + \int_0^{F_{u_2(Y^*)}(0)} \left[T_2(1-s) - 1\right]\; dF^{-1}_{u_2(Y^*)}(s) \\
&= \int_{F_{u_2(Y^*)}(0)}^1 \int_0^{1-s} T_2'(z)\, dz\; dF^{-1}_{u_2(Y^*)}(s) \\
&\qquad + \int_0^{F_{u_2(Y^*)}(0)} \int_0^{1-s} T_2'(z)\, dz\; dF^{-1}_{u_2(Y^*)}(s) - \int_0^{F_{u_2(Y^*)}(0)} dF^{-1}_{u_2(Y^*)}(s) \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z) \int_{F_{u_2(Y^*)}(0)}^{1-z} dF^{-1}_{u_2(Y^*)}(s)\; dz - \int_0^{F_{u_2(Y^*)}(0)} dF^{-1}_{u_2(Y^*)}(s) \\
&\qquad + \int_0^1 T_2'(z) \int_0^{\min\left[1-z, F_{u_2(Y^*)}(0)\right]} dF^{-1}_{u_2(Y^*)}(s)\; dz \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\; dz + F^{-1}_{u_2(Y^*)}(0) \\
&\qquad + \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z) \int_0^{F_{u_2(Y^*)}(0)} dF^{-1}_{u_2(Y^*)}(s)\; dz \\
&\qquad\qquad + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z) \int_0^{1-z} dF^{-1}_{u_2(Y^*)}(s)\; dz \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\; dz + F^{-1}_{u_2(Y^*)}(0) \\
&\qquad - F^{-1}_{u_2(Y^*)}(0) \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\; dz + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\; dz \\
&\qquad - F^{-1}_{u_2(Y^*)}(0) \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\; dz
\end{aligned}
$$

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$$
\begin{aligned}
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F_{u_2(Y^*)}^{-1}(1-z)\ dz + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\, F_{u_2(Y^*)}^{-1}(1-z)\ dz \\
&= \int_0^1 T_2'(z)\, F_{u_2(Y^*)}^{-1}(1-z)\ dz = \int_0^1 F_{u_2(Y^*)}^{-1}(t)\, T_2'(1-t)\ \mathrm{d}t \\
&= \int F_{u_2(Y^*)}^{-1}(\mathtt{U})\, T_2'(1-\mathtt{U})\, d\mathbb{P} = \int u_2\left(F_{Y^*}^{-1}\right)(\mathtt{U})\, T_2'(1-\mathtt{U})\, d\mathbb{P} \\
&= \int u_2\left(f^*(\mathtt{U})\right) T_2'(1-\mathtt{U})\, d\mathbb{P} = \int_0^1 u_2\left(f^*(t)\right) T_2'(1-t)\, \mathrm{d}t \geqslant \bar{u},
\end{aligned}
$$

where the equality $F_{u_2(Y^*)}^{-1} = u_2\left(F_{Y^*}^{-1}\right)$ follows from monotonicity of u_2, and the inequality follows from the feasibility of f^* for Problem (3.3). Hence, Y^* is feasible for Problem $(\mathcal{P}_{\bar{u}})$.

To show optimality of Y^* for Problem $(\mathcal{P}_{\bar{u}})$, let Y by any other feasible solution for Problem $(\mathcal{P}_{\bar{u}})$ and $f := F_Y^{-1}$ its quantile function. Then f is feasible for Problem (3.3) by feasibility of Y for Problem $(\mathcal{P}_{\bar{u}})$, since

$$
\bar{u} \leqslant U_2(Y) = \int u_2(Y)\, \mathrm{d}T_2 \circ \mathbb{P} = \int_0^1 u_2\left(f(t)\right) T_2'(1-t)\, \mathrm{d}t.
$$

Additionally, since $F_{-Y}(t) = -F_Y(1-t)$, for each $t \in \mathbb{R}$, a similar calculation as above yields, by monotonicity of u_1 that

$$
\begin{aligned}
U_1(W-Y) &= \int u_1(W-Y)\, \mathrm{d}T_1 \circ \mathbb{P} = \int F_{u_1(W-Y)}^{-1}(\mathtt{U})\, T_1'(1-U)\, \mathrm{d}\mathbb{P} \\
&= \int u_1\left(W + F_{-Y}^{-1}(\mathtt{U})\right) T_1'(1-U)\, \mathrm{d}\mathbb{P} = \int u_1\left(W - F_Y^{-1}(1-\mathtt{U})\right) T_1'(1-U)\, \mathrm{d}\mathbb{P} \\
&= \int u_1\left(W - f(\mathtt{U})\right) T_1'(\mathtt{U})\, \mathrm{d}\mathbb{P} = \int_0^1 u_1\left(W - f(t)\right) T_1'(t)\, \mathrm{d}t \\
&\leqslant \int_0^1 u_1\left(W - f^*(t)\right) T_1'(t)\, \mathrm{d}t = \int u_1(W - Y^*)\, \mathrm{d}T_1 \circ \mathbb{P} \\
&= U_1(W - Y^*),
\end{aligned}
$$

where the inequality follows from the optimality of f^* for Problem (3.3). Therefore, Y^* is optimal for Problem $(\mathcal{P}_{\bar{u}})$. □

For each $f \in \mathcal{Q}$, letting $v(t) = T_1^{-1}(t)$ and $z = v^{-1}(t) = T_1(t)$, for each $t \in [0,1]$ yields

$$
\int_0^1 u_2\left(f(t)\right) T_2'(1-t)\, \mathrm{d}t = \int_0^1 u_2\Big(f\left(v(z)\right)\Big) T_2'\left(1-v(z)\right) dv(z) = \int_0^1 u_2\left(q(t)\right) \Psi'(t)\, \mathrm{d}t,
$$

where $q = f \circ v$ and $\Psi(t) := 1 - T_2\left(1 - v(t)\right) = \widetilde{T}_2\left(v(t)\right)$, for each t, where $\widetilde{T}_2 : [0,1] \to [0,1]$ is the conjugate of the function T_2, given by $\widetilde{T}_2(t) = 1 - T_2(1-t)$, so that $\widetilde{T}_2'(t) := T_2'(1-t)$, for all $t \in [0,1]$.

A.2. **Step 2.** Recall Problem (3.4), after a change of variable, is based on Problem (3.3).

Lemma A.3. *If q^* solves Problem (3.4), then $f^* := q^* \circ T_1$ solves Problem (3.3).*

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Proof. Let q^* be optimal for Problem (3.4), and $f^* := q^* \circ T_1$. Then $f^* \in \mathcal{Q}$ and $q^* = f^* \circ v$, where $v = T_1^{-1}$. Letting $z = T_1(t)$ yields

$$\int_0^1 u_2\left(f_t^*\right) T_2'\left(1-t\right) \mathrm{d}t = \int_0^1 u_2\left(q^*_{T_1(t)}\right) T_2'\left(1-t\right) \mathrm{d}t = \int_0^1 u_2\left(q_z^*\right) \Psi'\left(z\right) \mathrm{d}z \geqslant \bar{u},$$

where the inequality follows from the feasibility of q^* for Problem (3.4). Hence f^* is feasible for Problem (3.3).

To show optimality of f^* for Problem (3.3), let f be feasible for Problem (3.3) and $q := f \circ v$. Then $q \in \mathcal{Q}$ and

$$\bar{u} \leqslant \int_0^1 u_2\left(f_t\right) T_2'\left(1-t\right) \mathrm{d}t = \int_0^1 u_2\left(q_t\right) \Psi_t' \mathrm{d}t.$$

Hence, q is feasible for Problem (3.4). Therefore, by optimality of q^* is optimal for Problem (3.4), it follows that

$$\begin{aligned}\int_0^1 u_1\left(W - f_t^*\right) T'(1-t)\mathrm{d}t &= \int_0^1 u_1\left(W - q^*_{T_1(t)}\right) \mathrm{d}T_1\left(t\right) = \int_0^1 u_1\left(W - q_z^*\right) dz \\ &\geqslant \int_0^1 u_1\left(W - q_z\right) dz = \int_0^1 u_1\left(W - f_t\right) T'(1-t)\mathrm{d}t.\end{aligned}$$

Hence, f^* is optimal for Problem (3.3). □

In light of Lemma A.3, we turn our attention to solving Problem (3.4). In order to do that, we will use a similar methodology to the one used by Xu [40], but adapted to the present setting. First, we recall the following result, due to He et al. [21, Appendix A].

Lemma A.4 (He et al. [21])**.** *Let f be a continuous real-valued function on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, and let g be its convex envelope on the interval $[0,1]$. Then the following holds:*

(1) *g is continuous and convex on $[0,1]$;*
(2) *$g\left(0\right) = f\left(0\right)$ and $g\left(1\right) = f\left(1\right)$;*
(3) *for all $x \in [0,1]$, $g\left(x\right) \leqslant f\left(x\right)$;*
(4) *g is affine on $\{x \in [0,1] : g\left(x\right) < f\left(x\right)\}$.*
(5) *if f is increasing, then so is g;*
(6) *if f is continuously differentiable on $(0,1)$, then so is g on $(0,1)$.*

Lemma A.5. *Let δ be the convex envelope of Ψ on $[0,1]$. Then for any $q \in \mathcal{Q}$,*

$$\int_0^1 u_2\left(q_t\right) \Psi_t' \mathrm{d}t \leqslant \int_0^1 u_2\left(q_t\right) \delta_t' \mathrm{d}t.$$

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Proof. Let δ be the convex envelope of the function Ψ on $[0,1]$. Since $\delta_t \leqslant \Psi_t$, for all $t \in [0,1]$, $\Psi_0 = \delta_0$, and $\Psi_1 = \delta_1$, it follows from Fubini's Theorem that

$$\begin{aligned}
0 &\geqslant \int_0^1 \left[(\Psi_1 - \delta_1) - \left(\Psi_y - \delta_y\right)\right] \mathrm{d}u_2\left(q_y\right) = \int_0^1 \int_y^1 \left[\Psi_x' - \delta_x'\right] \mathrm{d}x \; \mathrm{d}u_2\left(q_y\right) \\
&= \int_0^1 \int_0^x \left[\Psi_x' - \delta_x'\right] \mathrm{d}u_2\left(q_y\right) \; \mathrm{d}x = \int_0^1 \left[\int_0^x \mathrm{d}u_2\left(q_y\right)\right] \left[\Psi_x' - \delta_x'\right] \mathrm{d}x \\
&= \int_0^1 \left(u_2\left(q_t\right) - u_2\left(q_0\right)\right) \left[\Psi_t' - \delta_t'\right] \mathrm{d}t \\
&= \int_0^1 u_2\left(q_t\right) \left[\Psi'\left(t\right) - \delta_t'\right] \mathrm{d}t - u_2\left(q_0\right) \int_0^1 \left[\Psi_t' - \delta_t'\right] \mathrm{d}t \\
&= \int_0^1 u_2\left(q_t\right) \left[\Psi_t' - \delta_t'\right] \mathrm{d}t - u_2\left(q_0\right) \left(\Psi_1 - \delta_1 - \left(\Psi_0 - \delta_0\right)\right) \\
&= \int_0^1 u_2\left(q_t\right) \left[\Psi_t' - \delta_t'\right] \mathrm{d}t.
\end{aligned}$$

□

A.3. **Step 3.** Next, recall the relaxed Problem (3.5). We show that the solution is also optimal for Problem (3.4).

Lemma A.6. *If $q^* \in \mathcal{Q}$ satisfies $\int_0^1 \delta'\left(t\right) u_2\left(q_t^*\right) \mathrm{d}t = \bar{u}$, and if there is a $\lambda > 0$ such that for all $t \in (0,1)$, $q_t^* = \arg\max_y \left\{u_1\left(W - y\right) + \lambda u_2\left(y\right) \delta_t'\right\}$, then q^* is optimal for Problem (3.5).*

Proof. Let $q^* \in \mathcal{Q}$ be such that the two conditions above are satisfied. Then q^* is feasible for Problem (3.5). To show optimality, let $q \in \mathcal{Q}$ be any feasible solution for Problem (3.5). Then, by definition of q^*, it follows that for each $t \in (0,1)$,

$$u_1\left(W - q_t^*\right) - u_1\left(W - q_t\right) \geqslant \lambda \left[\delta_t' u_2\left(q_t\right) - \delta_t' u_2\left(q_t^*\right)\right].$$

Hence, $\int_0^1 u_1\left(W - q_t^*\right) \mathrm{d}t - \int_0^1 u_1\left(W - q_t\right) \mathrm{d}t \geqslant \lambda[\int_0^1 \delta_t' u_2\left(q_t\right) \mathrm{d}t - \bar{u}] \geqslant 0$. Consequently, it follows that

$$\int_0^1 u_1\left(W - q_t^*\right) \mathrm{d}t \geqslant \int_0^1 u_1\left(W - q_t\right) \mathrm{d}t,$$

and so q^* is optimal for Problem (3.5). □

Moreover, we obtain the following result. Recall that $m\left(x\right) := \frac{u_1'(W-x)}{u_2'(x)}$.

Lemma A.7. *For each $\lambda > 0$, define the function q^λ by*

$$q_t^\lambda := m^{-1}\left(\lambda \delta_t'\right), \text{ for all } t \in (0,1). \tag{A.1}$$

Then,

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(1) *for each* $\lambda > 0$, $q^\lambda \in \mathcal{Q}$;

(2) *there exists a* $\lambda^* > 0$ *such that* $\int_0^1 \delta_t' u_2\left(q_t^{\lambda^*}\right) \mathrm{d}t = \bar{u}$;

(3) *for all* $t \in (0,1)$ *and all* $\lambda > 0$, $q_t^\lambda = \arg\max_y \left\{u_1(W-y) + \lambda u_2(y)\,\delta_t'\right\}$.

Proof. Assumption 2.1 implies that u_1 and u_2 are decreasing and continuous. Therefore, the function m is continuously differentiable and

$$m'(x) = \frac{-u_1''(W-x)\,u_2'(x) - u_1'(W-x)\,u_2''(x)}{\left(u_2'(x)\right)^2} > 0.$$

Hence, the function m is continuous and increasing. Consequently, it is invertible, and its inverse is also increasing, by the Inverse Function Theorem. Thus, for each $\lambda > 0$, the convexity and continuity of δ imply that the function $q^\lambda = m^{-1}(\lambda\delta')$ is continuous and increasing. Therefore, for each $\lambda > 0$, $q_\lambda^* \in \mathcal{Q}$.

Now, for each $t \in (0,1)$ and $\lambda > 0$, the concavity of u_1 and u_2 yield the concavity of the function $y \mapsto \mathcal{M}_{\lambda,t}(y) := u_1(W-y) + \lambda u_2(y)\,\delta_t'$, since $\delta_t' \geqslant 0$. Therefore, first-order conditions yield a global maximum of the function $\mathcal{M}_{\lambda,t}$ at $y^* = m^{-1}(\lambda\delta_t')$. Finally, the existence of $\lambda^* > 0$ such that $\int_0^1 u_2(q_t^{\lambda^*})\delta_t' \mathrm{d}t = \bar{u}$ follows from the monotonicity and continuity properties of δ' (see Lemma A.4), and from the Intermediate Value Theorem. □

Proof of Proposition A.1: Lemmata A.5, A.6, and A.7 imply that for any $\lambda > 0$ and $q \in \mathcal{Q}$,

$$\int_0^1 u_1(W-q_t) + \lambda u_2(q_t)\,\Psi_t' \mathrm{d}t \leqslant \int_0^1 u_1(W-q_t) + \lambda u_2(q_t)\,\delta_t' \mathrm{d}t \leqslant \int_0^1 u_1(W-q_t^\lambda) + \lambda u_2(q_t^\lambda)\delta_t' \mathrm{d}t,$$

where q^λ is as in eq. (A.1). Now, for all $\lambda > 0$, we have $q^\lambda \in \mathcal{Q}$ by Lemma A.7, and

$$\mathrm{d}q_t^\lambda = \lambda\left(m^{-1}\right)'\left(\lambda\delta_t'\right)\mathrm{d}\delta_t'. \tag{A.2}$$

Letting $\mathcal{D} := \left\{t \in [0,1] : \delta_t \neq \Psi_t\right\} = \left\{t \in [0,1] : \delta_t < \Psi_t\right\}$, it follows that for any $\lambda > 0$,

$$\int_0^1 [\Psi_t - \delta_t]\,\mathrm{d}u_2(q_t^\lambda) = \int_{\mathcal{D}} [\Psi_t - \delta_t]\,\mathrm{d}u_2(q_t^\lambda).$$

But, since δ is affine on $\mathcal{D}$, $\mathrm{d}\delta' = 0$ on $\mathcal{D}$, and it follows from eq. (A.2) that $\mathrm{d}q_t^\lambda = 0$ on $\mathcal{D}$, for all $\lambda > 0$. Consequently, $\int_0^1 [\Psi_t - \delta_t]\,\mathrm{d}u_2\left(q_\lambda^*(t)\right) = 0$. Therefore, applying Fubini's Theorem yields

$$0 = \int_0^1 [\Psi_t - \delta_t]\,\mathrm{d}u_2(q_t^\lambda) = \int_0^1 u_2\left(q^\lambda(x)\right)\left[\Psi'(x) - \delta'(x)\right]\mathrm{d}x.$$

Consequently, $\int_0^1 u_2\left(q_t^\lambda\right)\Psi_t' \mathrm{d}t = \int_0^1 u_2\left(q_t^\lambda\right)\delta_t' \mathrm{d}t$. Therefore, for all $\lambda > 0$, $q \in \mathcal{Q}$,

$$\begin{aligned}\int_0^1 \left[u_1\left(W-q_t\right) + \lambda u_2(q_t)\,\Psi_t'\right]\mathrm{d}t &\leqslant \int_0^1 \left[u_1\left(W-q_t^\lambda\right) + \lambda u_2\left(q_t^\lambda\right)\delta_t'\right]\mathrm{d}t \\ &= \int_0^1 \left[u_1\left(W-q_t^\lambda\right) + \lambda u_2\left(q_t^\lambda\right)\Psi_t'\right]\mathrm{d}t.\end{aligned}$$

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Hence, if λ^* is chosen such that $\int_0^1 u_2(q_t^{\lambda^*})\Psi_t' \mathrm{d}t = \bar{u}$, then the optimal solution to Problem (3.5) is given by q^{λ^*}. Thus, by Lemmata A.2, A.3, A.6, and A.7, the function

$$Y^* = q_{\lambda^*}^* \left(T_1\left(\mathtt{U}\right)\right)$$

is optimal for Problem $(\mathcal{P}_{\bar{u}})$, where the function q_λ^* is given by eq. (A.1). □

A.4. **Step 4.**

Lemma A.8. *For a given $\bar{u} \in \mathbb{R}$, let Y_1 be optimal for Problem $(\mathcal{P}_{\bar{u}})$ and Y_2 be feasible for Problem $(\mathcal{P}_{\bar{u}})$. Then Y_2 is also optimal for Problem $(\mathcal{P}_{\bar{u}})$ if and only if for a.e. $t \in \mathbb{R}$,*

$$\mathbb{P}\left(\{s \in S : Y_1(s) > t\}\right) = \mathbb{P}\left(\{s \in S : Y_2(s) > t\}\right).$$

Proof. We start with the "if" statement. First suppose that for a.e. $t \in \mathbb{R}$,

$$\mathbb{P}\left(\{s \in S : Y_1(s) > t\}\right) = \mathbb{P}\left(\{s \in S : Y_2(s) > t\}\right).$$

Then by definition of the Choquet integral, it follows that

$$U_1\left(W - Y_1\right) = \int u_1\left(W - Y_1\right)\ \mathrm{d}T_1 \circ \mathbb{P} = \int u_1\left(W - Y_2\right)\ \mathrm{d}T_1 \circ \mathbb{P} = U_1\left(W - Y_2\right);$$

$$U_2\left(Y_1\right) = \int u_2\left(Y_1\right)\ \mathrm{d}T_2 \circ \mathbb{P} = \int u_2\left(Y_2\right)\ \mathrm{d}T_2 \circ \mathbb{P} = U_2\left(Y_2\right).$$

Therefore, Y_2 is also optimal for Problem $(\mathcal{P}_{\bar{u}})$. □

We proceed with the "only if" statement. Problem $(\mathcal{P}_{\bar{u}})$ has a solution due to Proposition A.1. Suppose that there are two solutions Y_1, Y_2 to Problem $(\mathcal{P}_{\bar{u}})$ such that it does not hold that for a.e. $t \in \mathbb{R}$, $\mathbb{P}\left(\{s \in S : Y_1(s) > t\}\right) = \mathbb{P}\left(\{s \in S : Y_2(s) > t\}\right)$. Since the probability space (S, Σ, P) is non-atomic, there exists a $Y_2^c \in \mathcal{X}$ such that Y_2^c, Y_2 are identically distributed and Y_2^c is comonotonic with Y_1. Thus,

$$\begin{aligned} U_1\left(W - Y_1\right) &= \int u_1\left(W - Y_1\right)\ \mathrm{d}T_1 \circ \mathbb{P} = \int u_1\left(W - Y_2\right)\ \mathrm{d}T_1 \circ \mathbb{P} \\ &= \int u_1\left(W - Y_2^c\right)\ \mathrm{d}T_1 \circ \mathbb{P} = U_1\left(W - Y_2^c\right). \end{aligned} \tag{A.3}$$

Let $\widetilde{\Sigma}$ be the σ-algebra on S generated by the random variable $\frac{1}{2}(Y_1 + Y_2^c)$. Define the probability measure $\mathbb{Q}^1$ on $(S, \widetilde{\Sigma})$ by $\mathbb{Q}^1(-\frac{1}{2}(Y_1+Y_2^c) > t) := T_1(\mathbb{P}(-\frac{1}{2}(Y_1+Y_2^c) > t))$, for $t \in \mathbb{R}$. Since the σ-algebra is generated by $\frac{1}{2}(Y_1+Y_2^c)$ and since T_1 is increasing and continuous, $\mathbb{Q}^1$ is indeed a probability measure on $(S, \widetilde{\Sigma})$. The three random variables $-Y_1, -Y_2^c, -\frac{1}{2}(Y_1+Y_2^c)$ are all $\widetilde{\Sigma}$-measurable and comonotonic with $-\frac{1}{2}(Y_1 + Y_2^c)$. Therefore, for $X \in \{-Y_1, -Y_2^c, -\frac{1}{2}(Y_1 + Y_2^c)\}$, it holds that

$$\begin{aligned} \int u_1\left(W + X\right)\ \mathrm{d}T_1 \circ \mathbb{P} &= \int_{-\infty}^0 \left(T_1 \circ \mathbb{P}(u_1(W + X) > z) - 1\right) \mathrm{d}z + \int_0^\infty T_1 \circ \mathbb{P}(u_1(W + X) > z)\mathrm{d}z \\ &= \int_{-\infty}^0 \left(\mathbb{Q}^1(u_1(W + X) > z) - 1\right) \mathrm{d}z + \int_0^\infty \mathbb{Q}^1(u_1(W + X) > z)\mathrm{d}z = \mathbb{E}^{\mathbb{Q}^1}[u_1(W + X)], \end{aligned}$$

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where $\mathbb{E}^{\mathbb{Q}^1}$ is the expectation under the probability measure $\mathbb{Q}^1$. Note that this relies on the useful fact that $-\frac{1}{2}(Y_1 + Y_2^c)$ and X are comonotonic. Then, since it does not hold that Y_1 and Y_2^c are equal in distribution, it follows that $\mathbb{P}(-Y_1 \neq -Y_2^c) > 0$. Define $\mathcal{A} \in \widetilde{\Sigma}$ such that $-Y_1(s) \neq -Y_2^c(s)$ for all $s \in \mathcal{A}$ and $-Y_1(s) = -Y_2^c(s)$ for all $s \in S \backslash \mathcal{A}$. Since the function T_1 is strictly increasing, it holds for $t \in \mathbb{R}$ that $\mathbb{Q}^1(-\frac{1}{2}(Y_1 + Y_2^c) > t)$ strictly decreases if and only if $\mathbb{P}(-\frac{1}{2}(Y_1 + Y_2^c) > t)$ strictly decreases, and thus $\mathbb{Q}^1$ is an equivalent probability measure to $\mathbb{P}$ on $\left(S, \widetilde{\Sigma}\right)$. It hence follows that $\mathbb{Q}^1(\mathcal{A}) := \int_{\mathcal{A}} d\mathbb{Q}^1(s) > 0$. By eq. (A.3), it follows that

$$\int_{\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 = \int_{\mathcal{A}} u_1\left(W - Y_2^c\right) d\mathbb{Q}^1.$$

Recall that $\int u_1\left(W - Y_1\right)\ \mathrm{d}T_1 \circ \mathbb{P} = \mathbb{E}^{\mathbb{Q}^1}[u_1\left(W - Y_1\right)]$. Moreover,

$$\begin{aligned}
&\int u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right)\ \mathrm{d}T_1 \circ \mathbb{P} = \mathbb{E}^{\mathbb{Q}^1}\left[u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right)\right] \\
&= \int u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right) d\mathbb{Q}^1 \\
&= \int_{\mathcal{A}} u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right) d\mathbb{Q}^1 + \int_{S\backslash\mathcal{A}} u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right) d\mathbb{Q}^1 \\
&= \int_{\mathcal{A}} u_1\left(W - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right) d\mathbb{Q}^1 + \int_{S\backslash\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 \\
&> \int_{\mathcal{A}} \left(\frac{1}{2}u_1\left(W - Y_1\right) + \frac{1}{2}u_1\left(W - Y_2^c\right)\right) d\mathbb{Q}^1 + \int_{S\backslash\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 \\
&= \frac{1}{2}\int_{\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 + \frac{1}{2}\int_{\mathcal{A}} u_1\left(W - Y_2^c\right) d\mathbb{Q}^1 + \int_{S\backslash\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 \\
&= \int_{\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 + \int_{S\backslash\mathcal{A}} u_1\left(W - Y_1\right) d\mathbb{Q}^1 \\
&= \int u_1\left(W - Y_1\right) d\mathbb{Q}^1 = \mathbb{E}^{\mathbb{Q}^1}[u_1\left(W - Y_1\right)] = \int u_1\left(W - Y_1\right)\ \mathrm{d}T_1 \circ \mathbb{P},
\end{aligned}$$

where the strict inequality follows from $u_1(\frac{1}{2}x + \frac{1}{2}y) > \frac{1}{2}u_1(x) + \frac{1}{2}u_1(y)$ whenever $x \neq y$, which follows from strict concavity of u_1. Via similar concavity arguments, we also find

$$\int u_2\left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\ \mathrm{d}T_2 \circ \mathbb{P} \geqslant \int u_2\left(Y_1\right) \mathrm{d}T_2 \circ \mathbb{P} \geqslant \bar{u}.$$

Thus, $\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c$ strictly improves Y_1, and so Y_1 is not a solution to $(\mathcal{P}_{\bar{u}})$, a contradiction. □

A.5. **Proof of Theorem 3.1.** Let $Y^* = m^{-1}\left(\lambda^* \delta'(T_1\left(\mathtt{U}\right))\right)$, where $\lambda^* > 0$, and where $\mathtt{U}$, m, and δ are as defined in Proposition A.1. Then by Proposition A.1, Y^* is optimal for Problem $(\mathcal{P}_{\bar{u}})$, with $\bar{u} := U_2\left(Y^*\right) = \int u_2\left(Y^*\right) \mathrm{d}T_2 \circ \mathbb{P}$. Consequently, $(W - Y^*, Y^*)$ is Pareto optimal.

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Conversely, suppose that $(W - Y^*, Y^*)$ is Pareto optimal. Then, Y^* solves Problem $(\mathcal{P}_{\bar{u}})$ with $\bar{u} := U_2(Y^*) = \int u_2(Y^*)\,\mathrm{d}T_2 \circ \mathbb{P}$. By Proposition A.1, a solution is given by

$$\hat{Y} = m^{-1}\left(\lambda^* \delta'\Big(T_1(\mathtt{U})\Big)\right),$$

where $\lambda^* > 0$ is chosen such that $U_2(\hat{Y}) = \bar{u}$, and where $\mathtt{U}$, m, and δ are as defined in Proposition A.1. By Lemma A.8, all solutions to Problem $(\mathcal{P}_{\bar{u}})$ have the same distribution as $\hat{Y}$. Hence, for every PO allocation $(W - Y^*, Y^*)$, there exists a $\lambda^* > 0$ such that Y^* has the same distribution as $m^{-1}(\lambda^* \delta'(T_1(\mathtt{U})))$. □

A.6. **A Remark.** We could have characterized the set of PO allocations using a different parametrization. Indeed, consider the following auxiliary problem: For $\Gamma_0 \in \mathbb{R}$,

$$(\overline{\mathcal{P}}_{\Gamma_0}) \qquad \sup_{Y \in \mathcal{X}} \Big\{ U_2(Y) : U_1(W - Y) \geqslant \Gamma_0 \Big\}.$$

The next result is standard.

Lemma A.9.

(i) *If the allocation (X_1^*, X_2^*) is PO, then X_2^* solves Problem $(\overline{\mathcal{P}}_{\Gamma_0})$ with $\Gamma_0 := U_1(X_1^*)$.*

(ii) *For a given $\Gamma_0 \in \mathbb{R}$, any solution Y^* to Problem $(\overline{\mathcal{P}}_{\Gamma_0})$ leads to an allocation $(W - Y^*, Y^*)$ that is PO.*

(iii) *If $Y^* \in \mathcal{X}$ solves Problem $(\overline{\mathcal{P}}_{\Gamma_0})$ for a given $\Gamma_0 \in \mathbb{R}$, then $U_1(W - Y^*) = \Gamma_0$.*

By a proof similar to that of Proposition A.1, we also obtain the following result.

Theorem A.10. *For a given $\Gamma_0 \in \mathbb{R}$, the random variable $\overline{Y}^* := \overline{m}^{\,-1}\left(\kappa^* \overline{\delta}'(\widetilde{T}_2(\mathtt{U}))\right)$ is optimal for Problem $(\overline{\mathcal{P}}_{\Gamma_0})$, where:*

- *$\mathtt{U}$ is a random variable on $(S, \Sigma, \mathbb{P})$ with a uniform distribution on $(0,1)$;*
- *$\overline{m}(x) := \frac{u_2'(x)}{u_1'(W-x)}$, for all $x \geqslant 0$;*
- *$\overline{\delta}$ is the concave envelope on $[0,1]$ of $\overline{\Psi}(t) := T_1(\widetilde{T}_2^{-1}(t))$;*
- *$\widetilde{T}_2(t) = 1 - T_2(1-t)$; and,*
- *$\kappa^* > 0$ is chosen such that $U_1\left(W - \bar{Y}^*\right) = \Gamma_0$.*

Likewise, by a proof similar to that of Lemma A.8, we also obtain the following result.

Lemma A.11. *For a given $\Gamma_0 \in \mathbb{R}$, let $\overline{Y}_1$ be optimal for Problem $(\overline{\mathcal{P}}_{\Gamma_0})$ and $\overline{Y}_2$ be feasible for Problem $(\overline{\mathcal{P}}_{\Gamma_0})$. Then $\overline{Y}_2$ is also optimal for Problem $(\overline{\mathcal{P}}_{\Gamma_0})$ if and only if for a.e. $t \in \mathbb{R}$,*

$$\mathbb{P}\left(\{s \in S : \overline{Y}_1(s) > t\}\right) = \mathbb{P}\left(\{s \in S : \overline{Y}_2(s) > t\}\right).$$

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Consequently, Lemma A.9, Theorem A.10, and Lemma A.11 provide an alternative way of characterizing the set of Pareto optima.

Appendix B. Other Proofs

Proof of Lemma 2.2: Integration in the sense of Choquet gives

$$\begin{aligned}\int u_i\left(Z\right)\,\mathrm{d}T_i\circ\mathbb{P} &:= \int_0^{+\infty} T_i\left(\mathbb{P}\left(\left\{s\in S: u_i\left(Z\left(s\right)\right)>t\right\}\right)\right)\,\mathrm{d}t \\ &\quad + \int_{-\infty}^{0}\left[T_i\left(\mathbb{P}\left(\left\{s\in S: u_i\left(Z\left(s\right)\right)>t\right\}\right)\right)-1\right]\,\mathrm{d}t \qquad \text{(B.1)}\\ &= -\int_{\mathbb{R}} u_i\left(t\right)\,dT_i\left(1-F_Z\left(t\right)\right) = \int_{\mathbb{R}} u_i\left(t\right)\,d\widetilde{T}_i\left(F_Z\left(t\right)\right).\end{aligned}$$

Since, T_i is differentiable, we obtain

$$\int u_i\left(Z\right)\,\mathrm{d}T_i\circ\mathbb{P} = \int_0^1 u_i\left(F_Z^{-1}\left(z\right)\right)T_i'\left(1-z\right)\,dz = \int_0^1 u_i\left(F_Z^{-1}\left(z\right)\right)\widetilde{T}_i'\left(z\right)\,dz.$$

□

Proof of Lemma 3.2: Define the state dependent utility $\widetilde{u}_i(x,t) := u_i\left(x\right)\,T_i'\left(t\right)$. Since $\widetilde{u}_i(\cdot,t)$ is concave for each t, as by assumption $T_i'\geqslant 0$, the claim follows by standard arguments for state dependent expected utility. Let $h\in F(f)$. By dominated convergence and the assumed differentiability of u_i, it follows that

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\widetilde{U}_i(f+\epsilon h)-\widetilde{U}_i(f)}{\epsilon} &= \lim_{\epsilon\to 0}\int_0^1\frac{u_i(f(s)+\epsilon h(s))-u_i(f(s))}{\epsilon}T_i'(s)\mathrm{d}s\\ &= \int_0^1 h(s)u_i'(f(s))T_i'(s)\mathrm{d}s = \langle\nabla\widetilde{U}_i(f),h\rangle.\end{aligned}$$

Note that dominated convergence can be applied as the concavity of u_i implies a non-increasing difference quotient and T_i maps into $[0,1]$. □

Proof of Corollary 3.3: Since $X=BV^l$ is a Banach space endowed with the norm of total variation, and since each $\widetilde{U}_i$ is concave by Lemma 3.2, we apply a standard Karush-Kuhn Tucker Theorem (e.g., Barbu and Prescapanu [3]) to obtain

$$\nabla\widetilde{U}_1(W-f^*) = \lambda^*\nabla\widetilde{U}_2(f^*).$$

By Lemma 3.2, $\nabla\widetilde{U}_i$ is explicit, which gives the equivalence to (3.6). □

Proof of Corollary 3.4: By the proof of Proposition 4.4 we have for each $x\in(0,1)$, $\frac{\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)} > \frac{T_i''(x)}{T_i'(x)}$. Thus Ψ is convex, and hence $\delta\equiv\Psi$. The rest follows from Theorem 3.1. □

Proof of Theorem 4.1:

(1) $\Longleftrightarrow$ (2), i.e., $\Psi_t\geqslant t$ for all $t\in[0,1]$ if and only if there exists a PO no-betting allocation.

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By virtue of Theorem 3.1, there exists a PO no-betting allocation if and only if $\delta_t = t$ for all $t \in [0,1]$. Thus, it is sufficient to show $\delta_t = t$ for all $t \in [0,1]$ if and only $\Psi_t \geqslant t$ for all $t \in [0,1]$. If $\delta_t = t$ for all $t \in [0,1]$, then it follows directly from Lemma A.4(3) that $\Psi_t \geqslant t$. Let $\Psi_t \geqslant t$. By Lemma A.4(2) it holds that $\delta_0 = \Psi_0 = 0$ and $\delta_1 = \Psi_1 = 1$. Since δ is convex, it holds that $\delta(t) \leqslant t$. The largest convex function is thus $\delta_t = t$, which satisfies $\delta_t \leqslant \Psi_t$. Thus, $\delta_t = t$ is the convex envelope.

(2) $\Longleftrightarrow$ (3). The fact that (3) implies (2) is immediate. We show that (2) implies (3), i.e., that the existence of a PO no-betting allocation implies that any PO allocation is a no-betting allocation. If there exists a PO no-betting allocation, this implies by Theorem 3.1 that there exists some $\lambda^* > 0$ such that $m^{-1}(\lambda^* \delta'(T_1(\mathtt{U})))$ has the same distribution as a deterministic random variable. The function m is strictly increasing, and thus m^{-1} is strictly increasing. Moreover, T_1 is assumed to be strictly increasing. Thus, it holds that there exists a $c > 0$ such that $\delta'_t = c$ for all $t \in [0,1]$ almost everywhere. Since $\delta_0 = 0$ and $\delta_1 = 1$ by Lemma A.4, this implies $\delta_t = t$. Hence, for any $\lambda > 0$, $m^{-1}(\lambda \delta'(T_1(\mathtt{U})))$ is deterministic. Consequently, by Theorem 3.1, any PO allocation is a no-betting allocation.

(2) $\Longleftrightarrow$ (4). The fact that (4) implies (2) is immediate. We show that (2) implies (4), i.e., that the existence of a PO no-betting allocation implies that every no-betting allocation is Pareto optimal. As argued above, if there exists a PO no-betting allocation, then Theorem 3.1 implies $\delta_t = t$, for each $t \in [0,1]$. Consequently, $\delta' \equiv 1$. Now, suppose that $Y = c \in \mathbb{R}$ induces a no-betting allocation. Letting $\lambda^* := m(c)$, where m is defined in Theorem 3.1, it follows that

$$Y = c = m^{-1}(\lambda^*) = m^{-1}\left(\lambda^* \delta'\big(T_1(\mathtt{U})\big)\right),$$

since $\delta' \equiv 1$. Consequently, it follows again from Theorem 3.1 that Y is Pareto optimal. □

Proof of Proposition 4.4: Let the functions Ψ and δ be as in Proposition A.1, and let the functions $\overline{\Psi}$ and $\overline{\delta}$ be as in Theorem A.10. For $i \in \{1,2\}$, let $\widetilde{T}_i$ be the conjugate of T_i, defined by $\widetilde{T}_i(z) := 1 - T_i(1-z)$, for each $z \in [0,1]$. Then letting $x = 1 - z$, it follows that for each $x \in [0,1]$ and for $i \in \{1,2\}$, $T_i(x) = 1 - \widetilde{T}_i(1-x)$. Therefore, for $i \in \{1,2\}$, and for each $x, z \in (0,1)$,

$$\widetilde{T}'_i(z) = T'_i(1-z) \quad \text{and} \quad T'_i(x) = \widetilde{T}'_i(1-x);$$
$$\widetilde{T}''_i(z) = -T''_i(1-z) \quad \text{and} \quad T''_i(x) = -\widetilde{T}''_i(1-x).$$

Suppose that there exits $i \in \{1,2\}$ such that for all $z \in (0,1)$,

$$\frac{\widetilde{T}''_i(z)}{\widetilde{T}'_i(z)} \leqslant \frac{T''_j(z)}{T'_j(z)},$$

where $j = 3 - i$. Then, for all $z = 1 - x \in (0,1)$,

$$\frac{-T''_i(x)}{T'_i(x)} = \frac{-T''_i(1-z)}{T'_i(1-z)} = \frac{\widetilde{T}''_i(z)}{\widetilde{T}'_i(z)} \leqslant \frac{T''_j(z)}{T'_j(z)} = \frac{-\widetilde{T}''_j(1-z)}{\widetilde{T}'_j(1-z)} = \frac{-\widetilde{T}''_j(x)}{\widetilde{T}'_j(x)},$$

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and so, for each $x \in (0,1)$,

$$\frac{\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)} \leqslant \frac{T_i''(x)}{T_i'(x)}.$$

Hence, without loss of generality, we can assume that $i = 1$ and $j = 2$, so that for each $x \in (0,1)$, $\frac{\widetilde{T}_2''(x)}{\widetilde{T}_2'(x)} \leqslant \frac{T_1''(x)}{T_1'(x)}$. This then implies that the function Ψ is concave and the function $\overline{\Psi}$ is convex. In turn, this implies that the functions δ and $\overline{\delta}$ are both linear, and that $\delta' \equiv \overline{\delta}' \equiv 1$. Consequently, the random variables Y^* and $\overline{Y}^*$ given in Proposition A.1 and Theorem A.10, respectively, are constants. That is, there exists a PO no-betting allocation. □

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