# Multi-period peer-to-peer risk sharing

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**Abstract**

Risk sharing has been practiced in various forms in financial industry. This paper is the first to study both dynamic and static risk-sharing mechanism for a group of participants over multiple periods. The design of risk-sharing strategies is based on the Pareto optimization of quadratic utilities of participants' reserves. Such a framework builds a connection between portfolio optimization in the finance literature and that for risk sharing in the insurance literature. Building on the most common form of reinsurance – pro-rata treaties, we propose a peer-to-peer (P2P) network for risk sharing. Assuming independent multivariate losses over time, we find that the optimal risk-sharing allocation exhibits a three-component structure with the long-term limit and two correction terms. This allows us to show convergence of the risk-sharing solution and the ratios of long-term reserves. Furthermore, we study the impact of actuarial fairness on various risk-sharing strategies and their long-term limits.

**Keywords:** risk management, peer-to-peer network, risk sharing, multi-period optimization, mean-variance objective, financial fairness.

## 1 Introduction

Risk sharing is an arrangement within a group of economic agents where all agents are committed to sharing the cost of compensating for losses incurred in the group. Sharing risk can be beneficial for all agents, as it allows for diversification of risks through pooling. By taking on risks from others, each agent passes on some of its own risks to the group and is expected to be better off by reducing the variability of the overall recovery costs. Mathematical and economic analysis of risk sharing dates back to the work of Borch (1962). In classical risk sharing, the set of feasible risk allocations is the class of $n$-dimensional multivariate random variables that add up to its aggregate risk. Borch (1962) uses as an objective to maximize a weighted sum of expected utilities. This has been extended to the use of mean-variance objectives (Acciaio, 2007; Simsek, 2013), expected utility (Wilson, 1968), monetary risk measures (Jouini et al., 2008), dual utilities (Boonen, 2015), and rank-dependent utilities (Jin et al., 2019; Boonen et al., 2021). Moreover, various notions of ambiguity aversion have been

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introduced to the problem to find optimal risk-sharing contracts (De Castro and Chateauneuf, 2011; Strzalecki and Werner, 2011; Rigotti and Shannon, 2012).

It has been shown in the classic risk-sharing literature that optimal risk-sharing schemes are often aggregate risk sharing, also known as "non-olet", a term coined by Borch (1968) after a Roman anecdote. The idea behind this is that all risks should be aggregated before being allocated to individual participants. The risk-sharing allocation depends only on the aggregate loss and not on the source of the loss. However, aggregate contracts rely on a central entity to facilitate payments. The concept of peer-to-peer (P2P) has become popular in recent years with the rise of P2P insurance originated in Germany and spreads out to many parts of the world. Applications of risk-sharing schemes for P2P insurance have been studied by Denuit and Dhaene (2012), Denuit (2020), Denuit and Robert (2021), Chen et al. (2022), and Levantesi and Piscopo (2022), Feng (2023). The nature of P2P cash transfers allows for the possibility of tracking contributors and recipients. Each participant may choose to take a different portion of every other participant's risk, instead of a portion of the aggregate risk of all. The differences between P2P risk sharing and aggregate risk sharing are studied in Abdikerimova and Feng (2021), Denuit et al. (2022), and Feng et al. (2022). More importantly, the P2P risk sharing enables the discussion of networks of P2P risk sharing where not all participants are fully connected. Discussions of such risk-sharing schemes can be found in Feng et al. (2023) and Charpentier et al. (2021).

The current literature focuses primarily on single-period P2P risk-sharing models. Participants can benefit from engaging in long-term risk sharing, which enables them to smooth fluctuations arising from risks and uncertainties over time. The literature on dynamic risk sharing has primarily focused on the objective of maximizing exponential utility of the wealth at the terminal period (see, e.g., Chen et al. (2021a)), while our focus is on quadratic utility functions of participants' terminal wealth.

In this paper, we seek risk allocations that are given by a proportional allocation of all individual risks. Such arrangement is easy to implement, widely used in practice and often known as quota-share treaties in two-party settings in the insurance industry. To obtain mathematically tractable results, the main assumption that we impose in this paper is the independence of multivariate losses over time, while losses of some participants are still allowed to be dependent with those of others and the multivariate distribution of future losses may be time varying. Despite its structural simplicity, little is known about this in the literature on general risk sharing, particularly for peer-to-peer risk sharing.

Quadratic utility preferences of an individual are generally modelled as the minimization of the weighted sum of expected loss minus the second moment of the loss. In this paper, we study the objective to minimize the weighted sum of quadratic utilities of all participants. This can be seen as the objective of a benevolent social planner. This social planner is able to determine the risk-sharing allocations at time 0 until a fixed future time $T$, and the participants commit to these allocations at time 0. This means that there is a cooperative solution and we do not model strategic behavior of the individual participants, although it turns out that the optimal risk-sharing allocation does not require any commitment of the participants. It is well-known that all solutions of a weighted-sum-minimization are Pareto optimal (see, e.g., Cohon (1978) and Miettinen (1999)). This means that the optimal risk-sharing allocation derived in this paper is Pareto optimal: there does not exist another risk allocation that has a larger utility for all participants, and a strictly larger utility for at least one participant.

We find that aggregate risk sharing contracts are not necessarily optimal. The optimal risk-sharing allocation derived in this model consists of three parts: one long-term allocation that is recipient-indifferent, and two correction terms. Recipient-indifference means that the risk-sharing allocation does not depend on who incurs the losses, and can thus be seen as an aggregate risk sharing contract. If multivariate losses are i.i.d. over time, then we show the convergence of the optimal allocations and that of relative reserves when the risk-sharing horizon $T$ goes to infinity. In a similar spirit as in dynamic equilibrium models in economics, we find that there is a long-term average that is disturbed on the short term, and correction terms that push the system to its long-term average (cf. Frankel, 1979). Moreover, we show how the allocation changes if the participants were to commit to a constant allocation over time.

We also solve the problem with the objective to minimize a weighted sum of variances under actuarial fairness constraints. Actuarial fairness refers to the condition under which the expected loss before and after risk sharing remains the same. The main motivation for risk sharing should be to decrease the variance. It should be noted that the actuarial fairness is interpreted differently from the financial fairness condition under which the value of allocated risk is the same as that of the original risk under some pricing measure. The financial fairness constraint is first studied by Bühlmann and Jewell (1979), and is further studied by Pazdera et al. (2017). Results in this paper can be extended to financial fairness. Nonetheless, we shall restrict discussions in this paper to actuarial fairness. The unconstrained problem has $2n$ weight parameters that can be chosen freely, where $n$ is the number of participants. Imposing $n$ actuarial fairness constraints can be achieved via constraints on the weights of the variance or via constraints on the weights of the expectation. We derive in the closed form how weights can be chosen to achieve actuarial fairness, and propose an alternative problem in which we add actuarial fairness constraints to the original problem.

Risk sharing is related to the optimal portfolio selection problem. For instance, Li and Ng (2000) study a mean-variance, dynamic portfolio selection problem, and therefore extending the static problem of Markowitz (1952) to a multi-period setting. A key difference between portfolio selection and risk sharing is the dimension of allocations. In portfolio selection problems, only one investor is considered to allocate his/her resource under management over multiple assets. There is no constraint on how much one invests in each type of asset. In risk sharing problems, there are multiple participants, each of whom allocates his/her own risk to others. However, the risks must remain within the system and be cleared. The allocation is multi-dimensional with an additional market clearing condition. Technically, this means that in this paper we extend Li and Ng (2000) to the case where the total risk exposure of participants is constrained via a market clearing condition. Our focus is on a finite, discrete-time setting, and we then solve the problem backwards in time via an updating of the problem's weight-parameters. To be precise, in every time-step backwards, the weights in the objective function are updated.

The contributions of this paper are three-fold: (1) The multi-period risk sharing model contributes to the existing risk sharing literature that focuses primarily on single-period models. We show that in the P2P setting the optimal dynamic risk sharing is horizon independent. The long term asymptotics of dynamic risk sharing strategies reveal dominating factors of decision-making for terminal reserve management. (2) The framework presented in this paper enables us to uncover the hidden connection between the well-studied portfolio selection problems in the mathematical finance literature and the risk sharing problems in the insurance and risk management literature. There is an analogy between the allocation of investment capital in different asset classes (portfolio selection) and that of loss among different risk takers (risk sharing). We use the dynamic programming approach developed initially for portfolio selection to identify optimal loss allocation strategies.

Even though we consider only quadratic utility optimization in this paper, one may be able to connect other portfolio optimization techniques to their counterparts in risk sharing. (3) Classic risk sharing literature tends to focus on spatial risk transfers within a group of economic agents, whereas the P2P risk sharing problem in this paper takes into account both the spatial and temporal effects of risk sharing. We hope this work will draw more interests in the literature on the temporal aspect of risk sharing.

This paper is set out as follows. Section 2 provides a motivating example of catastrophe risk pooling in a P2P setting. We consider both dynamic and static risk-sharing schemes with different objectives. In Section 3, we derive optimal risk-sharing allocations analytically. We show that risk-sharing allocations converge over time. In the limit, the allocation is dominantly influenced by the weights of the second moment in the objective function. We illustrate via examples the optimal time-dependent allocation. The actuarial fairness is discussed in the multi-period P2P setting with different interpretations in Section 4. A numerical example is provided in Section 5 to illustrate the convergence of allocation coefficients and the ratios of reserves. We conclude in Section 6 with a summary and discussion of future work.

## 2 Model

Risk sharing has become an important tool in the financial market for managing risks that are otherwise difficult to cover by traditional insurance. Take for example catastrophe risk pools such as the Caribbean catastrophe risk insurance facility (CCRIF). These countries are prone to natural disasters such as hurricanes and earthquakes. In 2004, Hurricane Ivan affected nine Caribbean countries and caused economic losses exceeding US$6 billion accross the region. As these countries are mostly developing countries with limited financial resources, it is much more difficult for the governments to cope with the devastating economic losses in the aftermath than it is for developed countries. There was clearly a demand for concerted effort by these countries to develop new protection mechanisms against natural hazards and rapid resources mobilization in the wake of disasters. As a result, the CCRIF was established in 2007 by the World Bank and 19 Caribbean and Central American countries as member states as a sovereign trust fund for disaster financing. Countries can pool risks in a diversified portfolio, retain some of the risk through joint capital, and transfer excess risk to the reinsurance and capital markets. Even though these countries are in close proximity, they are not always hit by the same major disaster within the same year. Hence, member states can share the financial burden of each other and mitigate the risks faced by individual states. Owing to its success, catastrophe risk pooling has seen rapid growth in other parts of the world, such as the Florida Hurricane Catastrophe Fund (United States), Flood Re (United Kingdom), African Risk Capacity (Bermuda), Pacific Catastrophe Risk Assessment and Financing Initiative (Pacific Islands), and others. A detailed analysis of catastrophe risk pooling can be found in Bollmann and Wang (2019).

The basic mechanism of such a risk pool can be described as follows, and this is also visualized in Figure 1. Suppose that there are a number of $n$ participating countries with ground-up losses denoted by $(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_n)$. Before entering the risk pool, each country is required to make three key decisions regarding their coverage selection. (1) Attachment point: the minimum severity of the event loss that triggers a claim. (2) Coverage limit: the severity of the event loss at or above which the maximum payment is allowed. (3) Ceding percentage : the percentage of event loss ceded to the risk pool, which may be carried by a reinsurer or all member states.

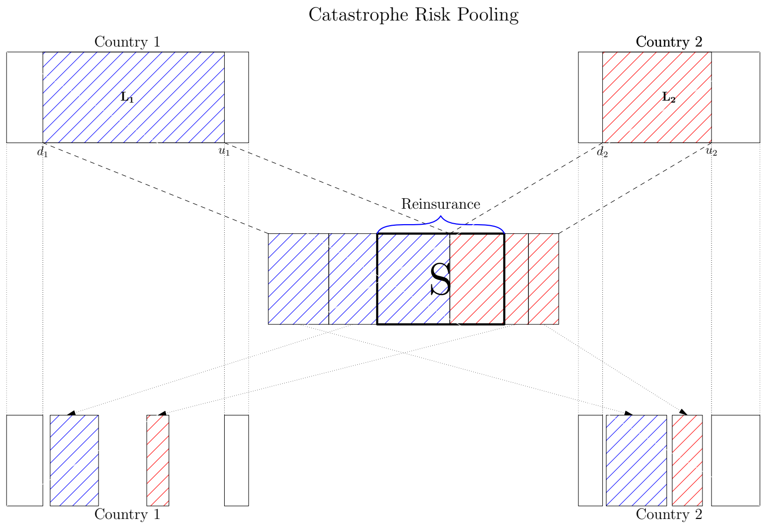


Figure 1: Mechanism of risk pooling. Note: the proportions in the figure are for illustrative purposes and do not represent the amount shared between two parties.

Denote by $d_j$ the per-event attachment point for country $j$, $u_j$ the per-event coverage limit, and by $\delta_j$ the ceding percentage. Therefore, the actual risk brought to the risk pool by country $j$, denoted by $L_j$, is given by

$$L_j = \delta_j \left( u_j \wedge (\tilde{L}_j - d_j)_+ \right),$$

where $Z_+ := \max\{Z, 0\}$. The aggregate loss in the risk pool is given by

$$S := \sum_{j=1}^{n} L_j.$$

It is typical that the risk pool purchases a reinsurance on the aggregate loss $S$ with attachment point $D$ and coverage limit $U$. The uncovered risks $S \wedge D$ and $(S - (D+U))_+$ are then carried by member countries. The allocation of reinsurance cost and uncovered losses to participating countries may be based on percentage contributions to the CAT pool. For example, a common approach is the *pro-rata allocation* in which country $j$ pays for $\alpha_j[S \wedge D + (S - (D+U))_+]$ where

$$\alpha_j = \frac{\mathbb{E}[L_j]}{\mathbb{E}[S]} \qquad \text{or} \qquad \frac{\mathbb{E}[L_j | S > D]}{\mathbb{E}[S | S > D]}.$$

In other words, each participating country carries a portion of the uncovered risks for other countries. The allocation coefficients are set to ensure some levels of actuarial fairness for participants. In other words, a country with historically higher economic losses is expected to pay more than another one with lower losses. In the top panel of Figure 1, the shaded areas in the boxes represent the risks of the participating countries that are ceded to the pool, which are the losses between the attachment point and the coverage limit. These two risks are then pooled to form the aggregate risk. A part of this aggregate risk is then covered by a reinsurance company, and this is visualized by the thickened black frame. The remaining portion (outside the black frame) of the aggregate risk is then split between the two countries. The bottom panel of the figure illustrates the risks carried by the participating countries after risk pooling. The empty boxes represent their retained risks and the shaded boxes are their portions of the pooled risks with the colors representing their origins.

While reinsurance may be an important component of the risk reduction, it is the cross-country risk sharing that is at the core of this mechanism. We take a minimalist approach and focus on only the very fundamental component of risk pooling without an insurer. We consider a completely decentralized risk pool where all risks are borne by member countries, i.e., $D = U = \infty$. All models to be discussed in this paper can be extended in the future with a risk transfer to a third party such as a traditional insurer. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^\top$ be the allocated loss for the countries. If the allocation of aggregate loss is based on the first pro-rata approach, then we can write the allocation in a matrix form,

$$\mathbf{Y} = \mathbf{AL},$$

where $\mathbf{A} = (\alpha_1 \mathbf{e}, \ldots, \alpha_n \mathbf{e})^\top$ and $\mathbf{e}$ is an $n \times 1$ vector of ones: $\mathbf{e} = (1, 1, \ldots, 1)'$. The catastrophe risk pooling is an example in a class of P2P risk sharing, which is a network where peers carry insurance coverage for each other. Some earlier work has been done in Feng et al. (2023). The connection of catastrophe risk pooling to other types of decentralized insurance schemes can be found in Feng et al. (2022).

### Aggregate versus P2P Risk Sharing

In most classical risk-sharing solutions, the optimal risk allocation is often carried out in Pareto settings. Under weak conditions, Pareto-optimal risk-sharing allocations have been proven to be functions of the aggregate risk, i.e., $Y_i = h_i(S)$ for some real-valued function $h_i$ and $S = \sum_{i=1}^n L_i$. P2P risk allocation rules are introduced in more general settings inspired by P2P business models in the sharing economy. Such risk-sharing allocations can be defined by non-aggregate methods, i.e., $Y_i = h_i(\mathbf{L})$ where $h_i$ is some vector-valued function. There are three reasons why P2P risk sharing may be preferred over aggregate risk sharing in practice.

- Aggregate risk sharing is generally based on a centralized model, which requires a well-functioning exchange or market maker to facilitate the market clearing, i.e., the aggregation and allocation of losses. The allocated risk is implicitly constructed as non-trivial functions of the original risk exposures $\{L_j\}_{j=1}^n$, and it requires a good understanding of the bivariate risk $(L_i, S)$, but this dependence structure is generally hard to approximate.

- As an alternative to aggregate risk sharing, companies could seek a risk-sharing contract with a peer company Over-The-Counter (OTC). In such contracts, a part of the underlying individual risk is ceded to a counterparty. The advantage of this is that the individual risk $L_{t,i}$ is better understood than the aggregate risk, and it is therefore much easier to implement such OTC risk-sharing contract.

- P2P risk sharing is a decentralized model where cash flows take place from peer to peer by ex-ante rules. If a counterparty fails to deliver on its promise to pay for others' losses, it does not necessarily affect bilateral transactions among other parties since the allocation rule of losses is predetermined.

The differences between aggregate and P2P risk-sharing schemes have been discussed in Denuit et al. (2022), Feng et al. (2022) and Feng and Li (2023). Readers are referred to these papers and references within.

**Spatial versus Temporal Effect**

The P2P risk sharing in a single-period model, such as the catastrophe risk pooling, is in essence a "pay-as-you-go" system, where all losses are distributed and settled when they occur, rather than before or after. A property of such a system is that there is no inter-generational risk transfer, which may be perceived as "the young carry the burden of the old". The effect of such a risk sharing is purely spatial risk diversification. In other words, the financial impact of losses occurred to some participants is spread out to all others. In this paper, we consider a multi-period P2P risk-sharing model in a "pay-as-you-go" setting. However, since the balance of risk sharing can be carried over from period to period, the P2P risk sharing can have a temporal effect, which provides some smoothing of losses over time. Given multiple periods on the horizon, it becomes an important issue to devise a mechanism not to spread the risk among participants but also manage the allocation to provide some level of financial stability over time.

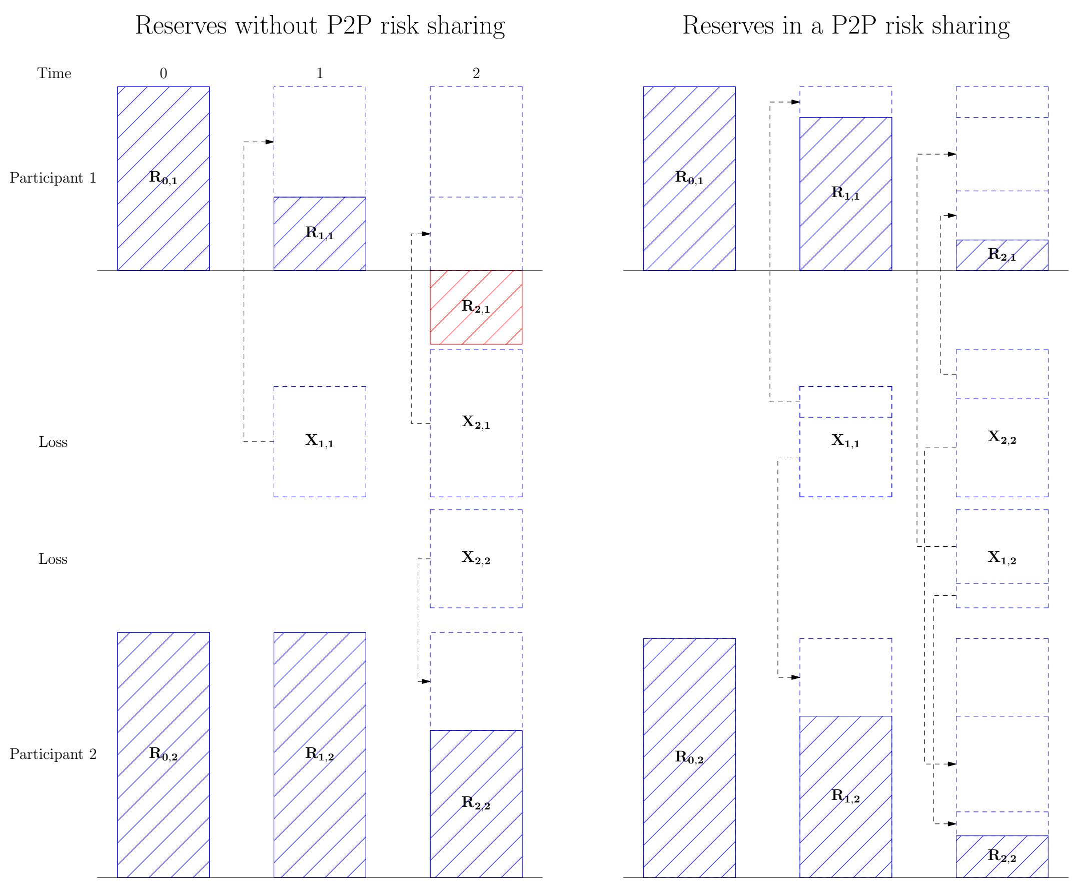

Figure 2: Reserves without P2P risk sharing vs. Reserves in a P2P risk sharing.

As illustrated in the top panel of Figure 2, when two participants carry their own losses independently over time, they are subject to volatile changes in terminal capital as the sizes of losses can vary drastically from period to period. The sizes of blue shaded boxes represent the magnitudes of remaining balance whereas the size of the red shaded box represents a deficit. The empty boxes in the middle section of the figure show the sizes of losses. The reserves on the top panel are diminished because of the losses. When a P2P risk-sharing scheme is in place, the two participants can trade losses with each other to provide some level of temporary relief. In the right panel of Figure 2, each empty box in the middle section is split between the two participants. If you follow the arrows, the shared losses for each participant are often smaller than their own losses. If we compare the magnitudes of reserves between the left panel and the right panel for each participant (each "row"), it is clear that reserves draw down slower in the P2P risk sharing than they do without the risk sharing. Hence, the mechanism exhibits temporal smoothing in this figure.

The purpose of this work is to understand the design of P2P risk sharing to control its temporal effect on the participants' long-term capital. One desirable property is the long-term stability of the reserve fund. The reserve fund often begins with an entity's asset dedicated for a particular line of business. As its liability becomes due, the asset is depleted over time. The ending balance often represents the surplus to be distributed as dividends or used for other purposes. For example, a stock company is expected to produce a steady stream of dividends, or is otherwise jeopardizing its share price. A nation state often appropriates a similar funding for disaster relief and may expect to maintain a stable level of funding or other physical assets for its national reserve. While the stability of surplus can be achieved through risk diversification in its liability portfolio, such a strategy is not always possible for non-insurers with a small client base. P2P risk sharing with other peers offers an alternative approach for stabilizing the terminal reserves.

**Model Formulation**

We consider a model similar to the setup of a catastrophe risk pool, where member state governments apportion funds for disaster relief purposes and pay upfront premiums in exchange of coverage from the catastrophe risk pool[1]. Or alternatively, these risk pools may start with grants from developed donor countries. The fund balances are expected to be drawn down as funds are used to pay claims. Thus, it is in the interests of such funds to retain as much balance as possible while minimizing their uncertainty.

There are $n$ participants who seek a P2P risk-sharing arrangement over a discrete and finite horizon $T$. Their risk exposure is given by $\mathbf{X}_t = (X_{t,1}, X_{t,2}, \dots X_{t,n})'$ for $t \in \{1, 2, \dots, T\}$, which is progressively measurable on the fixed, filtered probability space. Thus, $\mathbf{X}_t$ is modelled as an $n \times 1$ column vector of losses at time $t \in \{1, 2, \dots, T\}$, and the multivariate distribution of $\mathbf{X}_t$ is common knowledge among all $n$ participants. Throughout the paper, we assume that the risk exposure is independent over time with finite second moments. Such an assumption is most suitable for property and casualty applications, such as fire, flood, hurricane, etc. Temporal dependence can be introduced at the expense of losing some mathematical tractability. For brevity, we shall denote the expectation of the risk exposure by $\boldsymbol{\mu}_t := \mathbb{E}[\mathbf{X}_t]$ and the cross-moment by $\boldsymbol{\Xi}_t := \mathbb{E}[\mathbf{X}_t \mathbf{X}_t']$. In the case where all risk exposures are time-homogeneous, i.e., the distribution of losses $\mathbf{X}_t$ does not depend on the time index $t$, we shall suppress the subscript $t$ when no ambiguity arises. The losses are considered to be

[1] https://www.gfdrr.org/en/feature-story/what-makes-catastrophe-risk-pools-work

non-negative in general for practical applications. However, in this paper, they can be negative and interpreted as profit. This for instance can happen when one includes the revenue of participants in the distribution of $\mathbf{X}_t$.

We propose a proportional structure of the risk-sharing arrangement. The proportional P2P risk sharing is an extension of proportional reinsurance, which is one of the most commonly used treaties for reinsurance. It is also known to be the optimal form of insurance when a reinsurer aims to minimize its variance given the insurer's fixed variance. This result was extended to the case of $n$ participants for risk exchange in Beard et al. (1969). Hence we also adopt this setting for multi-period risk sharing in this paper. The parameter $\alpha_{t,i,j}$ represents the proportion of participant $j$'s risk to be borne by participant $i$ at time $t$. Participant $i$ keeps the amount $\alpha_{t,i,i}X_{t,i}$ of the initially endowed risk $X_{t,i}$ at time $t$. The risk $\alpha_{t,i,j}X_{t,j}$ will be ceded to the participant $i$. In this way, we define $\mathbf{A}_t$ as the $n \times n$ matrix of $\alpha_{t,\cdot,\cdot}$ for given time $t \in \{0,1,\ldots,T-1\}$. Thus, the $(i,j)$-th element of the matrix $\mathbf{A}_t$ is denoted by $\alpha_{t,i,j}$, and indicates the proportion of the risk $X_{t,j}$ that ceded to participant $i$. As all losses are expected to be absorbed by the participants, it holds that

$$\sum_{i=1}^{n} \alpha_{t,i,j} = 1,$$

for all $t = 0,1,\ldots,T-1$, and all $j = 1,\ldots,n$, or in matrix form:

$$\mathbf{e}'\mathbf{A}_t = \mathbf{e}', \tag{1}$$

where $t = 0,1,\ldots,T-1$. This is also known as the *zero-balance loss conservation* condition, or *market clearing* condition. In absence of any risk sharing, it holds that $\mathbf{A}_t = I_n$, where $I_n$ is the $n$-dimensional identity matrix.

Participant $i$ starts with a fixed initial reserve $R_{0,i} \in \mathbb{R}$, where $i = 1,\ldots,n$. The initial reserve amounts are exogenous and arbitrary. For instance, the initial reserve may be sufficient to cover expected future liabilities, i.e., $R_{0,i} \geq \sum_{t=1}^{T} \mathbb{E}[X_{t,i}]$. Let $R_{t,i}$ be the reserve for participant $i$ at time $t$. We assume that the risk-free rate is deterministic, and we evaluate all reserves and risk exposures as the time-0 value, i.e., the actual time-$t$ reserves and losses can be derived via a correction with the discount function. Thus, the reserves follow the following recursive relation, for all $t = 1,\ldots,T$,

$$\mathbf{R}_t = \mathbf{R}_{t-1} - \mathbf{A}_{t-1}\mathbf{X}_t. \tag{2}$$

The recursive relation states that the reserve in the current period $t$ is the reserve from the previous period $t-1$ minus the exchanged losses in the current period, for which the exchange rule $\mathbf{A}_{t-1}$ is known at time $t-1$. Note that the exchange rule $\mathbf{A}_t$ is determined dynamically to keep track of the current reserve and in anticipation of future losses. In (2), $\mathbf{A}_t$ and $\mathbf{R}_t$ are time-$t$ measurable, for $t = 0,1,\ldots,T-1$. This means that $\mathbf{A}_t$ is measurable at time $t$, and a random variable before time $t$. Thus, at time 0, $A_0$ is deterministic and $\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_{T-1}$ are all random variables.

In this paper, we take an objective of a benevolent social planner that maximizes a weighted sum of the expected reserves minus a weighted sum of the second moments of the reserves. This is equivalent with the maximization of the weighted sum of expected quadratic utilities, where every participant $i$ is endowed with a quadratic utility function: $u_i(w) = w - \beta_i w^2$ for wealth levels $w$ and $\beta_i > 0$. In other words, the planner aims to strike a balance between two quadratic utility objectives of the participants. A heavier weight could mean a higher priority for a particular participant. The maximization of a weighted sum of expected utility functions is popular in risk sharing, as every solution is Pareto optimal (see, e.g., Cohon (1978) and Miettinen (1999)). Here, Pareto optimality of a risk allocation is understood as follows. Every participant desires a risk sharing scheme that maximizes their own utility. It is typically not possible that all participants' utilities are maximized at the same time. The goal of a social planner is to find a best possible solution for the best interest of the entire group of these participants with competing interests. A risk sharing scheme is called Pareto optimal if there does not exist another scheme that allows for a higher quadratic utility for at least one participant and no less utility for all participants. The problem is formalized as follows.

**Problem 1** (Dynamic P2P Risk Sharing)**.** Let $\boldsymbol{\Omega}$ be an $n \times n$ diagonal matrix with positive elements on the diagonal and let $\boldsymbol{\lambda}$ be an $n \times 1$, non-negative vector. Solve

$$\max_{\{\mathbf{A}_t\}_{t=0}^{T-1}} \operatorname{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_T'\right] - \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_T\mathbf{R}_T'\right]\right), \tag{3}$$

under the constraints (1)-(2), where Tr is the trace (i.e., the sum of elements on the main diagonal).

Throughout this paper, $\mathbb{E}$ is understood as the expectation conditional on all information at time 0. Also, $\omega$ is understood as $n$-dimensional vector that takes the values on the diagonal of $\boldsymbol{\Omega}$, i.e., $\boldsymbol{\Omega}\mathbf{e} = \boldsymbol{\omega}$. Intuitively speaking, the aim of the optimization is to find the optimal exchange rule $\{\mathbf{A}_t\}_{t=0}^{T-1}$ that strikes a balance between maximizing the expected value of terminal reserves and minimizing their variations. The diagonal matrix $\boldsymbol{\Omega}$ and vector $\boldsymbol{\lambda}$ can be viewed as the relative weights in addressing these potentially conflicting objectives. The weights can be determined through certain influence metrics, such as the capitalization of participating companies, equity of the risk sharing consortium, etc.

As a comparison, we also study a similar problem in which we impose the extra condition that the optimal P2P risk-sharing coefficients are constant over time.

**Problem 2** (Static P2P Risk Sharing)**.** Solve the objective (3) subject to (1)-(2) and the additional constraint that $\mathbf{A}_t = \mathbf{A}$ for all $t = 0, \ldots, T-1$.

Solutions to Problem 2 are easy to implement as risk-sharing coefficients are all constant over time, and thus deterministic at time 0. The advantage of such a scheme is that the same risk-sharing allocation applies to all periods and provides the simplicity often preferred in practice. It is arguable that such an allocation is more appealing as participants usually want stability for their payment plans.

# 3 Optimal risk-sharing allocations

In this section, we analyze the various optimal P2P risk-sharing allocations and their economic interpretations.

## 3.1 Dynamic risk sharing

The optimization is carried out as follows. We find a recursive relationship in the objective function, and solve the objective function backwards. In each step backwards, we can update the weights in the objective, and solve for the optimal risk-sharing allocations in that period. Repeating this procedure leads to the following optimal solution to the dynamic P2P risk-sharing problem. The proof of Theorem 3.1 is in Appendix A.1.

**Theorem 3.1** (Dynamic P2P Risk Sharing)**.** The solution to Problem 1 is given by

$$\mathbf{A}_t = \mathbf{L} + \mathbf{C}_t + \mathbf{E}_t, \tag{4}$$

with three components:

$$\begin{aligned}
\mathbf{C}_t &= \left[\mathbf{I} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\right]\mathbf{R}_t \boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1}, \\
\mathbf{E}_t &= \frac{1}{2}\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1}, \\
\mathbf{L} &= \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}'),
\end{aligned}$$

where

$$\theta = \mathbf{e}'\boldsymbol{\Omega}^{-1}\mathbf{e} \quad \text{and} \quad \phi = \mathbf{e}'\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}.$$

The optimal solution to the dynamic P2P risk-sharing problem consists of three components.

(1) *Long-term risk-sharing allocation.* We will show later that $\mathbf{L}$ is a long-term risk-sharing allocation that depends only on the weights of the second moment of the multivariate risk. It is clear that, as a risk-sharing allocation on its own, $\mathbf{L}$ satisfies the zero-balance conservation condition.

(2) *Correction term for expected terminal reserves.* The term $\mathbf{E}_t$ is attributable to the objective of maximizing the expected value of ending balances, as it is the only term that depends on the weights of the first moments, $\boldsymbol{\lambda}$. Proposition 1 states that the correction term is removed if we apply equal weights in the objective the expected values of ending balances. The proof of Proposition 1 is in Appendix A.2. Observe that this correction term also has column sums of zero, i.e., $\mathbf{e}'\mathbf{E}_t = \mathbf{0}$ for all $t$.

(3) *Correction term for current reserves.* The component $\mathbf{C}_t$ is a correction factor attributable to the deviation of reserves from their long-term ratios. The ratios of reserves are considered being in a steady state if they satisfy the condition

$$\left[\mathbf{I} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\right]\mathbf{R}_t = \mathbf{0}.$$

This condition offers additional insight on the impact of reserves on the convergence of the dynamic risk-sharing allocation to its long-term form. Given that the initial reserves can start at arbitrary values, there is some structural "imbalance" in the early stage. As the time goes by and the ratio of reserves reaches some long-term steady state in which this condition holds. Then, the correction term disappears. In Proposition 2, we show that if the ratio of reserves is in a steady state and if the weights of the first moments do not depend on the participant, then the component $\mathbf{C}_t$ vanishes. Proof of Proposition 2 is in Section A.3. One should also note that this correction term has column sums of zero, i.e., $\mathbf{e}'\mathbf{C}_t = \mathbf{0}$.

**Definition 1.** The relation $\mathbf{x} \propto \mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ $(\mathbf{y} \neq 0)$ means that $\mathbf{x}$ lies in the one-dimensional subspace spanned by $\mathbf{y}$, or in other words, that there is a constant $\alpha \in \mathbb{R}$ such that $\mathbf{x} = \alpha\mathbf{y}$. This implies that if $\mathbf{x} \propto \mathbf{e}$, then $x_i = x_j$ for all $i, j = 1, 2, \ldots, n$.

**Proposition 1.** If $\boldsymbol{\lambda} \propto \mathbf{e}$, then $\mathbf{E}_t = 0$ for all $t = 1, \ldots, T-1$.

**Proposition 2.** If the reserves satisfy

$$\mathbf{R}_s \propto \boldsymbol{\Omega}^{-1}\mathbf{e}$$

and $\boldsymbol{\lambda} \propto \mathbf{e}$, then $C_t = 0$ for $t = s, s+1, \ldots, T-1$.

In a nutshell, the optimal allocation to the dynamic risk-sharing problem exhibits a clear structure attributable to three factors. When the risk is time-homogeneous, the matrix tends to converge to its long-term form, which is described in the first term. When pairwise ratios of the reserves are far from their steady state, the second term is a correction to the risk-sharing allocation caused by the "imbalance" in reserves. Given the competition between the two objectives on expectation and variation, the third term is a correction due to "uneven" weights on the expected value objective. As we shall see later in Section 3.3, the effect of "uneven" weights on expected value objectives and the "imbalanced" reserves shall eventually wear off. As the time goes by, the ratios of reserves shall converge to a long-term limit and the risk allocation matrix shall converge to the long-term risk-sharing allocation.

The optimal risk-sharing allocation $A_t$ in Theorem 3.1 does not depend on the length of the horizon $T$. This means that if the participants share their losses for 1 period or 10 periods, the optimal risk-sharing allocation at time 0, given by $\mathbf{A}_0$, remains the same. The optimal solution at time $t$ depends only on the reserves at time $t$. It is therefore also Markovian. Moreover, if the solution has been applied up to time $t$, the solution derived at time $t$ will apply as well. To be precise, for $t_0 < t_1 < T$, the social planner at time $t_0$ will find a solution $A_t$ for $t \in \{t_0, \ldots, T-1\}$, and the social planner at time $t_1$ will find the same solution allocation $A_t$ but then for $t \in \{t_1, \ldots, T-1\}$. No pre-commitment is therefore needed, and the solution is *time consistent*.

**Example 3.1** (One-period spatial-homogeneous case)**.** In a one-period model, the optimal risk-sharing allocation is given in Theorem 3.1 by

$$\mathbf{A}_0 = \left[\mathbf{I} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\right]\mathbf{R}_0\boldsymbol{\mu}_1'\boldsymbol{\Xi}_1^{-1} + \frac{1}{2}\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}_1'\boldsymbol{\Xi}_1^{-1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}'). \tag{5}$$

We say that the group under consideration is called spatially homogeneous if all peers have the same initial reserves, the same expected losses and all carry the same weights in the optimization objectives, i.e., $\mathbb{E}\left[X_{1,j}\right] = \mathbb{E}\left[X_{1,i}\right]$, $R_{0,i} = R_{0,j}, \omega_i = \omega_j$, $\lambda_i = \lambda_j$ for all $i, j = 1, 2, \ldots, n$. It is very natural to expect that the risk-sharing allocation would be uniform, which is to split all losses equally between each other. Observe that

$$\frac{\omega_i}{\omega_j} = 1, \quad \theta = \frac{n}{\omega}, \quad \frac{\phi}{\theta} = \lambda, \quad \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}' = \frac{\mathbf{e}\mathbf{e}'}{n},$$

and thus

$$\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0 = 0, \quad \frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda} = 0.$$

Hence, the optimal solution simplifies to

$$\mathbf{A}_0 = \frac{\mathbf{e}\mathbf{e}'}{n}.$$

This means that, indeed, all risks are shared equally by participants, and this can thus be seen as an egalitarian risk pooling.

This finding allows us to compare it with an optimal risk-sharing result in case participants are endowed with exponential utilities and $T = 1$: participants maximize $\mathbb{E}[\exp(-\lambda_i R_{1,i})]$ for risk-aversion parameter $\lambda_i$. Then, from Barrieu and El Karoui (2005), it follows that $\alpha_{0,i,j} = \frac{1/\lambda_i}{\sum_{k=1}^n 1/\lambda_k}$. Thus, this allocation does not depend on the recipient $j$, but rather on the relative risk-aversion of the participant to whom the risk is ceded.

## 3.2 Asymptotic behavior of the optimal risk-sharing allocations

In this section, we discuss the behavior of the optimal allocation, represented by $\mathbf{A}_t$, as $t \to T$ and $T \to \infty$. The allocation matrix $\mathbf{A}_t$ of Problem 1 can be written as

$$\mathbf{A}_t = M_t \left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right] \boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{2}M_t \left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right] \boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}'), \tag{6}$$

where

$$M_t = \prod_{i=1}^{t} \left(1 - \boldsymbol{\mu}_i \boldsymbol{\Xi}_i^{-1}\mathbf{X}_i\right).$$

The random variable $M_t$ converges in probability to zero over time. For this reason, the first two terms, which are influenced by the initial reserves and the objective on expectations of terminal reserves, diminishes in magnitude over time. The derivation of the above form of $\mathbf{A}_t$ and the proof of Theorem 3.2 are provided in Appendix A.4.

**Theorem 3.2.** The solution matrix $\mathbf{A}_t$ to Problem 1 converges in probability, as $t, T \to \infty$,

$$\mathbf{A}_t \longrightarrow \mathbf{L} = \frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}'),$$

where $\theta = \mathbf{e}'\boldsymbol{\Omega}\mathbf{e}$.

Note that $\mathbf{L}$ represents the first term in (4) and a long-run allocation as it does not depend on time $t$. It is an $n \times n$ matrix with all columns being identical. The economic interpretation of such a result is also straightforward. When all risks are independent, they are indistinguishable and different treatments of these participants can only come from their different weights in the objective function. The long-run allocation rule is *recipient indifferent*, i.e., $\mathbf{L} = (\alpha_{i,j}), \alpha_{i,j} = \alpha_i$ for all $j = 1, \ldots, n$. In other words, it does not matter anymore who incurs the losses, and all losses are split by factors determined by the participants' relative weight. In fact, their allocation coefficients are given respectively by harmonic weights representing their "significance" in the decision-making,

$$(\alpha_1, \ldots, \alpha_i, \ldots \alpha_n) = \left(\frac{1/\omega_1}{\sum_{j=1}^n 1/\omega_j}, \ldots, \frac{1/\omega_i}{\sum_{j=1}^n 1/\omega_j}, \ldots, \frac{1/\omega_n}{\sum_{j=1}^n 1/\omega_j}\right). \tag{7}$$

If the objective weight $\omega_i$ is large, the social planner pays great attention to avoid the variability of her payments. Then her *harmonic weight* $\hat{\omega}_i = (1/\omega_i)/(\sum_{j=1}^n 1/\omega_j)$ is small and consequently Participant $i$ carries a small portion of everybody else's losses. If all participants are given equal weight, i.e., $\omega_1 = \cdots = \omega_n$, then the risk-sharing allocation becomes uniform and all participants carry an equal amount of the losses of each other, i.e.,

$\alpha_1 = \cdots = \alpha_n = 1/n$. While the appearance of harmonic weights in the long-term pro-rata P2P risk-sharing allocation is surprising, they are known to play a role in capital/resources allocations, see for example Chong et al. (2023) and Chen et al. (2021b).

## 3.3 Asymptotic behavior of the reserves

Let the system be time-homogeneous, i.e., the loss variables $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$, are i.i.d. over time. Then reserves diverge to $-\infty$ over time if $\mu_t > 0$ and to $+\infty$ if $\mu_t < 0$. As time goes by, we however show that the pairwise ratios of reserves converge. The proof of Theorem 3.3 is in Appendix A.5.

**Theorem 3.3.** If the risk exposure is time-homogeneous in Problem 1 and $\boldsymbol{\lambda} \propto \mathbf{e}$, then the ratio of reserves for participants $i$ and $j$ converges in probability as follows, for $t, T \to \infty$,

$$\frac{R_{t,i}}{R_{t,j}} \longrightarrow \frac{\omega_j}{\omega_i} = \frac{\hat{\omega}_i}{\hat{\omega}_j}.$$

Theorem 3.3 implies that for large $t$, it holds that

$$R_{t,i} \approx \frac{\frac{1}{\omega_i}}{\sum_{j=1}^{n} \frac{1}{\omega_j}} \sum_{j=1}^{n} R_{t,j} = \frac{\frac{1}{\omega_i}}{\sum_{j=1}^{n} \frac{1}{\omega_j}} \left[ \sum_{j=1}^{n} R_{0,j} - \sum_{s=1}^{t} \sum_{j=1}^{n} X_{s,j} \right],$$

for all $i =, 1 \ldots, n$, where we note that the random variable $\sum_{j=1}^{n} R_{t,j}$ does not depend on the risk allocation itself. Theorem 3.3 states that if the multivariate losses are time-homogeneous, then the reserves have asymptotically a fixed ratio, and this fixed ratio is determined by the weights of the second-moment $\Omega$. This means that, while reserves may diverge, the reserve of Participant $i$ can asymptotically be written as a fraction of the total sum of reserves, which is equal to its harmonic weight $\hat{\omega}_i = (1/\omega_i)/(\sum_{k=1}^{n} 1/\omega_j)$. Moreover, if the initial reserves are such that

$$\mathbf{R}_s \propto \boldsymbol{\Omega}^{-1}\mathbf{e}, \text{ and } \lambda \propto \mathbf{e}$$

then the solution matrix immediately becomes

$$\mathbf{A}_t = \mathbf{L} = \frac{1}{\theta} \boldsymbol{\Omega}^{-1} (\mathbf{e}\mathbf{e}'),$$

for all $t = 0, 1, \ldots, T-1$, and the limit ratio of reserves in Theorem 3.3 holds true for every time period $t$.

**Remark 3.4.** The result of the theorem 3.3 can be intuitively explained by using the proofs of theorem 3.1 and theorem 3.2. In the proof of the theorem 3.1 we introduced a modified reserve, i.e., $\hat{\mathbf{R}}_t := \mathbf{R}_t - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}$. Then, in the proof of theorem 3.2, we showed that as $t, T \longrightarrow \infty$ one has

$$\left( \mathbf{I} - \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta} \right) \hat{\mathbf{R}}_t \longrightarrow 0.$$

Then, with an assumption of $\boldsymbol{\lambda} \propto \mathbf{e}$, one can show that reserves $\mathbf{R}_t \propto \boldsymbol{\Omega}^{-1}\mathbf{e}$.

**Remark 3.5.** Theorem (3.3) can be obtained without the assumption $\lambda_i = \lambda_j$ for all $i, j$, but with the assumption that the sum of expected losses at time $t$ must be non-zero. That is,

$$\mathbb{E}\left[\mathbf{e}'\mathbf{X_t}\right] = \sum_{i=1}^{n} \mu_{i,t} \neq 0.$$

Then, by using the law of large numbers, we can readily obtain the same result. Please refer to the proof of Theorem (3.3) for more details.

### 3.4 Static risk sharing

In contrast to the dynamic P2P risk-sharing problem, the static problem requires only a decision at the beginning and no further decision or change on the risk-sharing allocation in the rest of the periods. The main motivation for studying such a problem is that static allocations are fairly common in practice due to their simplicity. However, one should note that even in the dynamic setting the optimal risk-sharing allocation eventually converges over time to a long-term allocation if the multi-period model takes place on a long horizon. The proof of Theorem 3.6 is in Appendix A.6.

**Theorem 3.6** (Static P2P Risk Sharing)**.** The solution to Problem 2 is given by

$$\mathbf{A} = \mathbf{L} + \tilde{\mathbf{C}} + \tilde{\mathbf{E}}, \tag{8}$$

with the following two components:

$$\begin{aligned} \tilde{\mathbf{C}} &= \left[\mathbf{I} - \frac{1}{\theta}\mathbf{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\right]\mathbf{R}_0\mathbb{E}[\mathbf{Z}']\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}'], \\ \tilde{\mathbf{E}} &= \frac{1}{2}\left[\frac{\phi}{\theta}\mathbf{\Omega}^{-1}\mathbf{e} - \mathbf{\Omega}^{-1}\boldsymbol{\lambda}\right]\mathbb{E}[\mathbf{Z}']\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}'], \end{aligned}$$

where $\mathbf{Z} = \sum_{t=1}^{T}\mathbf{X}_t$, $\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}'] = (\mathbb{E}[\mathbf{Z}\mathbf{Z}'])^{-1}$, and where $\mathbf{L}, \theta$ and $\phi$ are defined in Theorem 3.1.

The optimal risk-sharing allocation of Theorem 3.6 is similar to the optimal solution of Theorem 3.1 with the same three-term structure. A comparison of (5) and (8) reveals that the static risk-sharing problem is in essence the same as a one-period risk-sharing problem when all risk exposures are lumped into a single period. If the risk exposures are also spatially homogeneous (recall Example 3.1 for the definition), then the optimal risk-sharing allocation also reduces to uniform risk sharing. In the case of one-period model, the dynamic and static problems are exactly identical.

### 3.5 Mean-variance risk sharing

While variations of terminal reserves are modelled by their second moments in the dynamic risk-sharing problem, it may also be desirable to consider explicitly the optimization of a mean-variance objective. This problem is formally stated in the following problem.

**Problem 3.** Let $\mathbf{\Omega}$ be an $n \times n$ diagonal matrix with positive elements on the diagonal and let $\boldsymbol{\lambda}$ be an $n \times 1$, non-negative vector. Solve

$$\max_{\{\mathbf{A}_t\}_{t=0}^{T-1}} \operatorname{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_T'\right] - \mathbf{\Omega}\operatorname{Cov}\left([\mathbf{R}_T)]\right), \tag{9}$$

under the constraints (1)-(2).

Problem 3 can have many practical applications. Note that the sum of expectations remains the same before and after risk sharing, while the sum of variances may decrease after risk sharing. Problem 3 seeks a risk-sharing arrangement that maximizes a weighted sum of mean-variance objectives. The proof of Theorem 3.7 is given in Section A.7.

**Theorem 3.7.** The solution to Problem 3 is given by

$$\mathbf{A}_t = \mathbf{L} + \frac{1}{2}\left(\mathbf{I} - \mathbf{L}\right)\mathbf{\Omega}^{-1}\boldsymbol{\lambda}\boldsymbol{\mu}_{t+1}\mathbf{\Sigma}_{t+1}^{-1},$$

where $\mathbf{L} = \frac{\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}$ and $\mathbf{\Sigma}_{t+1}^{-1} := \operatorname{Cov}^{-1}\left(\mathbf{X}_{t+1}\right)$.

A comparison of Theorem 3.1 and 3.7 shows that the reserve vector $\mathbf{R}_t$ does not matter for Problem 3. This is intuitive as initial reserves serve as constants and do not affect the variance of terminal reserves. This means that the asymptotic behavior of the solution of Problem 3 is straightforward, as the time-$t$ solution does not depend on the past. Also, if the losses are time-homogeneous, Theorem 3.7 states that the optimal solution to Problem 3 does not depend on time: $\mathbf{A}_t = \mathbf{A}$ for all $t$.

# 4 Actuarial fairness

To avoid adverse selection, it is often critical to ensure that all participants are treated in a fair manner. For example, in traditional insurance, a high-expected-risk individual is charged a higher premium than a low-expected-risk individual. As there is no ex-ante payment in a P2P risk-sharing setting, we instead require that the expectations of the losses after risk sharing coincide with the expectations of the original losses. In this way, we define *actuarial fairness* as the following constraint:

$$\mathbb{E}\left[\mathbf{R}_0 - \sum_{t=0}^{T-1}\mathbf{A}_t\mathbf{X}_{t+1}\right] = \mathbb{E}\left[\mathbf{R}_0 - \sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right],$$

or written in a different way

$$\sum_{t=0}^{T-1}\mathbf{A}_t\boldsymbol{\mu}_{t+1} = \sum_{t=0}^{T-1}\boldsymbol{\mu}_{t+1}. \tag{10}$$

That is, the expected value of terminal reserves is the same under the P2P risk sharing and without risk sharing. The right-hand-side of (10) shows the expected losses if there is no risk sharing. It should be noted that in a one-period model the fairness constraint reduces to $\mathbf{A}\boldsymbol{\mu} = \boldsymbol{\mu}$. Furthermore, if the pool is also spatially homogeneous, then the constraint says that the row sums of $\mathbf{A}$ should always be one, i.e., $\mathbf{A}\mathbf{e} = \mathbf{e}$.

There are two ways to apply fairness. (1) When multiple stakeholders have competing objectives, there are infinitely many ways in which they can achieve Pareto optimality and fairness is used to identify the set of weight parameters that yield an actuarially fair solution. (2) It could be assumed that stakeholders would only consider fair risk exchanges. Therefore, the fairness condition is used as a constraint in the feasible set of risk allocations in the definition of Pareto optimality.

## 4.1 Fair Pareto-optimal risk sharing

Here we revisit the dynamic risk exchange in Problem 1. For each set of parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\Omega}$, the problem selects a Pareto-optimal solution in which no participant can be better off without comprising the utility of at least one participant in the pool. We can find the Pareto-optimal solutions that satisfy the fairness condition (10). We can revisit the optimal solutions to Problem 1, and select the weights that yield solutions that are also actuarially fair over the multi-period horizon.

**Theorem 4.1** (Dynamic risk sharing)**.** Let the risk exposure be time-homogeneous and $\sum_{i=1}^{n} \mu_i \neq 0$. If the solution to Problem 1 satisfies eq. (10), then this solution converges to a recipient-indifferent risk-sharing allocation

$$\mathbf{A}_t \longrightarrow \mathbf{A} = \frac{1}{\mathbf{e}'\mathbf{M}\mathbf{e}}\mathbf{M}\mathbf{e}\mathbf{e}', \tag{11}$$

where $\mathbf{M}$ is a diagonal matrix with $\mu_i$ as the $i$-th diagonal element.

Observe that the long-term limit $\mathbf{A}$ represents a recipient-indifferent risk-sharing allocation where $\mathbf{A} = (\alpha_{i,j}), \alpha_{i,j} = \alpha_i$ for all $i, j = 1, \ldots, n$ and

$$(\alpha_1, \ldots, \alpha_n) = \left( \frac{\mu_1}{\sum_{j=1}^{n} \mu_j}, \ldots, \frac{\mu_n}{\sum_{j=1}^{n} \mu_j} \right). \tag{12}$$

The concept of "recipient-indifferent" risk sharing refers to the fact that the allocation does not distinguish between the receiving ends of the cash transfers. Regardless of who incurs the loss, each participant carries a portion of the loss in proportion to his/her expected loss. The long-term impact of the actuarial fairness condition is that all participants would eventually pay in proportion to their expected losses. This result clearly agrees with the fairness principle that high-risk participants should pay more than low risk ones. When current reserves have an impact on the optimal risk-sharing allocation in the short term, they would in the long run settle claims in proportion to their expected losses. Such a result provides a theoretical justification on the fact that pro-rata allocation (12) has been adopted in practice, such as the catastrophe risk pooling discussed at the beginning of Section 2.

It is also worthwhile pointing out that the long-term limit under the actuarial fairness condition is consistent with the long-term limit in (7) because the weights under the actuarial fairness are in inverse proportion to their expected losses in the long run. Recall that the limit of $A_t$ without imposing actuarial fairness is already given in Theorem 3.2, and this is given by the matrix $L$. Under financial fairness, it holds for large $t$ that $A_t \mu = \mu$, and this can only be satisfied in the limit when $\mu \propto \Omega^{-1} e$. This holds true when the matrix $M$ is proportional to $\Omega^{-1}$. In fact, for finite $T$, actuarial fairness holds under the following restrictions: for $i, j = 1, \ldots, n$ with $i \neq j$, it holds for $\lambda_i, \omega_i, \lambda_j, \omega_j$ that

$$\frac{\omega_i}{\omega_j} = \frac{T\mathbb{E}[\mathbf{X}_{t,j}] - \mathbb{E}[B_T]R_{0,j} + 0.5\mathbb{E}[B_T]\lambda_j/\omega_j}{T\mathbb{E}[\mathbf{X}_{t,i}] - \mathbb{E}[B_T]R_{0,i} + 0.5\mathbb{E}[B_T]\lambda_i/\omega_i}.$$

Thus, all parameters satisfying these restrictions yield actuarial fairness, which may contain infinitely many parameter tuples. Interested readers can find the evidence in the proof of Theorem 4.1, which is provided in Appendix A.8.

## 4.2 Actuarial fairness and restricted Pareto optimality

In the previous section we identify a fair risk-sharing allocation out of all Pareto-optimal risk-sharing allocations that are not necessarily fair. In this section, we only consider fair risk-sharing allocations in the Pareto setting. In this case, we present a slightly modified problem.

**Problem 4.** Let $\mathbf{\Omega}$ be an $n \times n$ diagonal matrix with positive elements on the diagonal, and $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ be an i.i.d. sequence. Solve

$$\min \operatorname{Tr}\left(\mathbf{\Omega}\mathbb{E}\left[\mathbf{R}_T\mathbf{R}_T'\right]\right),$$

subject to (10) and

$$\mathbf{R}_{t+1} = \mathbf{R}_t - \mathbf{A}_t\mathbf{X}_{t+1}, \qquad \mathbf{e}'\mathbf{A}_t = \mathbf{e}', \qquad \text{for } t = 0, 1, \ldots, T-1.$$

Note that the key difference between Problems 1 and Problem 4 is the inclusion of the extra fairness condition (10) as a constraint, which narrows down the feasibility set of allocations to the smaller class of fair risk-sharing allocations. The reason why expectations are not included in the objective function is that they are determined and fixed by the fairness condition, i.e., $\mathbb{E}[\mathbf{R}_T] = \mathbf{R}_0 - \boldsymbol{\mu}T$. Importantly, Problem 4 selects solutions that are Pareto-optimal in a restricted sense: no other *actuarially fair* solution exists that is weakly better for all participants and strictly better for at least one participant. Considering only alternative solutions that are actuarially fair, this restricted Pareto optimality is a weaker criterion than the (unrestricted) Pareto optimality in Section 4.1.

**Theorem 4.2.** The solution to Problem 4 is given by

$$\mathbf{A}_t^F = \frac{\prod_{k=1}^{t}(1-N_k)}{b_T}\left[\mathbf{I} - \frac{1}{\theta}\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'\right]\mathbb{E}\left[\mathbf{Z}\right]\boldsymbol{\mu}_{t+1}'\mathbf{\Xi}_{t+1}^{-1} + \frac{1}{\theta}\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}',$$

where

$$N_t = \boldsymbol{\mu}_t'\mathbf{\Xi}_t^{-1}\mathbf{X}_t, \quad b_T = \sum_{t=1}^{T}\boldsymbol{\mu}_t'\mathbf{\Xi}_t^{-1}\boldsymbol{\mu}_t\prod_{k=1}^{t-1}\left(1-\boldsymbol{\mu}_k\mathbf{\Xi}_k^{-1}\boldsymbol{\mu}_k\right) = 1-(1-\boldsymbol{\mu}_t'\mathbf{\Xi}_t^{-1}\boldsymbol{\mu}_t)^T \text{ and } \mathbb{E}\left[\mathbf{Z}\right] = \mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_t\right] = T\boldsymbol{\mu}_t.$$

As $t, T \to \infty$, the optimal risk-sharing allocations converge:

$$\mathbf{A}_t^F \longrightarrow \mathbf{L} = \frac{1}{\theta}\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'.$$

The proof of Theorem 4.2 is given in Appendix A.9. We remark that in Theorem 4.2, the parameters $\boldsymbol{\mu}_t$ and $\Xi_k$ do not depend on time $k$. Despite that the constraint equation (10) is part of an optimization problem, Theorem 4.2 states that the long-term limit of the optimal risk-sharing allocations is not affected by this constraint.

**Remark 4.3.** Suppose that Problem 2 is solved with a fairness constraint (10). The solution matrix is given by

$$\mathbf{A} = (\mathbf{I} - \mathbf{L}) \frac{\mathbb{E}[\mathbf{Z}]\,\mathbb{E}[\mathbf{Z}']\,\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}']}{\mathbb{E}[\mathbf{Z}']\,\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}']\,\mathbb{E}[\mathbf{Z}]} + \mathbf{L},$$

where $\mathbf{Z}$ is as in Theorem 3.6 and $\mathbf{L}$ as in Theorem 3.1. The derivation of the solution matrix is similar to the proof of Theorem 4.2, and thus omitted.

## 4.3 Minimum variance risk exchange

In this section, we discuss the optimal risk-sharing allocation that minimizes the sum of variances of final reserves. For this objective, we can find parallel results to those in Sections 4.1 and 4.2.

**Fair Pareto-optimal risk sharing**

**Corollary 4.4.** If the solution $\mathbf{A}_t$ to Problem 3 satisfies the fairness condition (10), then it is given by:

$$\mathbf{A}_t = \frac{1}{\mathbf{e}'\mathbf{M}\mathbf{e}} \mathbf{M}\mathbf{e}\mathbf{e}',$$

for $t = 0, 1, \ldots, T-1$, where $\mathbf{M}$ is defined in Theorem 4.1.

This result shows that the long-term limit of fair Pareto-optimal risk-sharing allocations in (11) is determined by the minimization of variance. This result again confirms an observation from Section 3.2 that the weights of the second moments in the objective function are dominating factors in the long term for the Pareto-optimal risk-sharing allocations.

**Restricted Pareto optimality**

Consider the restricted Pareto-optimal risk sharing with the class of actuarially fair risk-sharing allocations. Due to actuarial fairness, the expectation vector of terminal reserves is known. Therefore, Problem 4 can be written as

$$\min \operatorname{Tr}(\boldsymbol{\Omega} \operatorname{Cov}(\mathbf{R}_T)),$$

subject to (10), $\mathbf{R}_{t+1} = \mathbf{R}_t - \mathbf{A}_t \mathbf{X}_{t+1}$ and $\mathbf{e}'\mathbf{A}_t = \mathbf{e}'$, for $t = 0, 1, \ldots, T-1$. So, the risk-sharing allocation that minimizes the pooled variance is given by $\mathbf{A}_t^F$ in Theorem 4.2.

Observe that in a one-period model, the fair Pareto-optimal risk-sharing allocation solving the above problem is given by

$$\mathbf{A}_0^F = \frac{1}{\mathbf{b}_1} \left[ \mathbf{I} - \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta} \right] \boldsymbol{\mu}_1 \boldsymbol{\mu}_1' \Xi_1^{-1} + \frac{1}{\theta} \boldsymbol{\Omega}^{-1} \mathbf{e}\mathbf{e}',$$

where $\mathbf{b}_1 = \boldsymbol{\mu}_1' \Xi_1^{-1} \boldsymbol{\mu}_1$. This result is known and discussed in detail in the context of one-period P2P risk sharing in Feng et al. (2023).

**Pareto-optimal risk sharing with side payments**

We next propose an alternative approach to ensure actuarial fairness. The risk-sharing allocation is allowed to include zero-sum cash exchanges among the participants. A participant can take on more risk than it cedes to others in exchange for fixed fees from the others as compensations. In contrast to Problem 3, we aim to solve the following problem:

$$\min_{\mathbf{A}_t} \boldsymbol{\Gamma} \operatorname{Cov}(\mathbf{R}_T + \boldsymbol{\delta}), \tag{13}$$

under the constraints (1)-(2), with

$$\boldsymbol{\delta} = \mathbb{E}[\mathbf{R}_0 - \mathbf{Z}] - \mathbb{E}[\mathbf{R}_T]. \tag{14}$$

Note that $\mathbf{R}_0 - \mathbf{Z}$ is the risk exposure of all participants in the network when there is no risk sharing. As an alternative to (10), the condition (14) is also an actuarial fairness condition under which participants pay the same on average whether or not they participate in risk sharing. Here, $\boldsymbol{\delta}$ represents deterministic side-payments among participants. It is straightforward to see that the solution to the optimization problem in (13) is identical to the solution of Problem 3. Hence, the side-payments $\boldsymbol{\delta}$ can be determined by

$$\boldsymbol{\delta} = \left[\frac{\boldsymbol{\Gamma}^{-1}\mathbf{e}\mathbf{e}'}{\theta} - \mathbf{I}\right] \sum_{t=1}^{T} \boldsymbol{\mu}_t.$$

One should keep in mind that a common criticism of the approach with side-payments is that cash transfers may be required even if no participant incurs any loss.

## 5 Numerical implementation and examples

In this section, we provide numerical evidences to illustrate key takeaways from previous sections. Suppose that three participants participate in an optimal dynamic risk-sharing allocation for 7 periods. We assume that losses $(\mathbf{X}_1, \ldots, \mathbf{X}_T)$ are independent and identically distributed over time. They follow a multivariate normal distribution with the vector of expectations $\boldsymbol{\mu}' = [55, 70, 100]$ and the variance-covariance matrix given by:

$$\operatorname{Cov}(\mathbf{X}) = \begin{bmatrix} 225 & 50 & 20 \\ 50 & 121 & 30 \\ 20 & 30 & 625 \end{bmatrix}.$$

Note that losses from different participants are not identically distributed. A social planner selects the following weights given to individual participants:

$$\mathbf{\Omega} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } \boldsymbol{\lambda}' = [1, 1/2, 2].$$

Recall that a large value in $\mathbf{\Omega}$ means that a participant wants less volatility in the final reserve and a large value in $\boldsymbol{\lambda}$ means that a participant assigns a higher weight to the expected final reserve. Participants have the same initial reserves: $\mathbf{R}_0' = [1500, 1500, 1500]$.

## 5.1 Convergence of allocation coefficients

We use Monte Carlo simulations to generate $1,000$ sample paths of the realized losses over 7 periods. For each sample path, we calculate the solution matrix $\mathbf{A}_t$ given by (4) at the beginning of each period and all participants' reserves are updated at the end of each period using the recursive relation (2). While the optimal risk-sharing allocation $\mathbf{A}_0$ is determined and fixed at time 0, subsequent optimal risk-sharing allocations $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{T-1}$ depend on the then-current reserves $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_{T-1}$. Therefore, risk-sharing allocation matrices are random in general. In Figures 3–5, we display the 2.5% and 97.5% sample quantiles of all allocation coefficients as vertical intervals, with the sample mean represented by a dot. The long-term limits are displayed as blue lines. As shown earlier in Section 3.2, we expect that, as $t, T \longrightarrow \infty$, the allocation coefficients converge to their respective harmonic weights, i.e. $\alpha_{t,i,j} \longrightarrow \frac{1/\omega_i}{\sum_{k=1}^{3} 1/\omega_k}$, where $i, j = 1, 2, 3$.

In this example, we see a rapid convergence even for a relatively short time horizon. Not only does the randomness of allocation matrices diminish over time, but all coefficients also converge. For example, Figure 3 shows that $\alpha_{t,1,1}, \alpha_{t,1,2}, \alpha_{t,1,3}$ all approach $\frac{\omega_2 \omega_3}{\omega_2 \omega_3 + \omega_1 \omega_3 + \omega_1 \omega_2} = 0.6316$ as $t$ increases from 0 to 6, even though they start from vastly different initial points. This long-term limit is illustrated by a horizontal blue line in Figure 3. Similarly, we observe that $\alpha_{t,2,1}, \alpha_{t,2,2}, \alpha_{t,2,3}$ all approach $\frac{\omega_1 \omega_3}{\omega_2 \omega_3 + \omega_1 \omega_3 + \omega_1 \omega_2} = 0.2105$ as $t$ increases from 0 to 6. Figure 5 shows that $\alpha_{t,3,1}, \alpha_{t,3,2}, \alpha_{t,3,3}$ all approach $\frac{\omega_1 \omega_2}{\omega_2 \omega_3 + \omega_1 \omega_3 + \omega_1 \omega_2} = 0.1579$ as $t$ increases from 0 to 6. As shown in (6), the allocation matrices are initially affected by the initial reserves and unequal weights on the objectives of expected terminal reserves. These effects start to wear off as early as time 2. The same limits are reached for all allocation coefficients regardless of who receives the loss compensations. This is consistent with the observation in Theorem 3.2 that the limit risk-sharing allocation is recipient indifferent.

## 5.2 Convergence of reserve ratios

We next consider the impact of the optimal risk-sharing allocation on the ratios of reserves. Assume that the multivariate losses follow the same distribution as in Section 5.1. As an example, we compare the reserves of Participants 2 and 3. The time-0 reserves are known and the same for all three participants. In Figure 6, we present the ratio of reserves $R_{t,2}/R_{t,3}$ as a function of the time period $t$. The ratio of reserves for two participants at time $t = 0$ is 1. As the reserves are path-dependent, we show the 2.5% and 97.5% quantiles of the sample of reserve ratios. The size of the red line in Figure 6 at time $t = 1$ suggests that ratios of reserves can be far

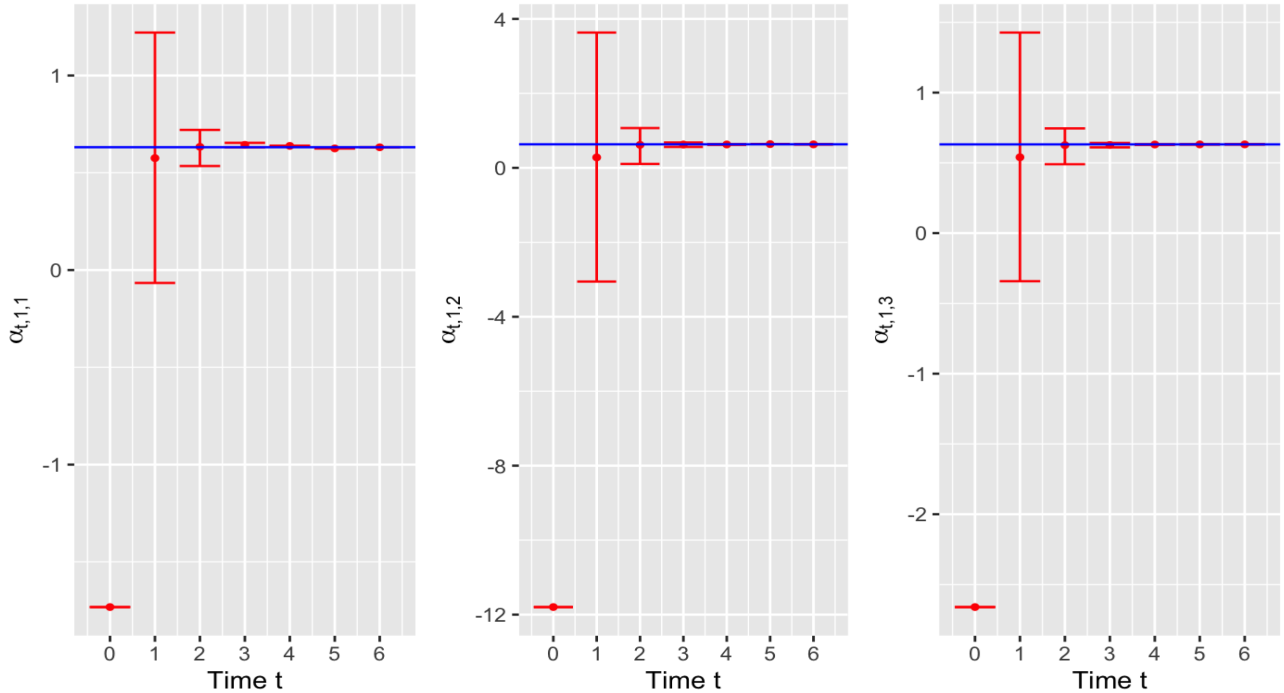


Figure 3: Optimal risk-sharing coefficients for Participant 1, $\alpha_{t,1,1}$, $\alpha_{t,1,2}$ and $\alpha_{t,1,3}$, for different periods $t$.

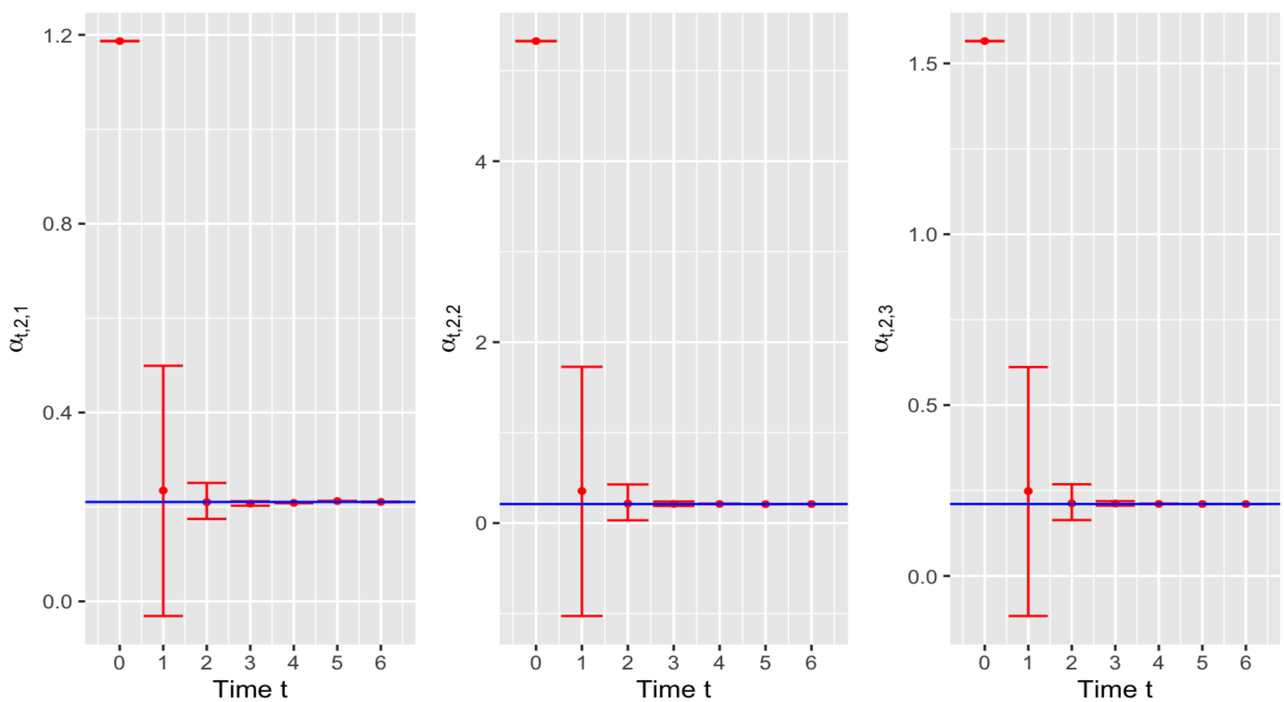


Figure 4: Optimal risk-sharing coefficients for Participant 2, $\alpha_{t,2,1}$, $\alpha_{t,2,2}$ and $\alpha_{t,2,3}$, for different periods $t$.

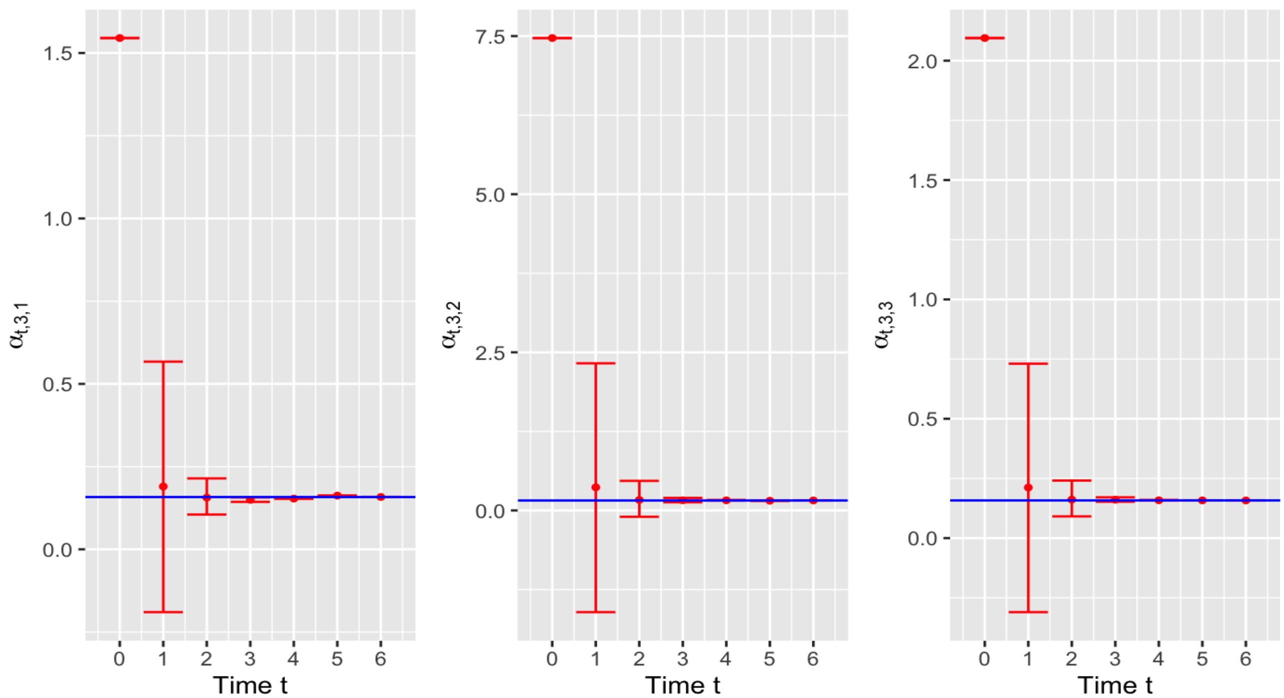


Figure 5: Optimal risk-sharing coefficients for Participant 3, $\alpha_{t,3,1}$, $\alpha_{t,3,2}$ and $\alpha_{t,3,3}$, for different periods $t$.

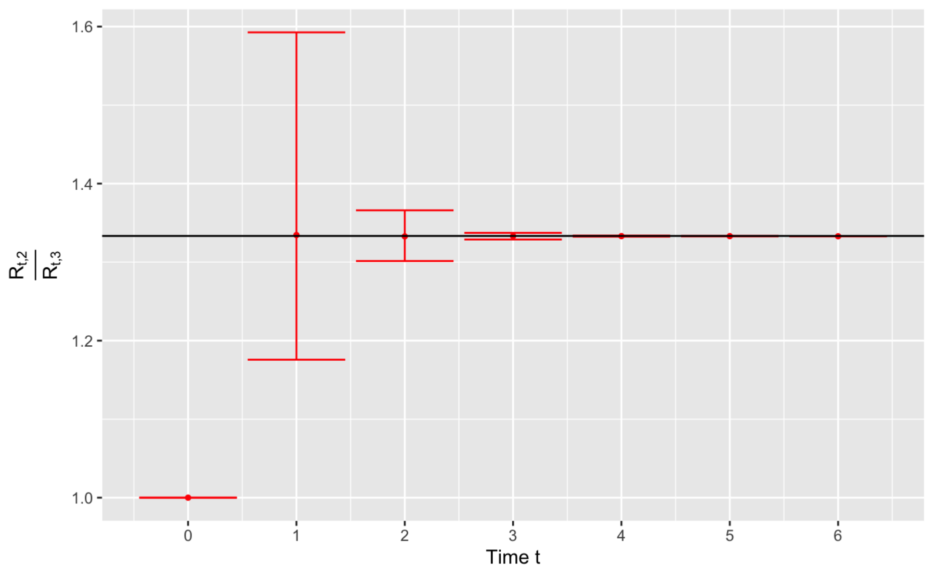


Figure 6: Ratio of $R_{t,2}$ to $R_{t,3}$ for the various values of $t = 0, 1, \ldots, T-1 = 6$.

from its long-term limit, which is displayed as the black line. The departure of the reserve ratio from its limit contributes to the deviations of allocation coefficients from their respective long-term limits at time 1 in Figures 3–5. Once the ratio of reserves converges to its limit, given by the ratio of their harmonic weights $\hat{\omega}_2/\hat{\omega}_3$, we see a similar convergence in the allocation coefficient matrices as well.

### 5.3 Actuarial fairness

We next demonstrate the impact of actuarial fairness on a Pareto-optimal risk-sharing allocation. As before we consider a three-party risk-sharing scheme with time-homogeneous risk exposures with expected losses $\boldsymbol{\mu}' = (55, 70, 100)$ and initial reserves $\mathbf{R}_0' = (1500, 1500, 1500)$. The covariance matrix and the weight-vector $\boldsymbol{\lambda}$ are the same as in the previous subsections.

Figure 7 shows various allocation coefficients for three participants of the Pareto-optimal risk-sharing allocations over time horizons of different length $T$. In contrast to Figures 3-5, where the allocation coefficients for different periods are compared for a fixed time horizon $T$, Figure 7 shows the allocation coefficient at a fixed time, in the penultimate period, in different models with varying maturity date $T$. The purpose is to show the effect of actuarial fairness over an increasingly long period of time. In each model with a fixed maturity date $T$, the weights $\boldsymbol{\Omega}$ are determined such that the optimal risk-sharing allocation satisfies the fairness condition (10).

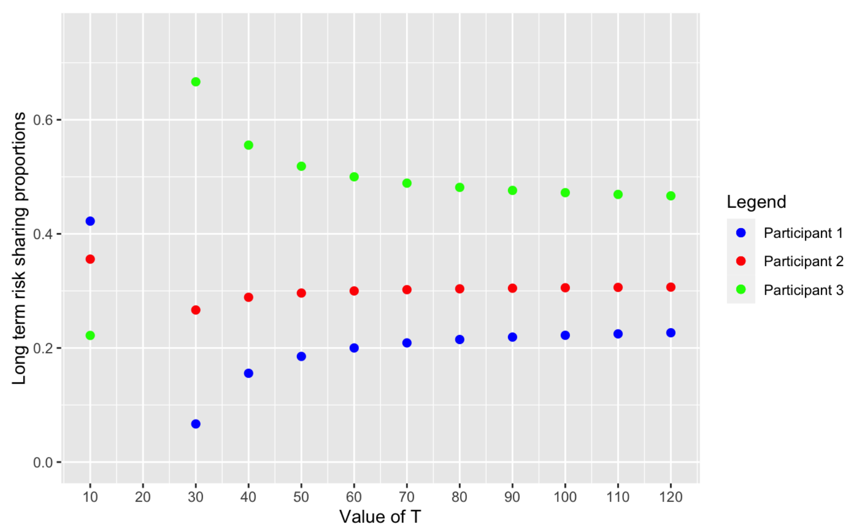


Figure 7: Fair Pareto-optimal risk-sharing allocations, $\alpha_{T-1,i,\cdot}$, that are recipient indifferent, as function of the terminal period $T$, where $i = 1, 2, 3$. That is, weight matrix $\boldsymbol{\Omega}$ is such that (10) is satisfied. The blue dots are $\alpha_{T-1,1,\cdot}$, the red dots are $\alpha_{T-1,2,\cdot}$ and the green dots are $\alpha_{T-1,3,\cdot}$. Note that mean of losses are around 20 times smaller than reserves, that is why to see the convergence effect after time $T = 20$. The allocations at time $T = 20$ are outside the range of $\alpha_{T-1,i,\cdot}$ that is selected on the y-axis.

The fair Pareto-optimal risk-sharing allocations are obtained in Section 4.1, and we next illustrate them for different time horizons, $T = 10, 20, \ldots, 110, 120$. Note that for each $T$ the weights $\Omega$ are re-calibrated to meet the fairness condition. We then derive the value of $\mathbf{A}_{T-1}$. In particular, we show the values of $\alpha_{T-1,i,1}$ for the three participants $i = 1, 2, 3$. We find in Figure 7 that for large values of $T$ the solution matrix converges to the proportion of expected losses. The choice of the cash recipient does not matter much, as coefficients $\alpha_{T-1,i,j}$ for $j = 1,2,3$ all converge to the same value. Observe that allocation coefficients for participants all converge to their respective ratios of expected losses over the expected loss of the entire group. That is, $\mathbf{A}_t \longrightarrow \frac{1}{\mathbf{e}'\mathbf{Me}}\mathbf{Mee}'$ when $t, T \to \infty$, which means the coefficient $\alpha_{T-1,1,1}$ converges to $\mu_1/(\sum_{i=1}^{3} \mu_i)$, the coefficient $\alpha_{T-1,2,1}$ converges to $\mu_2/(\sum_{i=1}^{3} \mu_i)$, and the coefficient $\alpha_{T-1,3,1}$ converges to $\mu_3/(\sum_{i=1}^{3} \mu_i)$. Participant 1 with the smallest expected loss is expected to pay the least compared with Participant 2, who is expected to pay less than Participant 3. Starting from $T = 50$, Participant 3 bears almost 0.44 of the total loss since this participant's expected loss is about 44 percent of the expected aggregate loss of the group. Participant 1 bears around 0.24 of the total loss and finally Participant 2 bears around 0.32 of the total loss. These results are consistent with (11).

One should note that Figure 7 is also closely related to Figures 3-5. Recall that as time goes by, the risk-sharing allocations converge to a long-term allocation $\mathbf{L}$ as discussed in Section 5.1.

# 6 Conclusion

Risk-sharing schemes have become increasingly common for managing catastrophic risks. However, most of the analysis on risk sharing has been done for one-period models. This paper fills the gap in the literature on the study of the temporal effect of risk sharing in a multi-period setting. In addition, we also consider P2P risk sharing, which has only been studied recently in the literature. The classic literature typically studies aggregate risk sharing, where all risks are aggregated before they are distributed among the participants. The P2P design enables us to keep track of risk providers and risk takers in a network. Compensations are provided pairwise between participants without the need for the aggregation of losses.

We consider proportional risk-sharing schemes with the objective to maximize the quadratic utility of terminal reserves. This analysis draws an analogy with portfolio selection problems in quantitative finance. In contrast to portfolio selection problems which often study the cross holding of assets, P2P risk sharing focuses on the sharing of liabilities with zero-balance loss conservation. In particular, due to the peer-to-peer nature of cost allocation, the risk-sharing problem is an extension of the multi-period portfolio selection problem by dimension and complexity. This model bridges the gap between the portfolio selection and the risk-sharing literature. The models in this paper can be extended to non-proportional schemes, likely requiring numerical solutions.

This paper shows the optimal risk-sharing allocation in a multi-period setting. The structure of the solution contains a long-term strategy and two correction factors, one of which is affected by initial reserves and the other by expected terminal reserves. We show that over the long term and under various quadratic utility objectives, optimal risk-sharing allocations always converge to recipient-indifferent risk-sharing allocations. Actuarial fairness is a critical concept for avoiding adverse selection when participants with different risk profiles participate in a heterogeneous pool. The impact of actuarial fairness is considered under various settings in the paper.

While the focus of this paper is on a benevolent social planner that optimizes a weighted sum of quadratic utilities, we suggest studying alternative preference measures in future work.One could further consider an extension of non-linear risk-sharing schemes in a multi-period setting, in which risk sharing only takes place between the attachment point and the coverage limit for each participant. However, such a problem is considerably more sophisticated and one may need to resort to numerical methods for a solution. While this paper focuses on the economic interpretations of the long-term effect of risk sharing, we hope to address more general setups in future work. For example, while initial reserves are exogenous, losses are independent over time, and there is no further injection of funding in this paper, this work can be extended to include more dynamic cash flows.

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# Appendix A

## A.1 Proof of Theorem 3.1

Define a modified reserve as $\hat{\mathbf{R}}_t = \mathbf{R}_t - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}$. Then, the objective function can be written as

$$\mathrm{Tr}\left[-\boldsymbol{\Omega}\mathbb{E}\left[\hat{\mathbf{R}}_T\hat{\mathbf{R}}_T'\right]\right] + \frac{1}{4}\boldsymbol{\lambda}'\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda} = \mathrm{Tr}\left(\mathbb{E}\left[\boldsymbol{\lambda}\mathbf{R}_T\right] - \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_T\mathbf{R}_T'\right]\right)$$

and the augmented Lagrangian function as

$$\mathrm{L} = \mathrm{Tr}\left(-\boldsymbol{\Omega}\mathbb{E}\left[(\hat{\mathbf{R}}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T)(\hat{\mathbf{R}}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T)'\right] + \frac{1}{4}\boldsymbol{\lambda}'\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right) + \left(\mathbf{e}' - \mathbf{e}'\mathbf{A}_{T-1}\right)\boldsymbol{\nu}.$$

The following equations for optimality are derived by setting the partial derivatives of the Lagrangian with respect to $\mathbf{A}_{T-1}$ and $\boldsymbol{\nu}$ to zero.

$$\begin{aligned} 2\boldsymbol{\Omega}\hat{\mathbf{R}}_{T-1}\mathbb{E}[\mathbf{X}_T'] - 2\boldsymbol{\Omega}\mathbf{A}_{T-1}\mathbb{E}[\mathbf{X}_T\mathbf{X}_T'] - \mathbf{e}\boldsymbol{\nu}' &= 0, \\ \mathbf{e}'\mathbf{A}_{T-1} &= \mathbf{e}'. \end{aligned}$$

Note that taking the second derivatives of the objective function with respect to $\mathbf{A}_t$ for $t \in \{0, 1, \ldots, T-1\}$ gives the following Hessian matrix:

$$-2\mathbb{E}\left[\boldsymbol{\Omega} \otimes \left(\mathbf{X_T}\mathbf{X_T}'\right)\right].$$

This matrix is strictly negative definite, and hence the objective function is a concave function. From the equation of the first-order derivative w.r.t. $\mathbf{A}_{T-1}$ we have

$$\mathbf{A}_{T-1} = \hat{\mathbf{R}}_{T-1}\boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1} - \frac{1}{2}\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\boldsymbol{\Xi}_T^{-1}.$$

Substituting this equation of $\mathbf{A}_{T-1}$ into the row-sum constraint gives

$$\mathbf{e}'\hat{\mathbf{R}}_{T-1}\boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1} - \frac{\mathbf{e}'}{2}\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\boldsymbol{\Xi}_T^{-1} = \mathbf{e}'.$$

Note that

$$\theta = \mathbf{e}'\boldsymbol{\Omega}^{-1}\mathbf{e} = \sum_{i=1}^{n}\frac{1}{\omega_i}.$$

Hence,

$$\begin{aligned}
\frac{\theta}{2}\boldsymbol{\nu}'\boldsymbol{\Xi}_T^{-1} &= \mathbf{e}'\hat{\mathbf{R}}_{T-1}\boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1} - \mathbf{e}', \\
\frac{\boldsymbol{\nu}'}{2}\boldsymbol{\Xi}_T^{-1} &= \frac{\mathbf{e}'}{\theta}\hat{\mathbf{R}}_{T-1}\boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1} - \frac{\mathbf{e}'}{\theta}.
\end{aligned}$$

Then, equation for $\mathbf{A}_{T-1}$ is given as

$$\mathbf{A}_{T-1} = \left[\hat{\mathbf{R}}_{T-1} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_{T-1}\right]\boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}').$$

Note that

$$\hat{\mathbf{R}}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T = (1-N_T)\hat{\mathbf{R}}_{T-1} + \frac{N_T}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_{T-1} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{X}_T = (1-N_T)\hat{\mathbf{R}}_{T-1} + \mathbf{D}_T,$$

where $N_T$ is the following 1-dimensional random variable:

$$N_T := \boldsymbol{\mu}_T'\boldsymbol{\Xi}_T^{-1}\mathbf{X}_T,$$

and $\mathbf{D}_T$ is the following $n \times 1$ random vector

$$\mathbf{D}_T := \frac{N_T}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_{T-1} - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{X}_T.$$

Note that $\mathbf{D}_T$ does not depend on $\mathbf{A}_{T-2}$, because $(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_{T-1} = (\sum_{i=1}^n \hat{R}_{T-1,i})\mathbf{e} = (\sum_{i=1}^n \hat{R}_{T-2,i} - \sum_{i=1}^n X_{T-1,i})\mathbf{e}$. However, $\mathbf{D}_T$ depends on both $\mathbf{X}_T$ and $\mathbf{X}_{T-1}$. Then, the objective function can be written as

$$\begin{aligned}
& \operatorname{Tr}\left(-\boldsymbol{\Omega}\mathbb{E}\left[[(1-N_T)\hat{\mathbf{R}}_{T-1} + \mathbf{D}_T][(1-N_T)\hat{\mathbf{R}}_{T-1} + \mathbf{D}_T]'\right]\right) \\
=\; & \operatorname{Tr}\left(-\boldsymbol{\Omega}\mathbb{E}\left[(1-N_T)^2\hat{\mathbf{R}}_{T-1}\hat{\mathbf{R}}_{T-1}' + 2(1-N_T)\mathbf{D}_T\hat{\mathbf{R}}_{T-1}' + \mathbf{D}_T\mathbf{D}_T'\right]\right) \\
=\; & \operatorname{Tr}\left(-\boldsymbol{\Omega}_{T-1}\mathbb{E}\left[\hat{\mathbf{R}}_{T-1}\hat{\mathbf{R}}_{T-1}'\right] - \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{D}_T\mathbf{D}_T'\right] - 2\mathbb{E}\left[\boldsymbol{\Omega}(1-N_T)\mathbf{D}_T\hat{\mathbf{R}}_{T-1}'\right]\right),
\end{aligned}$$

Note that the second and the third terms don't depend on $\mathbf{A}_{T-2}$. Since

$$\boldsymbol{\Omega}\mathbf{D}_T = \frac{1}{\theta}\left(N_T\mathbf{e}'\hat{\mathbf{R}}_{T-1} - \mathbf{e}'\mathbf{X}_T\right)\mathbf{e} = C_T\mathbf{e},$$

where $C_T$ is a scalar random variable that does not depend on $\mathbf{A}_{T-2}$. The third term does not depend on $\mathbf{A}_{T-2}$, because $\mathbf{e}'\hat{\mathbf{R}}_{T-1}$ does not have $\mathbf{A}_{T-2}$ term as mentioned earlier, i.e.

$$\mathrm{Tr}\left(\mathbb{E}\left[(1-N_T)C_T\mathbf{e}\hat{\mathbf{R}}'_{T-1}\right]\right) = \mathbb{E}\left[(1-N_T)C_T\mathbf{e}'\hat{\mathbf{R}}_{T-1}\right].$$

and thus the objective function at time $T-1$ has the same form as the one at time $T$, but with

$$\boldsymbol{\Omega}_{T-1} := \boldsymbol{\Omega}\mathbb{E}[1-N_T] = \boldsymbol{\Omega}\mathbb{E}[(1-N_T)^2].$$

The last equality follows from

$$\mathbb{E}[N_T^2] = \mathbb{E}\left[\boldsymbol{\mu}'_T\boldsymbol{\Xi}_T^{-1}\mathbf{X}_T\mathbf{X}'_T\boldsymbol{\Xi}_T^{-1}\boldsymbol{\mu}_T\right] = \mathbb{E}[N_T].$$

Thus, it follows that at time $T-1$ the objective function has the following form

$$\mathrm{Tr}\left[-\boldsymbol{\Omega}_{T-1}\mathbb{E}\left[\hat{\mathbf{R}}_{T-1}\hat{\mathbf{R}}'_{T-1}\right]\right] + \mathrm{Const.}$$

Therefore, the equation for $\mathbf{A}_{T-2}$ is

$$\mathbf{A}_{T-2} \quad = \quad \left[\hat{\mathbf{R}}_{T-2} - \frac{1}{\theta_{T-1}}\boldsymbol{\Omega}_{T-1}^{-1}(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_{T-2}\right]\boldsymbol{\mu}'_{T-1}\boldsymbol{\Xi}_{T-1}^{-1} + \frac{\boldsymbol{\Omega}_{T-1}^{-1}}{\theta_{T-1}}(\mathbf{e}\mathbf{e}').$$

The solution for any $t \in \{0,1,\ldots,T-1\}$ is given by

$$\mathbf{A}_t \quad = \quad \left[\hat{\mathbf{R}}_t - \frac{1}{\theta_{t+1}}\boldsymbol{\Omega}_{t+1}^{-1}(\mathbf{e}\mathbf{e}')\hat{\mathbf{R}}_t\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}_{t+1}^{-1} + \frac{\boldsymbol{\Omega}_{t+1}^{-1}}{\theta_{t+1}}(\mathbf{e}\mathbf{e}'),$$

where, for $t = T-1, T-2, \ldots, 1$,

$$\boldsymbol{\Omega}_t = \boldsymbol{\Omega}\prod_{j=t+1}^{T}\mathbb{E}\left([1-N_j]^2\right)$$

Now if we replace $\hat{\mathbf{R}}_t = \mathbf{R}_t - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}$ and write the solution matrix in terms of $\mathbf{R}_t$, the solution matrix $\mathbf{A}_t$ can be written as

$$\mathbf{A}_t = \left[\mathbf{R}_t - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_t\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}_{t+1}^{-1} + \frac{1}{2}\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}_{t+1}^{-1} + \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}.$$

Please note that

$$\boldsymbol{\Omega}_{t+1}^{-1} = \frac{1}{\prod_{j=t+2}^{T}\mathbb{E}[1-N_j]}\boldsymbol{\Omega}^{-1},$$

$$\theta_{t+1} = \mathbf{e}'\boldsymbol{\Omega}_{t+1}^{-1}\mathbf{e} = \mathbf{e}'\frac{1}{\prod_{j=t+2}^{T}\mathbb{E}[1-N_j]}\boldsymbol{\Omega}^{-1}\mathbf{e}.$$

**Remark**: Initially, the authors proved the result using $\mathbf{R}_t$. However, there was an excellent comment from an independent reviewer regarding the proof. The reviewer suggested to consider a reserve $\hat{\mathbf{R}}_t = \mathbf{R}_t - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}$.

Then, to observe that the objective function is simplified to

$$-\max_{\mathbf{A}_t}\mathbb{E}\left[\mathbf{\Omega}\hat{\mathbf{R}}_t\hat{\mathbf{R}}_t'\right]=\max_{\mathbf{A}_t}\mathrm{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_T'\right]-\mathbf{\Omega}\mathbb{E}\left[\mathbf{R}_T\mathbf{R}_T'\right]\right)-\frac{1}{4}\boldsymbol{\lambda}'\mathbf{\Omega}^{-1}\boldsymbol{\lambda}.$$

As a result of the suggestion, the proof became more concise and elegant.

## A.2 Proof of Proposition 1

Now if $\boldsymbol{\lambda}\propto\mathbf{e}$, then

$$\left[\frac{\phi}{\theta}\mathbf{\Omega}^{-1}\mathbf{e}-\mathbf{\Omega}^{-1}\boldsymbol{\lambda}\right]=\mathbf{0}.$$

Thus, $\mathbf{E}_t=\mathbf{0}$.

## A.3 Proof of Proposition 2

From Proposition 1 we have that if $\boldsymbol{\lambda}\propto\mathbf{e}$, then $\mathbf{E}_s=0$. Also $\frac{R_{s,i}}{R_{s,j}}=\frac{\omega_j}{\omega_i}$, for all $i,j=1,2,\dots,n$, implies that $\mathbf{C}_s=0$. Thus, the solution matrix $\mathbf{A}_s=\mathbf{L}_s$ and $\mathbf{R}_{s+1}=\mathbf{R}_s-\mathbf{L}_s\mathbf{X}_{s+1}$.

$$\begin{aligned}\mathbf{C}_{s+1} &= \left[\mathbf{I}_n-\frac{1}{\theta_{s+2}}\mathbf{\Omega}_{s+2}^{-1}(\mathbf{ee}')\right]\mathbf{R}_{s+1}\boldsymbol{\mu}_{s+1}'\mathbf{\Xi}_{s+1}^{-1}=\left[\mathbf{I}_n-\frac{1}{\theta_{s+2}}\mathbf{\Omega}_{s+2}^{-1}(\mathbf{ee}')\right](\mathbf{R}_s-\mathbf{L}_s\mathbf{X}_{s+1})\boldsymbol{\mu}_{s+1}'\mathbf{\Xi}_{s+1}^{-1}\\ &= -\left[\mathbf{I}_n-\frac{1}{\theta_{s+2}}\mathbf{\Omega}_{s+2}^{-1}(\mathbf{ee}')\right]\mathbf{L}_s\mathbf{X}_{s+1}\boldsymbol{\mu}_{s+1}'\mathbf{\Xi}_{s+1}^{-1}=-\mathbf{L}_s\mathbf{X}_{s+1}\boldsymbol{\mu}_{s+1}'\mathbf{\Xi}_{s+1}^{-1}+\frac{1}{\theta_{s+2}}\mathbf{\Omega}_{s+2}^{-1}(\mathbf{ee}')\mathbf{X}_{s+1}\boldsymbol{\mu}_{s+1}'\mathbf{\Xi}_{s+1}^{-1}=\mathbf{0}.\end{aligned}$$

For the second last equality we used $\mathbf{e}'\mathbf{L}_s=\mathbf{e}'$ and $\mathbf{L}_s=\frac{\mathbf{\Omega}_{s+2}^{-1}\mathbf{ee}'}{\theta_{s+2}}=\frac{\mathbf{\Omega}^{-1}(\mathbf{ee}')}{\theta}$.

## A.4 Proof of Theorem 3.2

The solution matrix $\mathbf{A}_t$ from Theorem 3.1 is

$$\mathbf{A}_t = \left[\mathbf{R}_t-\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}')\mathbf{R}_t\right]\boldsymbol{\mu}_{t+1}'\mathbf{\Xi}_{t+1}^{-1}+\frac{1}{2}\left[\frac{\phi}{\theta}\mathbf{\Omega}^{-1}\mathbf{e}-\mathbf{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}_{t+1}'\mathbf{\Xi}_{t+1}^{-1}+\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}'),$$

where

$$\phi=\mathbf{e}'\mathbf{\Omega}^{-1}\boldsymbol{\lambda}=\sum_{i=1}^{n}\frac{\lambda_i}{\omega_i}.$$

Let us rewrite the solution matrix in terms of $\hat{\mathbf{R}}_t$, defined in the proof of the Theorem 3.1. That is,

$$\mathbf{A}_t = \left[\hat{\mathbf{R}}_t-\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}')\hat{\mathbf{R}}_t\right]\boldsymbol{\mu}_{t+1}'\mathbf{\Xi}_{t+1}^{-1}+\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}')=\left(\mathbf{I}-\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}')\right)\hat{\mathbf{R}}_t\boldsymbol{\mu}_{t+1}'\mathbf{\Xi}_{t+1}^{-1}+\frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{ee}'),$$

where $\mathbf{I}$ is an $n \times n$ identity matrix. Now let us work on the first term of the solution matrix. Use the constraints $\hat{\mathbf{R}}_t = \hat{\mathbf{R}}_{t-1} - \mathbf{A}_{t-1}\mathbf{X}_t$ and $\mathbf{e}'\mathbf{A}_{t-1} = \mathbf{e}'$. Then, the term can be written as

$$\begin{aligned}
&\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_t\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1} = \left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\left(\hat{\mathbf{R}}_{t-1} - \mathbf{A}_{t-1}\mathbf{X}_t\right)\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1} \\
= &\left[\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_{t-1} - \mathbf{A_{t-1}X_t} + \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\mathbf{X}_t\right]\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1} \\
= &\left[\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_{t-1} - \left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_{t-1}N_t - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\mathbf{X}_t + \frac{\mathbf{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{X}_t\right]\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1} \\
= &\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_{t-1}(1-N_t)\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1}
\end{aligned}$$

Repeating the above steps will result in second term of the solution matrix being written down as

$$\prod_{i=1}^{t}(1-N_i)\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\right)\hat{\mathbf{R}}_0\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1}.$$

Hence, the matrix $\mathbf{A}_t$ becomes

$$\mathbf{A}_t = \prod_{i=1}^{t}(1-N_i)\left(\mathbf{I} - \frac{\mathbf{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\right)\hat{\mathbf{R}}_0\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}^{-1}_{t+1} + \frac{\mathbf{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'.$$

The expectation and variance of $N_t$ are given by:

$$\mu_{N_t} = \mathbb{E}[N_t] = \boldsymbol{\mu}'_t\mathbf{\Xi}^{-1}_t\boldsymbol{\mu}_t,$$

$$\begin{aligned}
\operatorname{Var}(N_t) &= \boldsymbol{\mu}'_t\mathbf{\Xi}^{-1}_t\operatorname{Var}(\mathbf{X_t})\mathbf{\Xi}^{-1}_t\boldsymbol{\mu}_t = \boldsymbol{\mu}'_t\mathbf{\Xi}^{-1}_t\left(\mathbf{\Xi}_t - \boldsymbol{\mu}_t\boldsymbol{\mu}'_t\right)\mathbf{\Xi}^{-1}_t\boldsymbol{\mu}_t \\
&= \left(\boldsymbol{\mu}'_t - \boldsymbol{\mu}'_t\mathbf{\Xi}^{-1}_t\boldsymbol{\mu}_t\boldsymbol{\mu}'_t\right)\mathbf{\Xi}^{-1}_t\boldsymbol{\mu}_t = \mu_{N_t} - \mu^2_{N_t}.
\end{aligned}$$

Since the variance is non-negative, we must have $0 < \mu_{N_t} \leq 1$. Particularly, it holds that $0 \leq \operatorname{Var}(N_t) \leq 0.25$. In order to find the expectation and the variance of $\prod_{i=1}^{t}(1-N_i)$ we will use the independence of losses over time. Thus,

$$\mathbb{E}\left[\prod_{i=1}^{t}(1-N_i)\right] = \prod_{i=1}^{t}\mathbb{E}(1-N_i) = \prod_{i=1}^{t}(1-\mu_{N_i}).$$

$$\begin{aligned}
\operatorname{Var}\left[\prod_{i=1}^{t}(1-N_i)\right] &= \mathbb{E}\left[(1-N_1)^2(1-N_2)^2\cdots(1-N_t)^2\right] - \left(\mathbb{E}\left[(1-N_1)(1-N_2)\cdots(1-N_t)\right]\right)^2 \\
&= \mathbb{E}\left[(1-N_1)^2\right]\mathbb{E}\left[(1-N_2)^2\right]\cdots\mathbb{E}\left[(1-N_t)^2\right] - \mathbb{E}^2\left[(1-N_1)\right]\mathbb{E}^2\left[(1-N_2)\right]\cdots\mathbb{E}^2\left[(1-N_t)\right] \\
&= \mathbb{E}\left[1-N_1\right]\mathbb{E}\left[1-N_2\right]\cdots\mathbb{E}\left[1-N_t\right] - \mathbb{E}^2\left[(1-N_1)\right]\mathbb{E}^2\left[(1-N_2)\right]\cdots\mathbb{E}^2\left[(1-N_t)\right] \\
&= \prod_{i=1}^{t}(1-\mu_{N_i}) - \prod_{i=1}^{t}(1-\mu_{N_i})^2.
\end{aligned}$$

Note that $0 \leq 1-\mu_{N_t} < 1$ for $t = 1,2,\ldots,T$ and also $\sum_{i=1}^{\infty} \mu_{N_i} = \infty$, from Rudin (1986) Theorem 15.5. The latter is true because losses for a given participant have identical distributions over time, i.e $X_{t,i} \stackrel{d}{=} X_{t+1,i}$ for $t \in \{1,2,\ldots,T\}$. Thus, both the expectation and variance of $\prod_{i=1}^{t}(1-N_i)$ approach zero as $t,T \longrightarrow \infty$. Therefore,

$$\prod_{i=1}^{t}(1-N_i) \longrightarrow 0 \quad \text{in probability as} \quad t,T \longrightarrow \infty,$$

which implies that

$$\mathbf{A}_t \longrightarrow \frac{1}{\theta}\mathbf{\Omega}^{-1}(\mathbf{e}\mathbf{e}') \quad \text{in probability s} \quad t,T \longrightarrow \infty.$$

**Remark**: The proof of the theorem has been updated and corrected after the independent reviewer's excellent comment.

## A.5 Proof of Theorem 3.3

From (2), we can write the time-$t$ reserve for participant $i$ as

$$R_{t,i} = R_{0,i} - \sum_{k=0}^{t-1} \alpha_{k,i,\cdot}\mathbf{X}_{k+1},$$

where $\alpha_{k,i,\cdot}$ is the $i$-th row of matrix $\mathbf{A}_k$ at time $k$. Then, with $\mathbf{A}_k$ as in the proof of Theorem 3.2, we get

$$\begin{aligned}\sum_{k=0}^{t-1} \alpha_{k,i,\cdot}\mathbf{X}_{k+1} &= \frac{1}{\omega_i\theta}\sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1} + \sum_{k=0}^{t-1} N_{k+1}\left(\prod_{m=1}^{k}(1-N_m)\right)\left(R_{0,i} - \frac{1}{\omega_i\theta}\sum_{j=1}^{n} R_{0,j}\right) \\ &+ \frac{1}{2}\sum_{k=0}^{t-1} N_{k+1}\left(\prod_{m=1}^{k}(1-N_m)\right)\left(\frac{1}{\omega_i\theta}\sum_{j=1}^{n}\frac{\lambda_j}{\omega_j} - \frac{\lambda_i}{\omega_i}\right).\end{aligned}$$

Thus, the reserve at time $t$ can be written as

$$\begin{aligned}R_{t,i} &= R_{0,i} - \frac{1}{\omega_i\theta}\sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1} - \sum_{k=0}^{t-1} N_{k+1}\left(\prod_{m=1}^{k}(1-N_m)\right)\left(R_{0,i} - \frac{1}{\omega_i\theta}\sum_{j=1}^{n} R_{0,j}\right) \\ &- \frac{1}{2}\sum_{k=0}^{t-1} N_{k+1}\left(\prod_{m=1}^{k}(1-N_m)\right)\left(\frac{1}{\omega_i\theta}\sum_{j=1}^{n}\frac{\lambda_j}{\omega_j} - \frac{\lambda_i}{\omega_i}\right).\end{aligned}$$

For convenience let

$$B_t := \sum_{k=0}^{t-1} N_{k+1}\prod_{m=1}^{k}(1-N_m).$$

Then, one has

$$\begin{aligned}1-B_t &= 1 - \sum_{k=0}^{t-1} N_{k+1}\prod_{m=1}^{k}(1-N_m) = (1-N_1)(1-B_{t-1}) \\ &= (1-N_1)(1-N_2)(1-B_{t-2}) = \ldots = \prod_{k=1}^{t}(1-N_k)\end{aligned}$$

From the proof of the Theorem 3.2 we know that $\prod_{k=1}^{t}(1-N_k) \longrightarrow 0$ in probability as $t,T \longrightarrow \infty$. Thus, $B_t \longrightarrow 1$ in probability as $t,T \longrightarrow \infty$. So, for large $t$ and $T$, the reserves will be

$$R_{t,i} \rightarrow \frac{1}{\omega_i \theta}\left(\sum_{j=1}^{n} R_{0,j} - \sum_{k=0}^{t-1} \mathbf{e}'\mathbf{X}_{k+1}\right) - \frac{1}{2}\left(\frac{1}{\omega_i \theta}\sum_{j=1}^{n}\frac{\lambda_j}{\omega_j} - \frac{\lambda_i}{\omega_i}\right).$$

Hence, ratio of $i$-th participant's reserve to $j$-th participant's reserve is

$$\frac{R_{t,i}}{R_{t,j}} \rightarrow \frac{\omega_j\left[\frac{1}{\theta}\left[\sum_{z=1}^{n} R_{0,z} - \sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1} - \frac{1}{2}\sum_{z=1}^{n}\frac{\lambda_z}{\omega_z}\right] + \frac{1}{2}\lambda_i\right]}{\omega_i\left[\frac{1}{\theta}\left[\sum_{z=1}^{n} R_{0,z} - \sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1} - \frac{1}{2}\sum_{z=1}^{n}\frac{\lambda_z}{\omega_z}\right] + \frac{1}{2}\lambda_j\right]}. \tag{15}$$

Furthermore, since we have $\boldsymbol{\lambda} \propto \mathbf{e}$, the above ratio of reserves simplifies to

$$\frac{R_{t,i}}{R_{t,j}} \rightarrow \frac{\omega_j}{\omega_i}.$$

Note that the only term that depend on time $t$ in (15) is $\sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1}$. Hence, the behavior of the above ratio, as $t \longrightarrow \infty$, is determined by the behavior of $\sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1}$ as $t \longrightarrow \infty$. If for $t = 1,\ldots T$, one has

$$\mathbb{E}\left[\mathbf{e}'\mathbf{X_t}\right] \neq 0,$$

then, by the law of large numbers, we get

$$\sum_{k=0}^{t-1}\mathbf{e}'\mathbf{X}_{k+1} \longrightarrow \infty.$$

That is, the expressions in square brackets in (15), both in numerator and denominator, will decrease at the same rate. Hence, the same result as that of the theorem follows.

## A.6 Proof of Theorem 3.6

The objective function can be written as

$$\max \operatorname{Tr}\left(\mathbb{E}\left[\boldsymbol{\lambda}(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)'\right] - \mathbb{E}\left[\boldsymbol{\Omega}(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)'\right]\right).$$

The augmented objective function is

$$\mathrm{L} = \operatorname{Tr}\left(\mathbb{E}\left[\boldsymbol{\lambda}(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)'\right] - \mathbb{E}\left[\boldsymbol{\Omega}(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)(\mathbf{R}_0 - \mathbf{A}\sum_{t=1}^{T}\mathbf{X}_t)'\right]\right) + \left(\mathbf{e}' - \mathbf{e}'\mathbf{A}\right)\boldsymbol{\nu}.$$

Let $\mathbf{Z} := \sum_{t=1}^{T}\mathbf{X}_t$. Then, taking a partial derivative of L with respect to $\mathbf{A}$ and $\boldsymbol{\nu}'$ gives

$$\begin{aligned} 2\boldsymbol{\Omega}\mathbf{R}_0\mathbb{E}[\mathbf{Z}'] - 2\boldsymbol{\Omega}\mathbf{A}\mathbb{E}[\mathbf{Z}\mathbf{Z}'] - \boldsymbol{\lambda}\mathbb{E}[\mathbf{Z}'] - \mathbf{e}\boldsymbol{\nu}' &= 0, \\ \mathbf{e}'\mathbf{A} &= \mathbf{e}'. \end{aligned}$$

Let $\mathbf{M} := \mathbb{E}[\mathbf{Z}\mathbf{Z}']$. From the first equation we obtain

$$\mathbf{A} = \left[\mathbf{R}_0 - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\mathbb{E}[\mathbf{Z}']\mathbf{M}^{-1} - \frac{1}{2}\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\mathbf{M}^{-1}.$$

Substituting the equation for $\mathbf{A}$ into the column-sum constraint yields

$$\mathbf{e}'\left[\mathbf{R}_0 - \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\mathbb{E}[\mathbf{Z}']\mathbf{M}^{-1} - \frac{\mathbf{e}'}{2}\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\mathbf{M}^{-1} = \mathbf{e}'.$$

Define

$$\theta := \mathbf{e}'\boldsymbol{\Omega}^{-1}\mathbf{e} = \sum_{i=1}^{n}\frac{1}{\omega_i} \quad \text{and} \quad \phi := \mathbf{e}'\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda} = \sum_{i=1}^{n}\frac{\lambda_i}{\omega_i}.$$

Then,

$$\nu' = \frac{2}{\theta}\mathbf{e}'\mathbf{R}_0\mathbb{E}[\mathbf{Z}'] - \frac{\phi}{\theta}\mathbb{E}[\mathbf{Z}'] - \frac{2\mathbf{e}'}{\theta}\mathbf{M}.$$

Substituting the above equation into the equation of $\mathbf{A}$ yields

$$\begin{aligned}\mathbf{A} &= \left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right]\mathbb{E}[\mathbf{Z}']\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}'] \\ &+ \frac{1}{2}\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\mathbb{E}[\mathbf{Z}']\mathbb{E}^{-1}[\mathbf{Z}\mathbf{Z}'] + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}').\end{aligned}$$

## A.7 Proof of Theorem 3.7

For a given $\mathbf{R}_{T-1}$ the objective function can be written as

$$\begin{aligned}&\mathrm{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_T'\right] - \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_T\mathbf{R}_T'\right] + \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_T\right]\mathbb{E}\left[\mathbf{R}_T'\right]\right) \\ &= \mathrm{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]' - \boldsymbol{\Omega}\mathbb{E}\left[\left(\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right)\left(\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right)'\right]\right. \\ &\qquad \left. + \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]'\right).\end{aligned}$$

The augmented Lagrangian function is then

$$\begin{aligned}\mathrm{L} &= \mathrm{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]' - \boldsymbol{\Omega}\mathbb{E}\left[\left(\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right)\left(\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right)'\right] + \right. \\ &\quad \left. \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]\mathbb{E}\left[\mathbf{R}_{T-1} - \mathbf{A}_{T-1}\mathbf{X}_T\right]'\right) + \left(\mathbf{e}' - \mathbf{e}'\mathbf{A}_{T-1}\right)\boldsymbol{\nu}.\end{aligned}$$

By taking the partial derivatives of L with respect to $\mathbf{A}_{T-1}$ and $\boldsymbol{\nu}$, one has

$$\begin{aligned}\frac{\partial \mathrm{L}}{\partial \mathbf{A}_{T-1}} &= \boldsymbol{\lambda}\mathbb{E}\left[\mathbf{X}_T'\right] - 2\boldsymbol{\Omega}\mathbf{A}_{T-1}\mathrm{Cov}(\mathbf{X}_T) - \mathbf{e}\boldsymbol{\nu}' = 0, \\ \frac{\partial \mathrm{L}}{\partial \boldsymbol{\nu}} &= \mathbf{e}'\mathbf{A}_{T-1} - \mathbf{e}' = 0.\end{aligned}$$

From $\frac{\partial \mathrm{L}}{\partial \mathbf{A}_{T-1}} = 0$ we get

$$\mathbf{A}_{T-1} = \frac{1}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\boldsymbol{\mu}_T'\boldsymbol{\Sigma}_T^{-1} + \frac{1}{2}\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\boldsymbol{\Sigma}_T^{-1},$$

where $\boldsymbol{\mu}_T = \mathbb{E}[\mathbf{X}_T]$ and $\boldsymbol{\Sigma}_T = \mathrm{Cov}(\mathbf{X}_T)$. Substituting the above expression of $\mathbf{A}_{T-1}$ in $\frac{\partial \mathrm{L}}{\partial \boldsymbol{\nu}} = 0$ gives

$$\mathbf{e}' = \frac{\mathbf{e}'\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\boldsymbol{\mu}_T'\boldsymbol{\Sigma}_T^{-1}}{2} + \frac{\mathbf{e}'\boldsymbol{\Omega}^{-1}\mathbf{e}\boldsymbol{\nu}'\boldsymbol{\Sigma}_T^{-1}}{2} = \frac{\phi}{2}\boldsymbol{\mu}_T'\boldsymbol{\Sigma}_T^{-1} + \frac{\theta}{2}\boldsymbol{\nu}'\boldsymbol{\Sigma}_T^{-1}.$$

Hence, we have

$$\frac{\boldsymbol{\nu}'}{2}\boldsymbol{\Sigma}_T^{-1} = \frac{\mathbf{e}'}{\theta} - \frac{\phi}{2\theta}\boldsymbol{\mu}_T'\boldsymbol{\Sigma}_T^{-1},$$

and consequently the solution matrix is

$$\mathbf{A}_{T-1} = \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta} + \frac{1}{2}\left(\mathbf{I} - \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}\right)\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\boldsymbol{\mu}_T'\boldsymbol{\Sigma}_T^{-1}.$$

In order to find $\mathbf{A}_{T-2}$ we substitute $\mathbf{A}_{T-1}$ in the objective function and then use $\mathbf{R}_{T-1} = \mathbf{R}_{T-2} - \mathbf{A}_{T-2}\mathbf{X}_{T-1}$. Note that $\mathbf{A}_{T-1}$ does not depend on $\mathbf{R}_{T-1}$, therefore, it does not depend on $\mathbf{A}_{T-2}$. The objective function becomes

$$\mathrm{Tr}\left(\boldsymbol{\lambda}\mathbb{E}\left[\mathbf{R}_{T-1}'\right] - \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_{T-1}\mathbf{R}_{T-1}'\right] + \boldsymbol{\Omega}\mathbb{E}\left[\mathbf{R}_{T-1}\right]\mathbb{E}\left[\mathbf{R}_{T-1}'\right]\right) + \mathrm{const}.$$

Since the above objective function has the same form as the previous one, we conclude that $\mathbf{A}_{T-2} = \mathbf{A}_{T-1}$. Therefore, we conclude that for $t \in \{0, 1, \ldots, T-1\}$ the solution is

$$\mathbf{A}_t = \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta} + \frac{1}{2}\left(\mathbf{I} - \frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}\right)\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\boldsymbol{\mu}_{t+1}'\boldsymbol{\Sigma}_{t+1}^{-1}.$$

This concludes the proof.

## A.8 Proof of Theorem 4.1

From the proof of Theorem 3.2 we get

$$\mathbf{A}_t = \left(\prod_{i=1}^{t}(1-N_i)\right)\left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right]\boldsymbol{\mu}_{t+1}'\boldsymbol{\Xi}_{t+1}^{-1} + \frac{1}{2}\left(\prod_{i=1}^{t}(1-N_i)\right)\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\lambda}\right]\boldsymbol{\mu}_{t+1}'\boldsymbol{\Xi}_{t+1}^{-1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}'), \tag{16}$$

Under the actuarial fairness condition (10), we obtain

$$\begin{aligned}\sum_{t=0}^{T-1}\mathbf{A}_t\mathbf{X}_{t+1} &= \left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] \\ &+ \frac{1}{2}\left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\lambda}\right] + \frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=1}^{T}\mathbf{X}_t.\end{aligned}$$

Then, the constraint equation (10) becomes

$$\mathbb{E}\left[\sum_{t=1}^{T} N_t \prod_{i=1}^{t-1}(1-N_i)\right]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] + \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} N_t \prod_{i=1}^{t-1}(1-N_i)\right]\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\lambda}\right]$$
$$+\frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=1}^{T}\mathbb{E}[\mathbf{X}_t] = \sum_{t=1}^{T}\mathbb{E}[\mathbf{X}_t] = T\mathbb{E}[\mathbf{X}_t].$$

Recall

$$\mathbb{E}[B_T] := \mathbb{E}\left[\sum_{t=1}^{T} N_t \prod_{i=1}^{t-1}(1-N_i)\right],$$

and thus we obtain that for $i, j = 1, \ldots, n$ with $i \neq j$:

$$\frac{\omega_i}{\omega_j} = \frac{T\mathbb{E}[\mathbf{X}_{t,j}] - \mathbb{E}[B_T]R_{0,j} + 0.5\mathbb{E}[B_T]\lambda_j/\omega_j}{T\mathbb{E}[\mathbf{X}_{t,i}] - \mathbb{E}[B_T]R_{0,i} + 0.5\mathbb{E}[B_T]\lambda_i/\omega_i}.$$

Note that the solution matrix $\mathbf{A}_t$ can be written as

$$\begin{aligned}
\mathbf{A}_t &= \left(\prod_{i=1}^{t}(1-N_i)\right)\left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{2}\left(\prod_{i=1}^{t}(1-N_i)\right)\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}') \\
&= \left(\prod_{i=1}^{t}(1-N_i)\right)\left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{2}\left(\prod_{i=1}^{t}(1-N_i)\right)\left[\frac{\phi}{\theta}\boldsymbol{\Omega}^{-1}\mathbf{e} - \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \\
&\quad \left(\frac{T\mathbb{E}[\mathbf{X}_{t,1}] - \mathbb{E}[B_T]R_{0,1} + 0.5B_T\lambda_1/\omega_1}{T\sum_{j=1}^{n}\mathbb{E}[\mathbf{X}_{t,j}] - \sum_{j=1}^{n}\mathbb{E}[B_T]R_{0,j} + 0.5\mathbb{E}[B_T]\sum_{j=1}^{n}\lambda_j/\omega_j}, \ldots, \right. \\
&\quad \left. \frac{T\mathbb{E}[\mathbf{X}_{t,n}] - \mathbb{E}[B_T]R_{0,n} + 0.5B_T\lambda_n/\omega_n}{T\sum_{j=1}^{n}\mathbb{E}[\mathbf{X}_{t,j}] - \sum_{j=1}^{n}\mathbb{E}[B_T]R_{0,j} + 0.5\mathbb{E}[B_T]\sum_{j=1}^{n}\lambda_j/\omega_n}\right)'\mathbf{e}'.
\end{aligned}$$

We have already shown in the proof of Theorem 3.2 that for i.i.d. losses and $t, T \longrightarrow \infty$, the first two terms of the matrix $\mathbf{A}_t$ go to zero. The third term reduces to

$$\left(\frac{\mathbb{E}[\mathbf{X}_1]}{\sum_{j=1}^{n}\mathbb{E}[\mathbf{X}_j]}, \ldots, \frac{\mathbb{E}[\mathbf{X}_n]}{\sum_{j=1}^{n}\mathbb{E}[\mathbf{X}_j]}\right)'\mathbf{e}'.$$

## A.9 Proof of Theorem 4.2

Let $\boldsymbol{\zeta} = \mathbb{E}\left[\mathbf{R}_0 - \sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right]$. The Lagrangian dual function can be written as

$$\begin{aligned}
L &= \mathrm{Tr}\{-\boldsymbol{\Omega}\mathbb{E}[\mathbf{R}_T\mathbf{R}'_T] + \text{constant}\} + (\mathbf{e}'\mathbf{A}_{T-1} - \mathbf{e}')\boldsymbol{\nu} + \boldsymbol{\gamma}'(\mathbb{E}[\mathbf{R}_T] - \boldsymbol{\zeta}) \\
&= \mathrm{Tr}\{\boldsymbol{\Gamma}\mathbf{e}(\mathbb{E}[\mathbf{R}_T] - \boldsymbol{\zeta})' - \boldsymbol{\Omega}\mathbb{E}[\mathbf{R}_T\mathbf{R}'_T]\} + (\mathbf{e}'\mathbf{A}_{T-1} - \mathbf{e}')\boldsymbol{\nu} + \text{constant},
\end{aligned}$$

where $\boldsymbol{\Gamma}$ is the diagonal matrix with the elements of $\boldsymbol{\gamma}$ on the diagonal. Note that the above Lagrangian dual function has the same form as the Lagrangian dual function in Theorem 3.1. Thus, using the proof of Theorem 3.2

we can write down the solution matrix as

$$\mathbf{A}_t = \left[\prod_{i=1}^{t}(1-N_i)\right]\left[\mathbf{R}_0 - \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}')\mathbf{R}_0\right]\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{2}\left[\prod_{i=1}^{t}(1-N_i)\right]\left[\frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \boldsymbol{\Omega}^{-1}\right]\boldsymbol{\Gamma}\mathbf{e}\boldsymbol{\mu}'_{t+1}\boldsymbol{\Xi}^{-1}_{t+1} + \frac{1}{\theta}\boldsymbol{\Omega}^{-1}(\mathbf{e}\mathbf{e}').$$

Then,

$$\begin{aligned}\sum_{t=0}^{T-1}\mathbf{A}_t\mathbf{X}_{t+1} &= \left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] \\ &+ \frac{1}{2}\left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \boldsymbol{\Omega}^{-1}\right]\boldsymbol{\Gamma}\mathbf{e} + \frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=0}^{T-1}\mathbf{X}_{t+1}.\end{aligned} \tag{17}$$

The constraint equation (10) can be written as

$$\mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{A}_t\mathbf{X}_{t+1}\right] = \mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right].$$

Then, using (17) we can write (10) as

$$\begin{aligned}&\mathbb{E}\left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] + \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right]\left[\frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \boldsymbol{\Omega}^{-1}\right]\boldsymbol{\Gamma}\mathbf{e} \\ &+\frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{X}_t\right] = \mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right].\end{aligned}$$

Recall

$$\mathbb{E}[B_T] := \mathbb{E}\left[\sum_{t=1}^{T}N_t\prod_{i=1}^{t-1}(1-N_i)\right],$$

so that it holds that

$$\mathbb{E}[B_T]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] + \frac{1}{2}\mathbb{E}[B_T]\left[\frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \boldsymbol{\Omega}^{-1}\right]\boldsymbol{\Gamma}\mathbf{e} + \frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=0}^{T-1}\mathbb{E}\left[\mathbf{X}_{t+1}\right] = \mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right],$$

and

$$\begin{aligned}\frac{\mathbb{E}[B_T]}{2}\left[\frac{\boldsymbol{\Omega}^{-1}\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \boldsymbol{\Omega}^{-1}\right]\boldsymbol{\Gamma}\mathbf{e} &= \mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right] - \frac{\boldsymbol{\Omega}^{-1}}{\theta}(\mathbf{e}\mathbf{e}')\sum_{t=0}^{T-1}\mathbb{E}\left[\mathbf{X}_{t+1}\right] - \mathbb{E}[B_T]\left[\mathbf{R}_0 - \frac{\boldsymbol{\Omega}^{-1}}{\theta}\mathbf{e}\mathbf{e}'\mathbf{R}_0\right] \\ \frac{\mathbb{E}[B_T]}{2}\left[\frac{\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \mathbf{I}\right]\boldsymbol{\Gamma}\mathbf{e} &= \boldsymbol{\Omega}\mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right] - \frac{(\mathbf{e}\mathbf{e}')}{\theta}\sum_{t=0}^{T-1}\mathbb{E}\left[\mathbf{X}_{t+1}\right] - \mathbb{E}[B_T]\left[\boldsymbol{\Omega}\mathbf{R}_0 - \frac{\mathbf{e}\mathbf{e}'}{\theta}\mathbf{R}_0\right] \\ \frac{\mathbb{E}[B_T]}{2}\left[\frac{\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \mathbf{I}\right]\boldsymbol{\Gamma}\mathbf{e} &= -\left[\frac{\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \mathbf{I}\right]\boldsymbol{\Omega}\mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right] + \mathbb{E}[B_T]\left[\frac{\mathbf{e}\mathbf{e}'\boldsymbol{\Omega}^{-1}}{\theta} - \mathbf{I}\right]\boldsymbol{\Omega}\mathbf{R}_0.\end{aligned}$$

Thus, the solution matrix $\mathbf{A}_t$ for $t \in \{0,1,\ldots,T-1\}$ is given by

$$\mathbf{A}_t = \frac{\prod_{i=1}^{t}(1-N_i)}{\mathbb{E}\left[\sum_{t=1}^{T} N_t \prod_{i=1}^{t-1}(1-N_i)\right]}\left[\mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right] - \frac{\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}\mathbb{E}\left[\sum_{t=0}^{T-1}\mathbf{X}_{t+1}\right]\right]\boldsymbol{\mu}'_{t+1}\mathbf{\Xi}_{t+1}^{-1} + \frac{\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}.$$

From Theorem 3.2, the solution matrix approaches $\frac{\mathbf{\Omega}^{-1}\mathbf{e}\mathbf{e}'}{\theta}$ as $t,T \longrightarrow \infty$. Hence, we conclude that in the limit the solution matrix $\mathbf{A}_t$ is not affected by the fairness constraint (10).