



# Implied correlation in the Lévy copula model<sup>1</sup>

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<sup>1</sup>Linders, D. & Schoutens, W. (2014) 'Basket option pricing and implied correlation in a Lévy copula model', Research report, AFI-1494, FEB KU Leuven.

# Outline

1. Introduction
2. The Lévy copula model
3. Approximate basket option pricing
4. Implied Lévy correlation

# Introduction

## The financial market

- The market is arbitrage-free and there exists a pricing measure  $\mathbb{Q}$ .
- The interest rate  $r$  is deterministic and constant.
- Price of stock  $j$  at time  $T$  is denoted by  $S_j(T)$ .
- The price of the basket at time  $T$  is denoted by  $S(T)$ :

$$S(T) = w_1 S_1(T) + \cdots + w_n S_n(T), \quad w_j \geq 0.$$

- The price of a **basket option** with strike  $K$  and maturity  $T$ :

$$C[K, T] = e^{-rT} \mathbb{E}_{\mathbb{Q}} [(S(T) - K)_+].$$

# Introduction

## Basket options

- Basket option pricing
  - ▶ Gaussian copula model
    - ★ Moment-matching: Brigo et al. (2004), Linders (2013).
    - ★ Upper and lower bounds: Dhaene et al. (2002), Deelstra et al. (2004), Carmona & Durrleman (2006).
    - ★ Geometric average: Korn & Zeytun (2013).
  - ▶ Elliptical copula model: Valdez, Dhaene, Maj & Vanduffel (2009).
  - ▶ Multivariate Variance Gamma model: Linders & Stassen (2014).
  - ▶ Jump diffusion model: Xu and Zheng (2014), Paletta et al. (2013).
  - ▶ Model-free upper bounds: Chen, Deelstra, Dhaene & Vanmaele (2008).
- We consider basket option pricing in a new multivariate stock price model and show how to calibrate this model using single-name and multi-name derivatives.

# Introduction

## Aim of the paper

- Lévy copula model

- ▶ Extend the Gaussian copula model to a Lévy setting.
- ▶ Based on the generic one-factor model introduced in Albrecher, Ladoucette & Schoutens (2007).

- Approximate basket option pricing

- ▶ Three-moments-matching approximation.
- ▶ Gaussian case: Brigo, Mercurio, Rapisarda & Scotti (2004).

- Implied Lévy correlation

- ▶ Index options are traded and their prices can be observed.
- ▶ Match the (approximate) model index option price with the market price
- ▶ The *Implied Lévy correlation smile* solves some issues of implied Gaussian correlation: Linders & Schoutens (2014a).

# The Lévy copula model

## Lévy processes

- Assume that  $L$  is *infinitely divisible* and has characteristic function  $\phi_L$ .
- Examples: Normal, Variance Gamma, NIG, Meixner, . . . .
- Define  $n + 1$  independent Lévy processes.
  - ▶  $X$  is a process based on the distribution  $L$ .
  - ▶  $X_1, X_2, \dots, X_n$  are independent processes based on  $L$ .
  - ▶ Characteristic function:

$$\mathbb{E} \left[ e^{iuX(t)} \right] = \mathbb{E} \left[ e^{iuX_j(t)} \right] = \phi_L(u)^t$$

- We assume that these are all standardized processes:
  - ▶  $\mathbb{E}[X(1)] = \mathbb{E}[X_j(1)] = 0$ ;
  - ▶  $\text{Var}[X(1)] = \text{Var}[X_j(1)] = 1$ .

# The Lévy copula model

## Building correlated Lévy processes

- The log return of stock  $j$  is modeled using the r.v.  $A_j$ :

$$A_j = X(\rho) + X_j(1 - \rho), \quad j = 1, 2, \dots, n,$$

- ▶  $\rho \in [0, 1]$ ,
  - ▶ log return = systematic component + idiosyncratic component
- Correlation:

- ▶ If  $i \neq j$  then

$$\text{Corr} [A_i, A_j] = \rho.$$

- ▶ The dependence is modeled by a single correlation parameter  $\rho$ .

# The Lévy copula model

## Properties

- $A_j$  is standardized:

$$\mathbb{E}[A_j] = 0, \quad \text{Var}[A_j] = 1.$$

- Distribution:

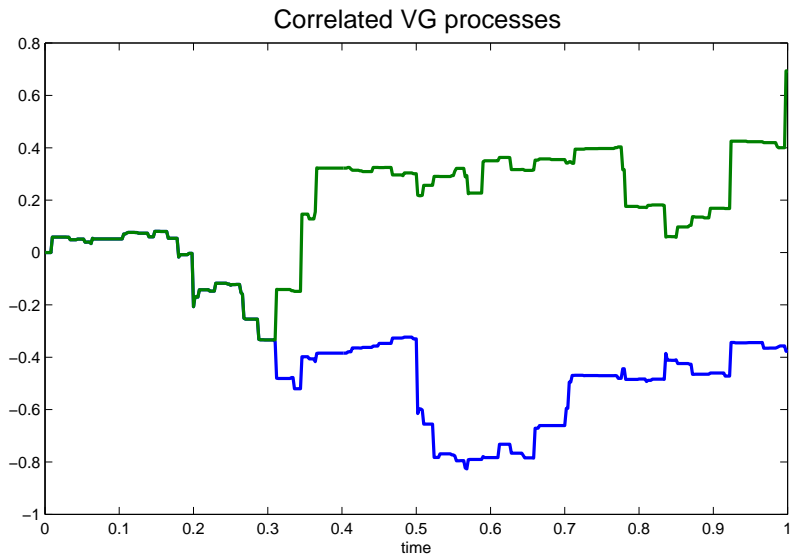
$$A_j \stackrel{d}{=} L.$$

- $A_j$  is independent from the time  $T$ .
- Conditioned on the systematic component, the stocks are independent.



# The Lévy copula model

## Correlated VG processes



# The Lévy copula model

## Stock price processes

- The Lévy copula model:

$$S_j(T) = S_j(0)e^{\mu_j T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \dots, n,$$

where  $\mu_j \in \mathbb{R}$  and  $\sigma > 0$ .

- If  $A_j$  is based on standard Brownian motions we find back the Gaussian copula case.
- $\sigma_j$  can be interpreted as the Lévy implied volatility; see also Corcuera, Guillaume, Leoni & Schoutens (2009).

# The Lévy copula model

## Risk-neutral stock price processes

- Choose  $\mu_j$  such that

$$\mathbb{E}[S_j(T)] = e^{(r-q_j)T} S_j(0).$$

- Risk-neutral dynamics:

$$S_j(T) = S_j(0) e^{(r-q_j-\omega_j)T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \dots, n, \quad (1)$$

- ▶  $\omega_j$  is a mean-correction:

$$\omega_j = \frac{1}{T} \log \phi_L \left( -i \sigma_j \sqrt{T} \right).$$

## Approximate basket option pricing

- The price of the basket at time  $T$  is denoted by  $S(T)$ :

$$S(T) = w_1 S_1(T) + \dots + w_n S_n(T), \quad w_j \geq 0.$$

- A **basket option** with strike  $K$  and maturity  $T$  pay at time  $T$ :

$$(S(T) - K)_+.$$

- The price is denoted by  $C[K, T]$  and given by:

$$C[K, T] = e^{-rT} \mathbb{E}_{\mathbb{Q}} [(S(T) - K)_+].$$

- In the Lévy copula model, the cdf of the r.v.  $S(T)$  is unknown.

# Approximate basket option pricing

## The approximate basket

- Define the Lévy process  $Y$  with mother distribution  $L$  and

$$A = Y(1).$$

- We replace  $S(T)$  by the more tractable r.v.  $\tilde{S}(T)$ .

$$\tilde{S}(T) = \bar{S}(T) + \lambda,$$

where  $\lambda \in \mathbb{R}$  and

$$\bar{S}(T) = S(0)e^{(\bar{\mu} - \bar{\omega})T + \bar{\sigma}\sqrt{T}A}.$$

- Black & Scholes case: Brigo et al. (2004).

# Approximate basket option pricing

## Three-moments-matching approximation

- The first three moments of  $S(T)$ :  $m_1, m_2$  and  $m_3$ .
  - ▶ Can be determined if  $\phi_L$  is known<sup>2</sup>.
- Determine  $\bar{\mu}$ ,  $\bar{\sigma}$  and  $\lambda$  such that

$$\begin{aligned}\mathbb{E} [\tilde{S}(T)] &= m_1, \\ \mathbb{E} [\tilde{S}(T)^2] &= m_2, \\ \mathbb{E} [\tilde{S}(T)^3] &= m_3.\end{aligned}$$

- Mean-correcting parameter:  $\bar{\omega} = \frac{1}{T} \log \phi_L \left( -i\bar{\sigma}\sqrt{T} \right)$ .

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<sup>2</sup>see Lemma 2 in Linders & Schoutens (2014b)

# Approximate basket option pricing

## The characteristic function

- Approximate  $C[K, T]$

- ▶ Replace the original basket  $S(T)$  by the approximation  $\tilde{S}(T)$ :

$$C^{MM}[K, T] = e^{-rT} \mathbb{E} \left[ (\tilde{S}(T) - K)_+ \right]$$

- ▶ Use that  $\tilde{S}(T) = \bar{S}(T) + \lambda$ :

$$C^{MM}[K, T] = e^{-rT} \mathbb{E} \left[ (\bar{S}(T) - (K - \lambda))_+ \right].$$

- New problem: price an option on  $\bar{S}(T)$  with shifted strike  $K - \lambda$ .
- Knowledge of the characteristic function  $\phi_{\log \bar{S}(T)}$  is needed:

$$\phi_{\log \bar{S}(T)}(u) = \exp \{ iu (\log S(0) + (\bar{\mu} - \bar{\omega})T) \} \phi_L \left( u \bar{\sigma} \sqrt{T} \right).$$

# The Carr-Madan formula

## Option pricing via the characteristic function

- Consider the r.v.  $X$  and assume that  $\mathbb{E} [X^{\alpha+1}]$  is finite for  $\alpha > 0$ .
- Then it was proven in Carr & Madan (1998):

$$e^{-rT} \mathbb{E} [(X - K)_+] = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^{+\infty} \exp \{-i v \log(K)\} g(v) dv,$$

where

$$g(v) = \frac{e^{-rT} \phi_{\log X}(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

- The key ingredient to determine option prices is a closed form of the characteristic function.
- Fast implementation using the FFT.



# Basket option pricing in the Lévy copula model

## Numerical illustration: VG copula model

- The Variance Gamma (VG) copula model

- ▶  $L$  has a VG distribution:

$$L \stackrel{d}{=} VG(\sigma = 0.57, \nu = 0.75, \theta = -0.95, \mu = 0.95).$$

- Consider a basket option paying at maturity  $T$ :

$$\left( \frac{1}{4} (S_1(T) + S_2(T) + S_3(T) + S_4(T)) - K \right)_+.$$

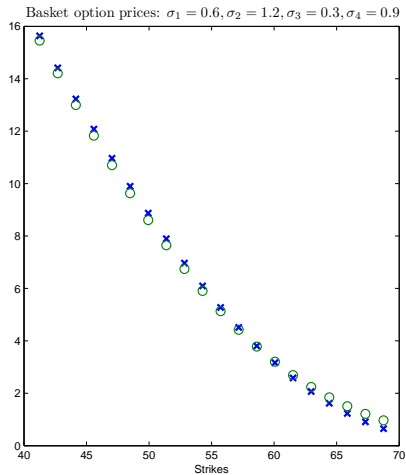
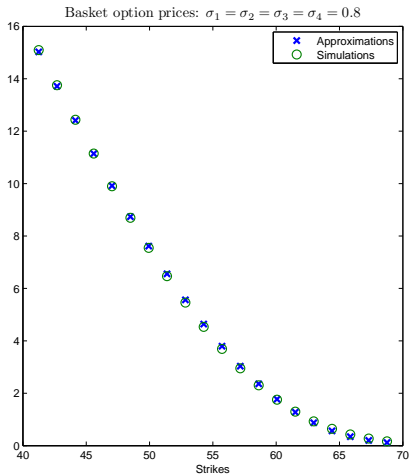
- ▶  $T = 0.5, \rho = 0$ ;

- ▶  $S_1(0) = 40, S_2(0) = 50, S_3(0) = 60, S_4(0) = 70$ .

- We compare simulated basket option prices with the corresponding approximations.

# Basket option pricing in the Lévy copula model

## Numerical illustration: VG copula model



# Pricing basket options in the Lévy copula model

## Calibrating the marginal parameters

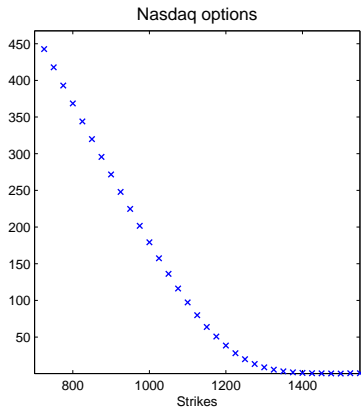
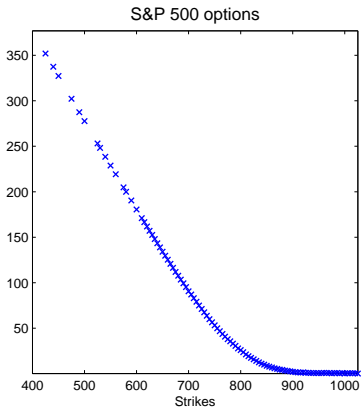
- Pricing date = February 19, 2009.
- We consider a basket option paying at maturity  $T$

$$(w_1 S_1(T) + w_2 S_2(T) - K)_+ .$$

- ▶  $T = 30$  days.
- ▶  $S_1(T)$  denotes the price of the S&P 500 and  $S_2(T)$  the price of the Nasdaq.
- ▶ The weights are chosen such that  $w_1 S_1(0) + w_2 S_2(0) = 100$ .
- **Question:** how can we determine basket option prices consistent with the observed option curves of the S&P 500 and the Nasdaq?

# Pricing basket options in the Lévy copula model

Market prices for S&P 500 and Nasdaq options with  $T = 30$  days



# Pricing basket options in the Lévy copula model

## Calibrating the marginal parameters

- Remember that in the Lévy copula model:

$$S_j(T) = S_j(0)e^{(r-q_j-\omega_j)T+\sigma_j\sqrt{T}A_j}, \quad j = 1, 2.$$

- ▶ The distributions  $A_1$  and  $A_2$  are described by the parameter vector  $\Theta$ .
  - ▶ The characteristic function  $\phi_{\log S_j(T)}$  is given in closed form.
- Model prices of vanilla options:

$$C_j^{model}[K, T; \Theta, \sigma_j] = e^{-rT} \mathbb{E} \left[ (S_j(T) - K)_+ \right].$$

- ▶  $C_j^{model}[K, T; \Theta, \sigma_j]$  can be determined using the Carr-Madan formula.

# Pricing basket options in the Lévy copula model

## Calibrating the marginal parameters

- Determine implied estimates for  $\Theta$  and  $\sigma_1, \sigma_2$  by minimizing the distance between the market and the model prices.
- Calibration has to be done simultaneously:
  - ▶ Common parameter vector  $\Theta$ .
  - ▶ Gaussian copula case:  $A_1$  and  $A_2$  are standard normal distributed and calibration can be done stock per stock.

# Pricing basket options in the Lévy copula model

## Calibrating the marginal parameters

Table: Lévy copula models: Calibrated model parameters

Model	Error	Model Parameters			Volatilities	
Normal	15.91%	$\mu_{normal}$	$\sigma_{normal}$		$\sigma_1$	$\sigma_2$
		0	1		0.2863	0.2762
VG	9.39 %	$\sigma_{VG}$	$\nu_{VG}$	$\theta_{VG}$	0.3876	0.3729
		0.3640	0.7492	-0.3123		
Meixner	9.33%	$\alpha_{Meixner}$	$\beta_{Meixner}$		0.4015	0.3833
		1.5794	-1.6235			
NIG	9.51 %	$\alpha_{NIG}$	$\beta_{NIG}$		0.4130	0.3941
		1.5651	-1.0063			

# Implied Lévy correlation

## Calibration using multi-name derivatives

- Dow Jones Industrial Average (DJ)
  - ▶ 30 underlying stocks.
  - ▶ Options are traded for each component.
  - ▶ The parameter vector  $\Theta$  and the volatility parameters  $\sigma_1, \sigma_2, \dots, \sigma_{30}$  can be calibrated.
- The only free parameter is the correlation  $\rho$ .
- Market prices for index options
  - ▶ Options on the Dow Jones are traded.
  - ▶  $\rho$  can be calibrated to traded Dow Jones index options.
- Only one traded index option is required for fixing  $\rho$ .



## Implied Lévy correlation

### Definition<sup>3</sup>

- We observe the market price  $C[K, T]$  for the index option with strike  $K$  and maturity  $T$ .
- Given the parameter  $\rho$ , we can determine the (approximate) model price:  $C^{model}[K, T; \underline{\sigma}, \Theta, \rho]$ .
- The Lévy copula framework accommodates models with Gaussian, Variance Gamma, Meixner, .... marginals.

### Definition (Implied correlation)

The implied correlation  $\rho[\pi]$ , with moneyness  $\pi = \frac{K}{S(0)}$  is defined such that:

$$C^{model}[K, T; \underline{\sigma}, \Theta, \rho[\pi]] = C[K, T].$$

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<sup>3</sup>see Linders & Schoutens (2014a)

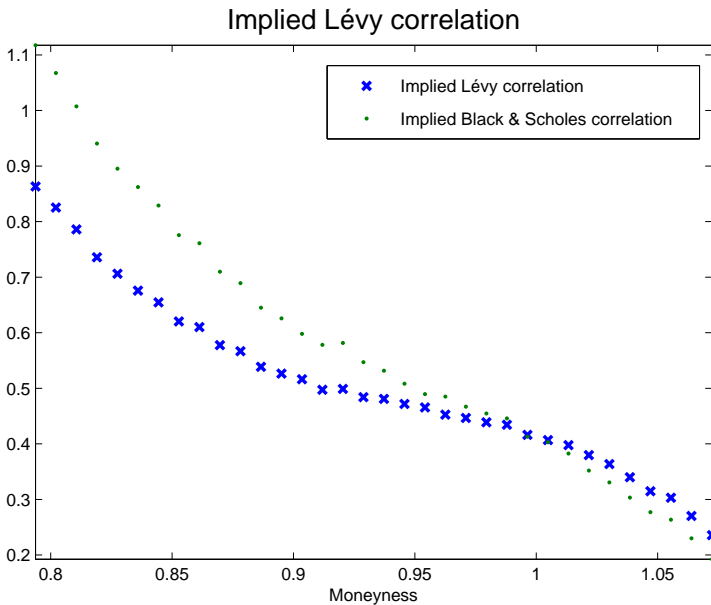
# Implied Lévy correlation

## The implied correlation smile

- The model price  $C^{model}[K, T; \underline{\sigma}, \Theta, \rho]$  is not given in closed form.
  - ▶ To have a fast calibration, a fast and efficient approximation is needed.
- For any strike  $K$  for which the market quote  $C[K, T]$  is given, we can determine an implied correlation estimate  $\rho \left[ \frac{K}{S(0)} \right]$ .
- Implied correlation smile
  - ▶ A single correlation parameter  $\rho$  is not capable of capturing the observed index option curve.
  - ▶  $\rho \left[ \frac{K}{S(0)} \right]$  is a decreasing function of  $K$ .
  - ▶ Similar to the *implied volatility smile*.
- We compare Dow Jones implied correlation in the *Gaussian* and the *Variance Gamma* copula models.

# Implied Lévy correlation

The DJ implied correlation smile: June 20, 2008 and  $T = 30$  days



# Implied Lévy correlation

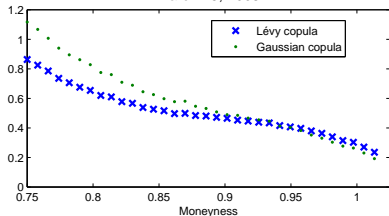
## The implied correlation smile

- Implied Gaussian correlation
  - ▶ For low moneyness, Gaussian implied correlation goes beyond 1.
  - ▶ The lognormal distribution is not capable of describing the marginals adequately.
- Implied Lévy correlation
  - ▶ Better fit of the marginals and correlations which are below 1.
  - ▶ The Lévy implied correlation smile is flatter than its Gaussian counterpart.
- At-the-money implied Gaussian and Lévy correlations are more or less equal.

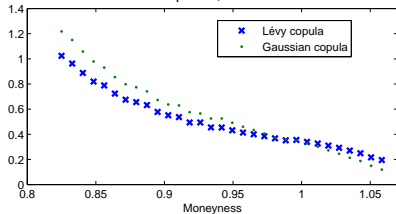
# Implied Lévy correlation

## The DJ implied correlation smile

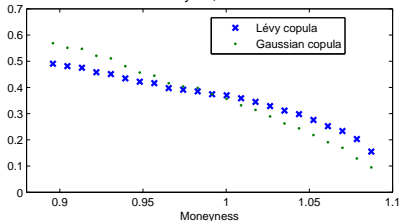
March 25, 2008



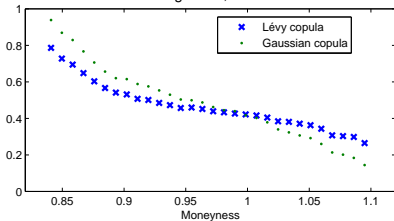
April 18, 2008



July 18, 2008



August 20, 2008



## Conclusions

- We extended the classical Gaussian copula model to a Lévy copula model.
- We considered the problem of pricing basket options.
- The model can be calibrated using single-name and multi-name derivatives.
- Implied Lévy correlation solves some issues of the classical Gaussian correlation.
- Literature:
  - ▶ Gaussian copula model: Skintzi & Refenes (2005), CBOE (2009), Linders & Schoutens (2014a).
  - ▶ Implied correlation defined using spread options: Tavin (2013).
  - ▶ Lévy base correlation: Garcia, Goossens, Masol & Schoutens (2009).

**Thank you for your attention!**

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