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TEST FOR HOMOGENEITY IN GAMMA MIXTURE MODELS  
USING LIKELIHOOD RATIO

by

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# Test for homogeneity in gamma mixture models using likelihood ratio

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## Abstract

We study a testing problem of homogeneity in gamma mixture models. It is found that under consistency of the penalized maximum likelihood estimator there is a proportion of the penalized likelihood ratio test statistic that degenerates to zero. The limiting distribution of the statistic is found to be the chi-bar-square distributions. The degeneration is due to the negative definiteness of a complicated random matrix, depending on the shape parameter under the null hypothesis. In light of this dependency, bounds on the distribution are introduced and a weighted average procedure is proposed. Simulation suggests that the results are accurate and consistent, and that the asymptotic result applies to the maximum likelihood estimator, obtained via an Expectation-Maximization algorithm.

Key Words: Chi-bar-square distributions, gamma mixture, likelihood ratio, maximum likelihood, negative definite

## 1 Introduction

In recent years, gamma mixture models have seen a surge of applications in many fields. Craig and Strassels (2010) examined the out-of-pocket prices of analgesic medications using a two-component gamma mixture model. See also Mayrose et al. (2005) for applications in bioinformatics and the references in Liu et al. (2003). Due to their importance, developing effective and handy statistical procedures for gamma mixture models is an imperative task, in particular for the test of homogeneity. An obvious way of approaching the problem is to use the ordinary likelihood ratio test (LRT). One of the few

results available is Liu et al. (2003). The authors showed that when the range of some parameters is unbounded, the LRT statistic diverges to infinity at a rate of  $\log \log n$  and that its asymptotic behaviour is of extreme-value type through a highly complex piece of stochastic analysis. However, their simulation results suggested that the limiting distribution is far from converging to the extreme value distribution and that a possible solution is to simulate the finite-sample null distribution. The peculiar behaviour of the statistic arises because the maximum likelihood estimator (MLE) of a component of the mixture model fails to converge. See, for example, the asymptotic result for  $R_n(\varepsilon; I)$  in Chen and Chen (2001). Related problems in general mixture models were also addressed by Ghosh and Sen (1985) and Chen and Chen (2001). These authors showed that the asymptotic distribution involves the supremum of a Gaussian process. See also Liu and Shao (2004) in normal mixture models. However, there are several shortfalls of the above results. Firstly, the results lose their appeal because the functional form of the statistic with respect to the stochastic process is difficult to compute (Chen et al., 2001). Secondly, the divergence to infinity is so slow that it is not detected in simulation and that the convergence to the extreme value distribution is hardly detectable (Liu et al., 2003; Liu and Shao, 2004). Lastly, Hall and Stewart (2005) provided a theoretical analysis on the reduction of power against alternative hypotheses regarding the above results.

The consistency of the MLE in the testing problem has not been solved until the introduction of a clever penalized procedure proposed by Chen et al. (2001). The authors innovated the modified likelihood ratio test (MLRT) by incorporating a penalty function. The MLRT was also developed by Chen and Kalbfleisch (2005) in normal mixture models and further extended to an EM-test by Li et al. (2009) and Chen and Li (2009). Exact theoretical results have been obtained in some special cases. For densities with a single parameter of interest, the MLRT statistic has the limiting distribution  $0.5\chi_0^2 + 0.5\chi_1^2$  (Chen et al., 2001; Li et al., 2009). For the normal mixture model, the statistic has  $\chi_2^2$  when the means and the variances are unequal and unknown (Chen and Li, 2009). Conceivably, the MLRT falls into the type II likelihood ratio problem (Lindsay, 1995, Section 4.4) which generates the chi-bar-square distributions of which some are parameter-dependent limiting null distributions. The above result in the normal mixture models returns to the  $\chi_2^2$  distribution due to parameter identifiability. Qin and Smith (2006) investigated an extension of the MLRT in multivariate normal mixture models. The authors showed the asymptotic null distribution being the

chi-bar-square distributions and suggested it must be found using numerical methods. In many other mixture models, such as the gamma mixture models, the results  $0.5\chi_0^2 + 0.5\chi_1^2$  or  $\chi_2^2$  cannot be applied directly without theoretical justifications. The general testing problem has not been completely solved and remains as a long-standing open problem. Charnigo and Sun (2004) acknowledged the generalization of the MLRT to higher dimensional problems and suggested the corresponding null distribution be simulated. However, the extension is not at all straight forward as presented in this paper and simulation of the null distribution in the absence of a closed-form expression should no longer be tolerated. A clear guideline has been long overdue for practitioners in the rejection or retention of the homogeneity assumption. The purposes of the paper are to fill this research gap in gamma mixture models and to explore how the limiting null distribution depends on the parameters.

Motivated by the above needs and the importance of the gamma mixture models, this paper aims at investigating the limiting distribution of the MLRT statistic. Given consistent penalized MLE we obtain the condition under which the MLRT statistic degenerates to zero and determine the proportion of degeneration. Then, we can show that the asymptotic null distribution has parameter-dependent chi-bar-square distributions. This subsequently establishes a foundation for quick model selection using the  $\chi_2^2$  distribution in practice. Moreover, in light of the popular Expectation-Maximization (EM) algorithm for parameter estimation in finite mixture models, we demonstrate through intensive simulation studies that our results can be applied to the likelihood ratio statistic evaluated at the MLE obtained via the EM algorithm.

The article is organized as follows: In Section 2, we present the asymptotic results. Section 3 lists a number of considerations in the applications of the results. The asymptotic analysis is supplemented by simulation in Section 4. Section 5 presents two data examples and Section 6 gives a conclusion.

## 2 Asymptotic Results

We consider a two-parameter gamma density function

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

where  $\alpha > 1$  and  $\beta > 0$  are shape and scale parameters, respectively. Given a set of independent and identically distributed data, we are interested in testing the homogeneity hypothesis  $H_0$  against the alternative hypothesis of a two-component gamma mixture model  $H_1$  where

$$\begin{aligned} H_0 : f(x) &= f(x; \alpha, \beta); \\ H_1 : f(x) &= \pi f(x; \alpha_1, \beta_1) + (1 - \pi) f(x; \alpha_2, \beta_2), \end{aligned}$$

and  $0 < \pi < 1$  is a mixing proportion. In this paper, we study a very general testing problem that the parameters under the hypotheses are all unknown and unequal. This is completely different from the setting in Liu et al. (2003). For parametric hypothesis testing problems it is customary to use the ordinary LRT based on the statistic which is defined as

$$LR_n = 2 \left\{ L(\hat{\pi}, \hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2) - L(0.5, \hat{\alpha}, \hat{\beta}, \hat{\alpha}, \hat{\beta}) \right\},$$

where

$$L(\pi, \alpha_1, \beta_1, \alpha_2, \beta_2) = \sum_{i=1}^n \log \{ \pi f(x_i; \alpha_1, \beta_1) + (1 - \pi) f(x_i; \alpha_2, \beta_2) \} \quad (1)$$

is the log-likelihood function and  $\hat{\theta}$  is the MLE of parameter  $\theta$ . It is well known that the consistency of the MLE, obtained by maximizing (1) directly, is not guaranteed. See for example Ghosh and Sen (1985); Hathaway (1985); Chen and Chen (2001). This motivates a penalized procedure coined by Chen et al. (2001) based on a modified log-likelihood function

$$L^p(\pi, \alpha_1, \beta_1, \alpha_2, \beta_2) = L(\pi, \alpha_1, \beta_1, \alpha_2, \beta_2) + c \log \{ 4\pi(1 - \pi) \}, \quad (2)$$

where  $c$  is a positive constant corresponding to the level of modification. An alternative penalty function  $c \log(1 - |1 - 2\pi|)$  was suggested by Li et al. (2009). However, this suggestion seems to fail in our study as the penalty is too severe as found in simulations. More details will be given in Section 4. Denote by  $\hat{\theta}^p$  the penalized MLE of  $\theta$  obtained by maximizing (2) given a suitable value of  $c$ . We refer to Theorem 1 in Chen et al. (2001) for the consistency of  $\hat{\theta}^p$ . The MLRT statistic is

$$LR_n^p = 2 \left\{ L^p(\hat{\pi}^p, \hat{\alpha}_1^p, \hat{\beta}_1^p, \hat{\alpha}_2^p, \hat{\beta}_2^p) - L(0.5, \hat{\alpha}, \hat{\beta}, \hat{\alpha}, \hat{\beta}) \right\}. \quad (3)$$

We study the asymptotic distribution of  $LR_n^p$  which can be expressed as  $LR_n^p = LR_{1n}^p - LR_{0n}$  in terms of the true parameter  $(\alpha_0, \beta_0)$  under  $H_0$ , where

$$\begin{aligned} LR_{0n} &= 2 \left\{ L(0.5, \hat{\alpha}, \hat{\beta}, \hat{\alpha}, \hat{\beta}) - L(0.5, \alpha_0, \beta_0, \alpha_0, \beta_0) \right\}; \\ LR_{1n}^p &= 2 \left\{ L^p(\hat{\pi}^p, \hat{\alpha}_1^p, \hat{\beta}_1^p, \hat{\alpha}_2^p, \hat{\beta}_2^p) - L(0.5, \alpha_0, \beta_0, \alpha_0, \beta_0) \right\}. \end{aligned}$$

An immediate asymptotic approximation for  $LR_{0n}$  is

$$\left( n^{-1/2} \sum_{i=1}^n Y_i^T \right) \left( n^{-1} \sum_{i=1}^n Y_i Y_i^T \right)^{-1} \left( n^{-1/2} \sum_{i=1}^n Y_i \right) + o_p(1),$$

where  $Y_i$  is a random vector given by

$$Y_i = \begin{Bmatrix} -\Gamma^{(1)}(\alpha_0) + \log \beta_0 + \log X_i \\ \alpha_0 \beta_0^{-1} - X_i \end{Bmatrix} \quad (4)$$

and  $\Gamma^{(k)}(\alpha) = d^k \ln \Gamma(\alpha) / d^k \alpha$ . In Appendix A, we derive the following asymptotic approximation for  $LR_{1n}^p$

$$\begin{aligned} & \left( n^{-1/2} \sum_{i=1}^n Y_i^T \right) \left( n^{-1} \sum_{i=1}^n Y_i Y_i^T \right)^{-1} \left( n^{-1/2} \sum_{i=1}^n Y_i \right) \\ & + \left( n^{-1/2} \sum_{i=1}^n W_i^T \right) \left( n^{-1} \sum_{i=1}^n W_i W_i^T \right)^{-1} \left( n^{-1/2} \sum_{i=1}^n W_i \right) + o_p(1) \end{aligned} \quad (5)$$

if  $n^{-1/2} \sum_{i=1}^n U_i$  is non-negative definite, where

$$W_i = \left( \sum_{i=1}^n Z_i \tilde{\gamma}_2 Y_i^T \right) \left( \sum_{i=1}^n Y_i Y_i^T \right)^{-1} Y_i - Z_i \tilde{\gamma}_2,$$

$\tilde{\gamma}_2 \neq 0$  is the solution to  $(\sum_{i=1}^n W_i W_i^T)^{-1} \sum_{i=1}^n U_i = I_2$ ,  $I_2$  is the two-dimensional identity matrix,

$$U_i = Z_i - V_i Z_i, \quad V_i = \left( \sum_{j=1}^n Y_j^T \right) \left( \sum_{j=1}^n Y_j^T Y_j \right)^{-1} Y_i, \quad (6)$$

and  $Z_i$  is a symmetric random matrix whose elements on the top left, top right and bottom right are, respectively

$$\begin{aligned} Z_{i[11]} &= -\Gamma^{(2)}(\alpha_0) + \{-\Gamma^{(1)}(\alpha_0) + \log \beta_0 + \log X_i\}^2; \\ Z_{i[12]} &= \beta_0^{-1} + \{-\Gamma^{(1)}(\alpha_0) + \log \beta_0 + \log X_i\} (\alpha_0 \beta_0^{-1} - X_i); \\ Z_{i[22]} &= -\alpha_0 \beta_0^{-2} + (\alpha_0 \beta_0^{-1} - X_i)^2. \end{aligned} \quad (7)$$

The random quantity  $V_i$  is scalar. If  $n^{-1/2} \sum_{i=1}^n U_i$  is negative-definite,  $W_i$  are taken to be zero resulting in  $LR_n^p = o_p(1)$ . Under  $H_0$ ,  $Y_i$  and  $Z_i$  are random with mean zero. Then, by the central limit theorem,  $n^{-1/2} \sum_{i=1}^n W_i$  converges to a bivariate normal random vector with mean zero. We summarize the results in the following. Under  $H_0$ , the asymptotic distribution of  $LR_n^p$  degenerates to zero with a weight  $0 < p < 1$  and has a  $\chi_2^2$  distribution with a weight  $1 - p$  where  $p$  is the probability that the matrix  $n^{-1/2} \sum_{i=1}^n U_i$  is negative-definite and  $U_i$  is defined in (6). That is,

$$LR_n^p \sim p + (1 - p) \chi_2^2 \quad (8)$$

for large  $n$ . The limiting distribution in (8) is known as the chi-bar-square distributions (Johnson et al., 1994, pg. 454). A more precise expression for  $p$  will be derived in Section 3.2. Hence, the above result will be restated by (10) indicating clearly the dependency on the shape parameter.

### 3 Practical Considerations

#### 3.1 Estimating $p$

From the definition of  $U_i$  in (6), we observe its dependence on the random vector  $Y_i$  and the random matrix  $Z_i$  given by (4) and (7), respectively, which are related to the parameter  $(\alpha_0, \beta_0)$  under  $H_0$ . In addition, the estimate of  $p$  may also depend on  $n$  as the random matrix concerned involves a summation over  $n$  random matrices. As a rough visualization of the relations between these variables, we simulate  $n$  random variables from  $f(x; \alpha_0, \beta_0)$ , compute  $Y_i$ ,  $Z_i$  and  $U_i$ , and evaluate the proportions in 10000 replications that  $n^{-1/2} \sum_{i=1}^n U_i$  is negative definite. Denote by  $p_s(\alpha_0, \beta_0, n)$  such a proportion. Fig. 1 displays two plots of  $p_s(\alpha_0, \beta_0, n)$  at some selected values of  $\alpha_0$ ,  $\beta_0$  and  $n$ . The left panel shows three series of  $p_s(\alpha_0, \beta_0, n)$  against  $\alpha_0$  at  $n = 1000$ , each series corresponding to different values of  $\beta_0$ . There is a decreasing trend of  $p_s(\alpha_0, \beta_0, n)$  as  $\alpha_0$  increases, this trend being invariant in  $\beta_0$ . The right panel shows another three series of  $p_s(\alpha_0, \beta_0, n)$  against  $\alpha_0$  at  $\beta_0 = 2$ , each series corresponding to different values of  $n$ . Similar decreasing trend of  $p_s(\alpha_0, \beta_0, n)$  against  $\alpha_0$  is observed. In addition, the values of  $p_s(\alpha_0, \beta_0, n)$  get lower at larger sample sizes and seem to converge to some certain level as  $n$  grows. Overall,  $p_s(\alpha_0, \beta_0, n)$  seems to decrease as  $\alpha_0$  increases, but remains constant as  $\beta_0$  varies. Its possible convergence as

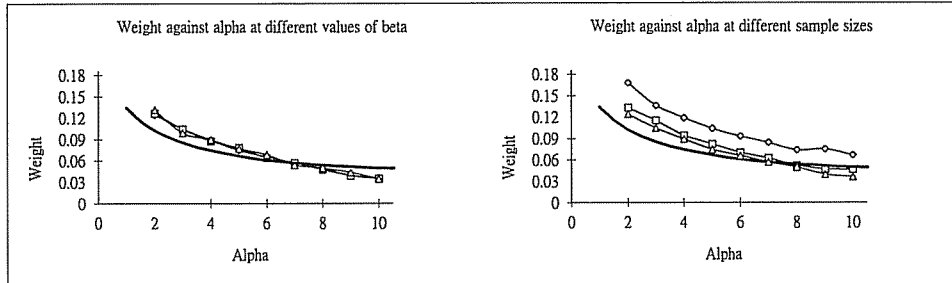


Figure 1: Plots of weight  $p_s(\alpha_0, \beta_0, n)$  against  $\alpha_0$ . The left panel shows three series of  $\beta_0 = 2$  ( $\diamond$ ),  $\beta_0 = 6$  ( $\square$ ) and  $\beta_0 = 2$  ( $\triangle$ ). The right panel shows three series of  $n = 100$  ( $\diamond$ ),  $n = 500$  ( $\square$ ) and  $n = 1000$  ( $\triangle$ ). The solid line depicts the asymptotic weight  $p(\alpha_0)$ .

$n$  tends to infinity motivates further investigation. Last, it is worth pointing out some merits of the simulation technique. Apart from quick and easy construction of the weight estimate, its use in the construction of a lower bound in finite samples will be outlined in Section 4.

### 3.2 Asymptotic $p$

We require some general conditions on  $Y_i$ ,  $Z_i$  and the products of their elements. In particular,

$$n^{-1} \sum_{i=1}^n Y_i Y_i^T \rightarrow M, \quad n^{-1} \sum_{i=1}^n Y_{i[j]} Z_i \rightarrow v_j$$

in probability for  $j = 1, 2$ , where  $Y_{i[j]}$  denotes the  $j$ th element of vector  $Y_i$ . The expression of each of the elements in matrices  $M$  and  $v$  are given in Appendix B. Denote by  $U_{i[11]}$ ,  $U_{i[12]}$  and  $U_{i[22]}$ , respectively the elements on the top left, top right and bottom right of  $U_i$ . By the central limit theorem, the vector on the left-hand side below

$$n^{-1/2} \begin{pmatrix} \sum_{i=1}^n U_{i[11]} \\ \sum_{i=1}^n U_{i[12]} \\ \sum_{i=1}^n U_{i[22]} \end{pmatrix} \rightarrow N_3 \left\{ 0, \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \right\}$$

converges, as  $n \rightarrow \infty$ , to a random vector denoted by  $S^T = (S_1, S_2, S_3)$  having a trivariate normal distribution  $N_3$  with zero mean vector and covariance



matrix whose elements are

$$\begin{aligned}
\sigma_{11} &= 2 \{ \Gamma^{(2)}(\alpha_0) \}^2 + \Gamma^{(4)}(\alpha_0) + \alpha_0 \{ \Gamma^{(3)}(\alpha_0) \}^2 \{ 1 - \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1}; \\
\sigma_{12} &= \left[ -2\Gamma^{(2)}(\alpha_0) + \Gamma^{(3)}(\alpha_0) \{ 1 - \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1} \right] \beta_0^{-1}; \\
\sigma_{13} &= \left[ 2 + \alpha_0 \Gamma^{(3)}(\alpha_0) \{ -1 + \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1} \right] \beta_0^{-2}; \\
\sigma_{22} &= \left[ -1 + \Gamma^{(2)}(\alpha_0) \{ -1 + \alpha_0^2 \Gamma^{(2)}(\alpha_0) \} \right] \{ -1 + \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1} \beta_0^{-2}; \\
\sigma_{23} &= \left[ -2\alpha_0 + \{ -1 + \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1} \right] \beta_0^{-3}; \\
\sigma_{33} &= \alpha_0 \left[ 2 + 2\alpha_0 + \{ 1 - \alpha_0 \Gamma^{(2)}(\alpha_0) \}^{-1} \right] \beta_0^{-4}.
\end{aligned}$$

Hence, the negative-definiteness condition implies that  $p$  can be approximated by the following probability asymptotically

$$P(\{S_1 < 0\} \cap \{S_1 S_3 - S_2^2 > 0\}) = \int_{-\infty}^0 \int_{-\infty}^{\infty} \int_{-\infty}^{s_2^2 s_1^{-1}} g(s_1, s_2, s_3) ds_3 ds_2 ds_1, \quad (9)$$

where  $g(s_1, s_2, s_3)$  is the density function of the above trivariate normal distribution. The probability can be easily evaluated by numerical integration using, for example, Wolfram Mathematica<sup>®</sup>. It is important to observe that the integral is independent of  $\beta_0$ . This can be easily verified by simple transformation in the integration. Hence, the probability may precisely be denoted by  $p(\alpha_0)$  and the result in (8) is more appropriately written as

$$LR_n^p \sim p(\alpha_0) + \{1 - p(\alpha_0)\} \chi_2^2. \quad (10)$$

A more precise description of the relation of  $p(\alpha_0)$  as  $\alpha_0$  varies can be obtained by (9). Fig. 1 overlays a curve of  $p(\alpha_0)$  against  $\alpha_0$  for  $1 \leq \alpha_0 \leq 10$ . Its match with the simulated weight  $p_s(\alpha_0, \beta_0, n)$  suggests that the mysterious dependency of the proportion of degeneration on the shape parameter is further illustrated. An astounding observation is that  $p(\alpha_0)$  is small at some large values of  $\alpha_0$ . An example  $p(12) = 0.0497$  suggests that the  $\chi_2^2$  distribution is quite accurate to approximate the asymptotic null distribution. In addition, as an empirical rule of thumb, we may use that  $p(\alpha_0) \leq p(1) = 0.1345$  and  $p(\alpha_0) > 0$  to develop a lower bound  $D_L$  and an upper bound  $D_U$  for the statistic  $LR_n^p$

$$D_L \leq LR_n^p \leq D_U, \quad D_L \sim 0.1345 + 0.8655 \chi_2^2, \quad D_U \sim \chi_2^2 \quad (11)$$

as a quick guideline. Given a significance level,  $H_0$  is retained if the observed MLRT statistic falls below the critical value evaluated by the above lower

bound, and is rejected if it is above the critical value based on the upper bound.

### 3.3 Weighted Average Procedure

The previous subsections demonstrate the theoretical analysis to the testing problem. However, practical implementation of (10) encounters a drawback that the value  $\alpha_0$  is unknown. A possible solution is to substitute this value by the parameter estimate, for instance, the maximum likelihood estimate  $\hat{\alpha}$  under  $H_0$ . Then, the weight  $p(\alpha_0)$  is estimated by  $\hat{\alpha}$  through (9) and the asymptotic null distribution is established as in (10). However, the substitution may suffer a certain degree of bias because all prior beliefs are placed on  $\hat{\alpha}$ . Lindsay (1995) suggested the use of the least favourable null distribution by employing the least favourable critical value within a confidence interval for  $\alpha_0$ . However, the problem remains unsolved if the observed test statistic falls below this least favourable critical value.

In light of the above difficulties, we propose a weighted average procedure to accommodate the estimation error. It is well-known that  $\hat{\alpha} - \alpha_0$  is asymptotically normal with mean zero and variance  $v(\alpha_0) = n^{-1}\alpha_0\{-1 + \alpha_0\Gamma^{(2)}(\alpha_0)\}^{-1}$ . Then,  $r$  candidates of  $\alpha_{0,k}$  can be obtained from the normal distribution through

$$\frac{k}{r+1} = \int_{-\infty}^{\alpha_{0,k}} \{2\pi v(\hat{\alpha})\}^{-\frac{1}{2}} \exp\left\{-\frac{(x - \hat{\alpha})^2}{2v(\hat{\alpha})}\right\} dx, \quad k = 1, \dots, r$$

provided that  $\alpha_{0,k} \geq 1$ . Each of these  $\alpha_{0,k}$  forms an asymptotic null distribution given by (8). The assignment of an equal weight to each  $\alpha_{0,k}$  leads to the asymptotic null distribution

$$p_w(\hat{\alpha}, r) + \{1 - p_w(\hat{\alpha}, r)\} \chi_2^2, \quad p_w(\hat{\alpha}, r) = \frac{1}{r} \sum_{k=1}^r p(\alpha_{0,k}).$$

As illustrated in Fig. 1 the concavity of the weight in the shape parameter, the weighted average procedure will give a weight slightly larger than the direct substitution does. The effect of this finite-sample refinement is illustrated in Table 1. The weights  $p_w(\hat{\alpha}, r)$  using  $n = 100$  are slightly larger than those using  $n = 1000$  which are very close to the value obtained by direct substitution  $p(\hat{\alpha})$ . Hence, this procedure tends to protect the null hypothesis

Table 1: Weight  $p$  against  $\alpha$  using weighted average procedure  $p_w(\alpha, r)$  and direct substitution  $p(\alpha)$ . Different numbers of candidates  $r$  and the effect of sample size  $n$  for the weighted average procedure are shown.

$\alpha$	$p_w(\alpha, 10)$		$p_w(\alpha, 20)$		$p(\alpha)$
	$n = 100$	$n = 1000$	$n = 100$	$n = 1000$	
2	0.1023	0.1023	0.1023	0.1023	0.1023
4	0.0744	0.0743	0.0744	0.0743	0.0743
6	0.0612	0.0611	0.0613	0.0611	0.0608
8	0.0540	0.0536	0.0541	0.0536	0.0536
10	0.0509	0.0497	0.0512	0.0497	0.0496

less when information from the sample is scarce. It is possible to impose more protection on  $H_0$  by adjusting the assignment of weight. Moreover, the input  $r$  seems less important compared to the sample size. We shall fix  $r = 10$  in data analysis in Section 5.2.

### 3.4 MLE obtained via EM Algorithm

Mixture models are getting popular in the statistics literature because of its wide range of applications, including examination of homogeneity of populations, assessment of unimodality and identifications of clusters or outliers. The introduction of the EM algorithm has further pushed up its popularity. One of the appealing features of the ordinary LRT under regularity conditions is the direct use of the two maximized log-likelihood functions under each of the hypotheses. The MLRT does not have such convenience as it will normally not be considered for parameter estimation of a mixture model. Although the penalized procedure preserves the consistency of the estimators under  $H_0$ , the loss of the above convenience may eclipse the use of the MLRT. Another problem inherited in the MLRT is the possible reduction of power under  $H_1$  that arises from the penalty function. The discussion about power will be deferred to Section 4. In light of this, we may use the following conventional likelihood ratio as an alternative statistic

$$LR_{itr,n}^{EM} = 2 \left\{ L \left( \hat{\pi}^{EM}, \hat{\alpha}_1^{EM}, \hat{\beta}_1^{EM}, \hat{\alpha}_2^{EM}, \hat{\beta}_2^{EM} \right) - L \left( 0.5, \hat{\alpha}, \hat{\beta}, \hat{\alpha}, \hat{\beta} \right) \right\}, \quad (12)$$

where  $\hat{\theta}^{EM}$  is the MLE of  $\theta$  obtained via the EM algorithm and  $itr$  is the number of EM algorithm iterations given a suitable initial guess. This statis-

tic not only preserves the convenience as the ordinary LRT does, but is part of it also very common in the formation of AIC and BIC in mixture model selection. We put forward this discussion because the EM algorithm may share the possible estimator inconsistency as inherited in the ordinary likelihood ratio statistic. Therefore, the use of the aforementioned information criteria may tend to reject  $H_0$  more often than they do under regularity conditions. This is due to the peculiar behaviour of the log-likelihood function under  $H_0$ . However, there are advantages in the use of  $LR_{itr,n}^{EM}$  over the ordinary LRT which we should not overlook. They are outlined and a new interpretation of result is proposed in the following.

In the rest of this subsection, we assume without loss of generality that  $\pi \geq 0.5$ . The argument in Chen and Chen (2001) points to the problem that in the ordinary LRT under  $H_0$ , the products  $(1 - \hat{\pi}^{EM}) \hat{\alpha}_2^{EM}$  and  $(1 - \hat{\pi}^{EM}) \hat{\beta}_2^{EM}$  are consistent but not  $\hat{\alpha}_2^{EM}$  and  $\hat{\beta}_2^{EM}$ . The EM algorithm suffers similar problem except that it can never reach the boundary point of  $\pi$  and that the iterations will either slowly merge  $\hat{\alpha}_1^{EM}$  with  $\hat{\alpha}_2^{EM}$  and  $\hat{\beta}_1^{EM}$  with  $\hat{\beta}_2^{EM}$ , or force  $\hat{\pi}^{EM}$  towards one (Lindsay, 1995, Section 3.4). Denote by  $E_I$  the former event that individual estimators are consistent and by  $E_{II}$  the latter event that individual estimators are not consistent. The advantage of  $LR_{itr,n}^{EM}$  is on the extremely slow convergence of the EM algorithm under  $H_0$ . The occurrence of  $E_I$  or  $E_{II}$  can be easily observed as the iterations proceed. If  $E_{II}$  is observed, we may retain  $H_0$  in the absence of a tolerable significance level; otherwise, large values of  $LR_{itr,n}^{EM}$  may suggest rejection of  $H_0$  according to (10) conditional on  $E_I$ . Precisely,

$$LR_{itr,n}^{EM} | E_I \sim p(\alpha_0) + \{1 - p(\alpha_0)\} \chi_2^2 \quad (13)$$

as  $n \rightarrow \infty$ . The number of iteration  $itr$  may be determined based on some stopping rules as outlined in Lindsay (1995). Our simulation results suggest that when a suitable initial guess, such as the penalized maximum likelihood estimate, is adopted, the increase in the likelihood function is not significant as the iterative run proceeds. Hence, practitioners may pick a number of  $LR_{itr,n}^{EM}$  values after the EM algorithm has insignificant changes and conclude to retain or to reject  $H_0$  if these values yield consistent results. The above suggestions essentially preserve the convenience in the use of likelihood-ratio-type tests and avoid power deterioration in applications. The arguments and suggestions in this subsection will be supplemented by the material lifetimes example in Section 5.2.

Table 2: Simulation results at selected nominal levels of 0.9, 0.95 and 0.99 using two homogenous gamma models.

Empirical significance levels for $H_0 : f(x; 2, 1)$						
$n$	0.90		0.95		0.99	
	$LR_n^p$	$LR_{10,n}^{EM}$	$LR_n^p$	$LR_{10,n}^{EM}$	$LR_n^p$	$LR_{10,n}^{EM}$
100	0.8856	0.8715	0.9402	0.9299	0.9862	0.9824
200	0.8943	0.8883	0.9476	0.9438	0.9888	0.9865
500	0.8975	0.8966	0.9441	0.9431	0.9889	0.9886
1000	0.9028	0.9026	0.9529	0.9524	0.9897	0.9896
Empirical significance levels for $H_0 : f(x; 8, 1)$						
$n$	0.90		0.95		0.99	
	$LR_n^p$	$LR_{10,n}^{EM}$	$LR_n^p$	$LR_{10,n}^{EM}$	$LR_n^p$	$LR_{10,n}^{EM}$
100	0.8927	0.8793	0.9437	0.9338	0.9879	0.9842
200	0.8956	0.8886	0.9440	0.9400	0.9857	0.9839
500	0.8947	0.8936	0.9468	0.9460	0.9907	0.9903
1000	0.8953	0.8948	0.9465	0.9459	0.9894	0.9893

## 4 Simulation

We have conducted an extensive simulation study to evaluate the accuracy of the results. Due to the dependency of  $p(\alpha_0)$  on  $\alpha_0$ , it is interesting to conduct simulations using different values of  $\alpha_0$  and holding  $\beta_0 = 1$  with a number of sample sizes. The first statistic under study is  $LR_n^p$  given by (3). It is the MLRT statistic with  $c = \log 50$  in the penalty function in accordance with the recommendations in Chen et al. (2001). We also consider the penalty function  $\log 50 \log(1 - |1 - 2\pi|)$  suggested by Li et al. (2009). The penalty appears to be so severe that even when the constant is reduced from  $\log 50$  to 0.1, convergence is not always successful. There are failure loops that the initial guesses iterate to itself. For example, when  $\alpha_0 = 2$  and  $n = 500$ , the failure rate is 0.5354. The second statistic  $LR_{itr,n}^{EM}$  given by (12) uses the likelihood ratio evaluated at the MLE obtained via the EM algorithm. The extremely slow convergence in the EM algorithm makes simulation studies tedious. Lindsay (1995) pointed out that the solution of the likelihood equations can depend greatly on the initial values. Therefore, we use the penalized MLEs as initial guesses and carry out ten iterations.

We report the empirical sizes obtained from 10000 replications. Two sets

Table 3: Proportions of zero statistics and simulated weights  $p_s(\alpha_0, \beta_0, n)$  using four homogenous gamma models. The asymptotic weights for the four cases are  $p(2)=0.1023$ ,  $p(6)=0.0576$ ,  $p(8)=0.0536$  and  $p(12)=0.0497$ .

$n$	$H_0 : f(x; 2, 1)$			$H_0 : f(x; 6, 1)$		
	$LR_n^p$	$LR_{10,n}^{EM}$	$p_s(2, 1, n)$	$LR_n^p$	$LR_{10,n}^{EM}$	$p_s(6, 1, n)$
100	0.1195	0.1195	0.1636	0.0516	0.0516	0.0925
200	0.1258	0.1258	0.1596	0.0484	0.0484	0.0839
500	0.1206	0.1206	0.1382	0.0542	0.0542	0.0763
1000	0.1225	0.1225	0.1301	0.0545	0.0545	0.0649
$n$	$H_0 : f(x; 8, 1)$			$H_0 : f(x; 12, 1)$		
	$LR_n^p$	$LR_{10,n}^{EM}$	$p_s(8, 1, n)$	$LR_n^p$	$LR_{10,n}^{EM}$	$p_s(12, 1, n)$
100	0.0465	0.0465	0.0731	0.0349	0.0349	0.0578
200	0.0470	0.0470	0.0674	0.0324	0.0324	0.0439
500	0.0384	0.0384	0.0592	0.0336	0.0336	0.0357
1000	0.0444	0.0444	0.0473	0.0297	0.0297	0.0259

of simulation are illustrated in Table 2. Other sets using different values of  $\alpha_0$  share similar results and hence are not reported. The agreement between the theoretical results and the simulation results is obvious. Improvements are generally obtained when we increase the sample size. The simulation also shows the dependency of  $p(\alpha_0)$  on  $\alpha_0$ . In Table 3, we report the proportions of zero statistics obtained from the simulation and the weight  $p_s(\alpha_0, \beta_0, n)$  obtained from simulation in Section 3.1, and the asymptotic weight  $p(\alpha_0)$  is in the caption. First, it is interesting that both statistics  $LR_n^p$  and  $LR_{10,n}^{EM}$  result in the same figures. This implies that the EM algorithm no longer increases the likelihood value under the occurrence of degeneration. Second, it is obvious that the asymptotic analysis leading to  $p(\alpha_0)$  agrees quite well with the simulation results of  $LR_n^p$  and  $LR_{10,n}^{EM}$ . This consistently justifies one of the main results of this paper that the degeneration arises from the negative-definiteness of the random matrix. The relatively weak approximation in the sample size of 100 can be explained by the relatively weak second-order approximation given by (5). Lastly, the value  $p_s(\alpha_0, \beta_0, n)$  is the largest when the sample size less than 1000. We may replace the lower bound given by (11) by  $p_s(\alpha_0, \beta_0, n)$  if smaller as a more refined benchmark in finite-sample situation.

Some insights on the power of the tests can be gained. We consider a

Table 4: Average values of test statistics under alternative hypotheses.

	$H_{101}$	$H_{102}$	$H_{103}$	$H_{104}$	$H_{105}$	$H_{106}$
$LR_n^p$	576.1	487.5	780.4	43.54	61.82	61.82
$LR_{10,n}^{EM}$	884.3	783.7	781.2	43.79	76.23	76.21
	$H_{107}$	$H_{108}$	$H_{109}$	$H_{110}$	$H_{111}$	$H_{112}$
$LR_n^p$	470.8	129.0	341.4	62.63	124.9	61.82
$LR_{10,n}^{EM}$	558.5	140.8	342.2	62.72	125.6	74.64

number of gamma mixture models which are either entirely different in mixing proportion, shape and scale parameters or with some of these parameters being equal. Each of the following alternative hypotheses is formulated to test against  $H_0$

$$\begin{aligned}
 H_{101} &: 0.2f(x; 8, 1) + 0.8f(x; 2, 4); & H_{102} &: 0.8f(x; 8, 1) + 0.2f(x; 2, 4); \\
 H_{103} &: 0.5f(x; 8, 1) + 0.5f(x; 2, 4); & H_{104} &: 0.5f(x; 8, 4) + 0.5f(x; 2, 1); \\
 H_{105} &: 0.2f(x; 8, 1) + 0.8f(x; 2, 1); & H_{106} &: 0.2f(x; 8, 4) + 0.8f(x; 2, 4); \\
 H_{107} &: 0.2f(x; 8, 1) + 0.8f(x; 8, 4); & H_{108} &: 0.2f(x; 2, 1) + 0.8f(x; 2, 4); \\
 H_{109} &: 0.5f(x; 8, 1) + 0.5f(x; 8, 4); & H_{110} &: 0.5f(x; 2, 1) + 0.5f(x; 2, 4); \\
 H_{111} &: 0.5f(x; 8, 1) + 0.5f(x; 2, 1); & H_{112} &: 0.5f(x; 8, 4) + 0.5f(x; 2, 4).
 \end{aligned}$$

Every simulation experiment consists of 10000 replications, each of sample size 1000. We find that the upper bound  $\chi_2^2$  given by (11) is extremely efficient in the testing process. Almost all simulated test statistics of  $LR_n^p$  and  $LR_{10,n}^{EM}$  are greater than the critical values of the  $\chi_2^2$  distribution. The powers are all equal to one at significance levels 0.9, 0.95 and 0.99 except for the test of  $H_{104}$ . In this particular case, the statistic  $LR_n^p$  gives powers of 0.9999, 0.9997 and 0.9985 at the corresponding significance levels, whereas  $LR_{itr,n}^{EM}$  yields powers of 0.9999, 0.9997 and 0.9987. A moderate improvement in terms of power is obtained using the EM algorithm which increases the likelihood value at each cycle in the iterative sequence (Dempster et al., 1977). Meanwhile, the penalty function in the MLRT may suffer despite slight reduction in power. A more concrete picture about power can be obtained by comparing the values of the test statistics. In Table 4, we report the average test statistics in 10000 replications for each of the alternative hypotheses. The averages of  $LR_{10,n}^{EM}$  are always higher than the use of  $LR_{10,n}^{EM}$  achieves higher power.

## 5 Data Example

### 5.1 Danish Fire Loss

This example is based on the Danish fire loss data set which consists of 2157 losses exceeding one million Danish Krone from the years 1980 to 1990 inclusive. It is well known that the data set has a heavy right tail in the extreme value literature (Embrechts et al., 1997). The adequacy of the homogenous gamma model which has a moderate tail is suspected. We apply the proposed results and methods to see if a two-component gamma mixture model will improve the fitting with further verifications, justified by some goodness-of-fit measures. McNeil (1997) provided a time series plot to check for clustering of large losses and a sample mean excess function to determine heavy-tailed behaviour. The results suggest the validity of the independence assumption and the possibility in modeling excesses over high thresholds using the generalized Pareto distribution. Recently, Wong and Li (2010) proposed a threshold model incorporating the generalized Pareto distribution for excesses and a Weibull distribution for the rest of the observations. This threshold model flexibly gives a global fit and an appropriate tail modeling. These two findings suggest that the loss data are likely to be independent but from a heterogeneous population.

The maximum likelihood estimate of a gamma model is  $(\hat{\alpha}, \hat{\beta}) = (1.299, 0.382)$  with a corresponding maximized log-likelihood of -4752. In the gamma mixture model, the penalized procedure and the EM algorithm give

$$\begin{aligned} \left( \hat{\pi}^p, \hat{\alpha}_1^p, \hat{\beta}_1^p, \hat{\alpha}_2^p, \hat{\beta}_2^p \right) &= (0.5098, 15.68, 10.09, 1.226, 0.2040); \\ \left( \hat{\pi}^{EM}, \hat{\alpha}_1^{EM}, \hat{\beta}_1^{EM}, \hat{\alpha}_2^{EM}, \hat{\beta}_2^{EM} \right) &= (0.2816, 1.256, 0.1619, 10.19, 6.036) \end{aligned}$$

which yield the values of test statistic of 1829 and 1978, respectively. The evidence for the mixture model is overwhelming as both statistics are far greater than the critical values of the  $\chi_2^2$  distribution at any reasonable significance level. Further support for this is a goodness-of-fit assessment based on probability-probability plots as shown in Fig. 2. The gamma mixture model provides a much better fit as the plot exhibits obviously a straight line pattern. This example lends further support to the asymptotic distribution in (10), improvement in power through the use of  $LR_{itr,n}^{EM}$ , and demonstrates the simplicity in implementation.



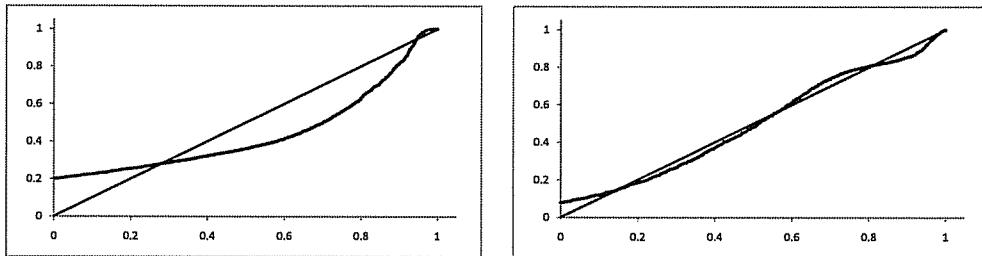


Figure 2: The probability-probability plots. The left panel shows the plot of the fitted gamma model and the right panel shows that of the fitted gamma mixture model using the EM algorithm. A 45-degree straight line is given for reference.

## 5.2 Material Lifetimes

Gamma distributions give useful representations of a number of physical situations such as random processes in time. We consider a set of 101 observations for the lifetime of an aluminum sheet under maximum stress of 21,000 psi. A brief description and the data listed in increasing order are available in Birnbaum and Saunders (1958). The authors demonstrated a realistic adjustment to exponential models in representing lifetimes in a life-testing situation. Therefore, it is interesting to check the redundancy of a mixture structure in the representation. More insight may be gained by applying our results in studying the data set.

The parameter estimate of a gamma model is  $(\hat{\alpha}, \hat{\beta}) = (11.86, 0.008462)$ . In the gamma mixture model, the penalized maximum likelihood estimate of

$$(\hat{\pi}^p, \hat{\alpha}_1^p, \hat{\beta}_1^p, \hat{\alpha}_2^p, \hat{\beta}_2^p) = (0.5001, 22.82, 0.01457, 10.26, 0.008307)$$

leads to a value of the MLRT statistic of 2.387. A lower bound of the  $p$ -value evaluated through (11) is 0.3032, larger than any reasonable size of a statistical test. In the absence of  $\alpha_0$  under  $H_0$ , direct substitution and the weighted average procedure yields  $p(\hat{\alpha})=0.0495$  and  $p_w(\hat{\alpha}, 10)=0.0518$ , respectively. The corresponding  $p$ -values are 0.3393 and 0.3377. The use of the penalized estimates initiates the EM iterative sequence. A series of 1000 iteration steps seems to indicate the occurrence of event  $E_{II}$  that individual estimators are not consistent as shown in the left panel of Fig. 3 in which

Table 5: Estimation results of the EM algorithm for the Material Life Data. The  $p$ -values are calculated using (13) based on (a) direct substitution and (b) weighted average procedure.

$itr$	Parameter estimate $(\hat{\pi}^{EM}, \hat{\alpha}_1^{EM}, \hat{\beta}_1^{EM}, \hat{\alpha}_2^{EM}, \hat{\beta}_2^{EM})$	$LR_{itr,n}^{EM}$	$p$ -value	
			(a)	(b)
10	(0.5048, 22.73, 0.01452, 10.25, 0.008307)	2.389	0.3390	0.3374
50	(0.5222, 22.20, 0.01425, 10.15, 0.008262)	2.397	0.3378	0.3362
100	(0.5446, 21.53, 0.01389, 9.999, 0.008175)	2.408	0.3362	0.3346

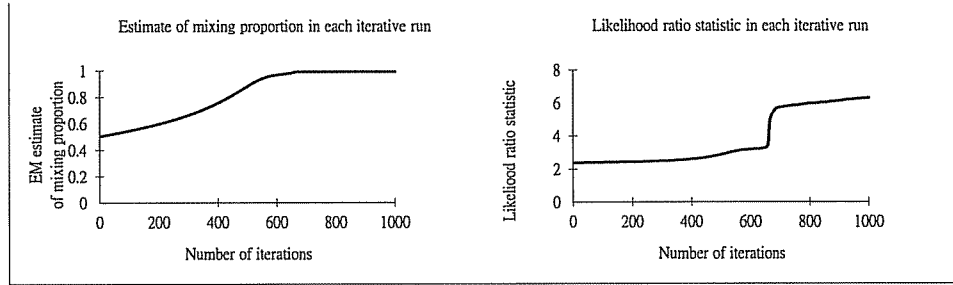


Figure 3: The EM estimate of the mixing proportion (Left) and the statistic  $LR_{itr,n}^{EM}$  (Right) in each iterative run.

$\hat{\pi}^{EM}$  increases slowly to one as the iteration moves on. Evidence in favour of  $H_0$  is obvious. On the other hand, the behavior of  $LR_{itr,n}^{EM}$  is agonizing, in particular as  $\hat{\pi}^{EM}$  is closer to one that a jump in the test statistic is observed in the right panel of Fig. 3. The AIC criterion starts to reject  $H_0$  in the 833th iteration whereas the BIC criterion and the statistic  $LR_{itr,n}^{EM}$  at 5% significance level consistently suggest retention of  $H_0$  in all 1000 iterations. However, the insignificant increase of the statistic in the first 600 iterations suggests early termination of the EM algorithm. Therefore, the suggestions in Section 3.4 are useful. We can consider a number of the test statistics in different iterative steps and apply (13). The results reported in Table 5 consistently suggest the retention of  $H_0$  in agreement with the method of MLRT.

## 6 Conclusion

We investigate the modified likelihood ratio test for homogeneity in two-component gamma mixture models. We have found that the limiting distribution of the test statistic is the parameter-dependent chi-bar-square distributions given by a degeneration to zero with weight  $p(\alpha_0)$  and a chi-square distribution with two degrees of freedom with weight  $1 - p(\alpha_0)$ . This weight is related to the negative-definiteness of a complicated random matrix dependent on the shape parameter of the homogenous gamma model. An asymptotic approximation using a trivariate normal probability has been developed for  $p(\alpha_0)$ . All these theoretical results have been revealed through an extensive simulation to be very accurate and reliable.

In applications, the shape parameter is unknown. Based on the behaviour of  $p(\alpha_0)$ , we have developed a lower bound for the retention of the homogenous hypothesis and an upper bound for the rejection. The bounds have been proved to be extremely useful in simulation and in two real examples. In a rare occasion that the observed test statistic falls between the bounds or if practitioners require an evaluation of  $p$ -value, we recommend the weighted average procedure which takes into account the estimation error of the shape parameter. This procedure has yielded consistent results in a study of the Material Life data.

Due to the popularity of the EM algorithm in the analysis of mixture models, we recommend the likelihood ratio test statistic evaluated at the maximum likelihood estimates obtained via the EM algorithm. There are some appealing advantages including the preservation of the convenience of the conventional likelihood ratio test procedure and in the prevention of power reduction. The fact that the EM iterative sequence converges slowly allows the selection of a number of observed test statistics. Decision may therefore be based on these statistics by comparing the derived asymptotic null distribution conditional on the consistency of individual estimators. We have demonstrated that this approach has a better power in simulation. Its simplicity and convenience has also been illustrated in the real examples.

A number of interesting insights have been obtained on the form of the asymptotic null distribution and on its practical implementation. We believe many other mixture models share similar characteristics and this deserves further research and discussion. In particular, the parameter-dependent structure of the limiting distribution may not be as simple as our situation in which only the shape parameter is involved. Developing simple decision cri-

teria such as bounds appears to be very challenging.

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## A Approximation for $LR_{1n}^p$

Define

$$Y_i = \frac{1}{f(X_i; \alpha_0, \beta_0)} \left\{ \begin{array}{c} \frac{\partial f(X_i; \alpha_0, \beta_0)}{\partial \alpha} \\ \frac{\partial f(X_i; \alpha_0, \beta_0)}{\partial \beta} \end{array} \right\}; \quad Z_i = \frac{1}{f(X_i; \alpha_0, \beta_0)} \left\{ \begin{array}{cc} \frac{\partial^2 f(X_i; \alpha_0, \beta_0)}{\partial^2 \alpha} & \frac{\partial^2 f(X_i; \alpha_0, \beta_0)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 f(X_i; \alpha_0, \beta_0)}{\partial \alpha \partial \beta} & \frac{\partial^2 f(X_i; \alpha_0, \beta_0)}{\partial^2 \beta} \end{array} \right\}$$

whose expressions for the gamma model are given by (4) and (7), respectively. We follow Section 2.3 of Chen et al. (2000) that the resulting characterization of  $LR_{1n}^p$  involves the maximum of the following function

$$2 \sum_{i=1}^n \delta_i - \sum_{i=1}^n \delta_i^2$$

plus  $o_p(1)$ , where

$$\begin{aligned} \delta_i = & \pi \begin{pmatrix} \alpha_1 - \alpha_0 \\ \beta_1 - \beta_0 \end{pmatrix}^T Y_i + \pi \begin{pmatrix} \alpha_1 - \alpha_0 \\ \beta_1 - \beta_0 \end{pmatrix}^T Z_i \begin{pmatrix} \alpha_1 - \alpha_0 \\ \beta_1 - \beta_0 \end{pmatrix} \\ & + (1 - \pi) \begin{pmatrix} \alpha_2 - \alpha_0 \\ \beta_2 - \beta_0 \end{pmatrix}^T Y_i + (1 - \pi) \begin{pmatrix} \alpha_2 - \alpha_0 \\ \beta_2 - \beta_0 \end{pmatrix}^T Z_i \begin{pmatrix} \alpha_2 - \alpha_0 \\ \beta_2 - \beta_0 \end{pmatrix}. \end{aligned}$$

A re-parametrization using vector parameters  $\gamma_1$  and  $\gamma_2$ , where

$$\gamma_1 = \left\{ \begin{array}{c} \pi(\alpha_1 - \alpha_0) + (1 - \pi)(\alpha_2 - \alpha_0) \\ \pi(\beta_1 - \beta_0) + (1 - \pi)(\beta_2 - \beta_0) \end{array} \right\}; \quad \gamma_2 = \{0.5\pi(1 - \pi)\}^{\frac{1}{2}} \begin{pmatrix} \alpha_1 - \alpha_2 \\ \beta_1 - \beta_2 \end{pmatrix}$$

leads to

$$LR_{1n}^p = \max_{\gamma_1, \gamma_2} q(\gamma_1, \gamma_2) + o_p(1),$$

where

$$q(\gamma_1, \gamma_2) = 2 \sum_{i=1}^n \left( \gamma_1^T Y_i + \frac{1}{2} \gamma_1^T Z_i \gamma_1 + \gamma_2^T Z_i \gamma_2 \right) - \sum_{i=1}^n \left( \gamma_1^T Y_i + \frac{1}{2} \gamma_1^T Z_i \gamma_1 + \gamma_2^T Z_i \gamma_2 \right)^2.$$

From Lemma 1 in Charnigo and Sun (2004) and Lemma A2 in Li et al. (2009) that  $\hat{\gamma}_1^p = O_p(n^{-1/2})$  and by the strong law of large numbers that  $n^{-1} \sum_{i=1}^n Z_i = o_p(1)$ . It follows that

$$LR_{1n}^p = \max_{\gamma_1, \gamma_2} q^*(\gamma_1, \gamma_2) + o_p(1),$$

where

$$q^*(\gamma_1, \gamma_2) = 2 \sum_{i=1}^n (\gamma_1^T Y_i + \gamma_2^T Z_i \gamma_2) - \sum_{i=1}^n (\gamma_1^T Y_i + \gamma_2^T Z_i \gamma_2)^2.$$

The maximum value of  $q^*(\gamma_1, \gamma_2)$  is (5) excluding the term  $o_p(1)$ .

## B Asymptotic weight $p(\alpha_0)$

Denote the matrices  $M$  and  $v_j$  by

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}^{-1}; \quad v_j = \begin{pmatrix} v_{j11} & v_{j12} \\ v_{j12} & v_{j22} \end{pmatrix},$$

where

$$\begin{aligned} m_{11} &= \frac{\alpha_0}{-1 + \alpha_0 \Gamma^{(2)}(\alpha_0)}; & m_{12} &= \frac{\beta_0}{-1 + \alpha_0 \Gamma^{(2)}(\alpha_0)}; & m_{22} &= \frac{\beta_0^2 \Gamma^{(2)}(\alpha_0)}{-1 + \alpha_0 \Gamma^{(2)}(\alpha_0)}; \\ v_{111} &= \Gamma^{(3)}(\alpha_0); & v_{112} &= 0; & v_{122} &= \frac{1}{\beta_0^2}; \\ v_{211} &= 0; & v_{212} &= \frac{1}{\beta_0^2}; & v_{222} &= -\frac{2\alpha_0}{\beta_0^3}. \end{aligned}$$

Then, we express  $U_{i[11]}$ ,  $U_{i[12]}$  and  $U_{i[22]}$  by

$$\begin{aligned} U_{i[11]} &= Z_{i[11]} - Y_{i[1]} (m_{11} v_{111} + m_{12} v_{211}) - Y_{i[2]} (m_{12} v_{111} + m_{22} v_{211}); \\ U_{i[12]} &= Z_{i[12]} - Y_{i[1]} (m_{11} v_{112} + m_{12} v_{212}) - Y_{i[2]} (m_{12} v_{112} + m_{22} v_{212}); \\ U_{i[22]} &= Z_{i[22]} - Y_{i[1]} (m_{11} v_{122} + m_{12} v_{222}) - Y_{i[2]} (m_{12} v_{122} + m_{22} v_{222}). \end{aligned}$$

The result in Section 3.2 follows from the central limit theorem and the covariance matrix is obtained by

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} = \left\{ \begin{array}{ccc} E(U_{1[11]} U_{1[11]}) & E(U_{1[11]} U_{1[12]}) & E(U_{1[11]} U_{1[22]}) \\ E(U_{1[11]} U_{1[12]}) & E(U_{1[12]} U_{1[12]}) & E(U_{1[12]} U_{1[22]}) \\ E(U_{1[11]} U_{1[22]}) & E(U_{1[12]} U_{1[22]}) & E(U_{1[22]} U_{1[22]}) \end{array} \right\}.$$

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