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Inversion of Bayes Formula for Events

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In standard notation, let A be a non-void event and $\{H_1, H_2, \cdots, H_m\}$ be a set of events which are (i) non-void, (ii) mutually exclusive (i.e. $H_j \cap H_k = \emptyset$ and hence $P(H_j \cup H_k) = P(H_j) + P(H_k)$ for $j \neq k$), and (iii) collectively exhaustive, $P(\cup_{i=1}^m H_j) = 1$. The **Bayes formula** is

$$P(H_j|A) = \frac{P(A|H_j)P(H_j)}{P(A)} = \frac{P(A|H_j)P(H_j)}{\sum_{k=1}^m P(A|H_k)P(H_k)}, \quad j = 1, \cdots, m, \tag{1}$$

where the last substitution is by virtue of the so-called **formula of total probability**,

$$P(A) = \sum_{k=1}^{m} P(A|H_k)P(H_k), \tag{2}$$

which is valid due to the assumptions (i) to (iii) of $\{H_1, H_2, \cdots, H_m\}$.

In Bayesian inference, which is the first paradigm of statistical inference in history, $\{H_1, H_2, \cdots, H_m\}$ are antecedent events viewed as competing hypotheses and $P(A|H_j)$ is the probability that the event A occurs as an outcome of the jth hypothesis. The investigator assigns $P(H_j)$, called the **prior probability**, to the jth hypothesis based on available information to him/her or in accordance with his/her belief on the odds of the competing hypotheses. Given that A occurs, Bayes formula (1) gives the revised probability, called the **posterior probability**, of the jth competing hypothesis.

Now consider the question whether we can prescribe the posterior probabilities that we want and work out the prior probabilities to get them using Bayes formula. The question amounts to expressing $P(H_j)$ in terms of $P(H_j|A)$, $P(A|H_j)$ (and hence $P(\bar{A}|H_j)$). If the inversion from posterior to prior is in the form of a formula, we shall call it an **IBF** (for **Inverse Bayes Formula**), following Ng (1995a, 1995b, 1997a) in the original setting of probability density function (pdf). In case the inversion is in the form of an algorithm instead of a single formula, we shall call it **IBFA**, meaning **Inversion of Bayes Formula Algorithm**. For the practical need leading to the question in pdf setting, please see Tanner and Wong (1987), Tanner (1996), Ng (1997b) and Tan, Tian and Ng (2009, page 2, 9).

If all IBF or IBFA are correctly derived based on the assumption, called **compatibility** or **consistency**, that the given values of $P(H_j|A)$ and $P(A|H_j)$ are really from an existing set of joint probabilities, one can construct $P(\bar{A} \cap H_j)$ and $P(A \cap H_j)$ and then reconstruct all $P(H_j|A)$ and $P(A|H_j)$, based on the results from the IBF or IBFA. So checking compatibility is a minor issue, being equivalent to checking whether or not the reconstructed probabilities from the IBF or IBFA concur with those we have used as input. *With this said once and for all, the issue of checking compatibility need not be explicitly mentioned in the lemmas and propositions in this article to avoid repetitiveness.* Outside the Bayesian context, it seems apt to call an IBF a **de-conditioning formula (DCF)**, and an IBFA as **DCA**, as it calculates unconditional probabilities from supposed conditional probabilities and checks compatibility.

We shall need the following facts in this article, which are easy to verify.

Lemma 1 (Equivalent statements for non-void events)
If A and B are non-void events, the following statements are equivalent: $P(A|B) = 1$, $P(\bar{A}|B) = 0$, $P(B|\bar{A}) = 0$, $P(B \cap \bar{A}) = 0$, $P(B \cap A) = P(B)$, and $P(B|A) = P(B)/P(A)$.

Lemma 2 (Probability ratios)
If all the probabilities involved in each identity are positive, we have

$$\frac{P(A_1)}{P(A_2)} = \frac{P(A_1|A_2)}{P(A_2|A_1)}; \quad \frac{P(A_1)}{P(A_2)} = \frac{P(A_1|B)}{P(B|A_1)} \Big/ \frac{P(A_2|B)}{P(B|A_2)} = \frac{P(A_1|B)P(B|A_2)}{P(A_2|B)P(B|A_1)}. \tag{3}$$

Probability ratios (or the **odds** in a general sense) are the key quantities in IBF and IBFA. If the probability ratio for A_1 over A_2 is formed through B as in the second identity of Lemma 2, we say that B is a **ratio bridge** (or simply a **bridge** if the ratio-context is clear) of A_1 and A_2 , or simply that B **bridges** A_1 and A_2, or that A_1 **and** A_2 **are ratio-bridged by** B. The "bridge" element is actually displayed in the cyclic appearance of A_1, B, B, A_2 in the numerator and in reverse order in the denominator of the last ratio of the lemma. Furthermore, if A_2 is ratio-bridged to A_3 by another event C, we can form the probability ratio between A_1 and A_3 by multiplication or division of the relevant pair of ratios with the relationship $P(A_1)/P(A_3) = (P(A_1)/P(A_2))(P(A_2)/P(A_3))$, and we shall say in this case that A_1 and A_3 are **ratio-connected.** A set of events is said to be ratio-connected, or **connected** in abbreviation, if every pair is either ratio-bridged or ratio-connected.

Proposition 1 (Inversion of Bayes Formula for Dichotomous Outcomes)
Let A be non-void and $\{H_1, H_2, \cdots, H_m\}$ satisfy the conditions (i) to (iii).

(a) If $P(A|H_j)$ and $P(\bar{A}|H_j)$ be both positive for some j, then the following are true:

$$P(A) = \left\{1 + \frac{P(\bar{A}|H_j)P(H_j|A)}{P(H_j|\bar{A})P(A|H_j)}\right\}^{-1}, \quad P(\bar{A}) = \left\{1 + \frac{P(A|H_j)P(H_j|\bar{A})}{P(H_j|A)P(\bar{A}|H_j)}\right\}^{-1} \tag{4}$$

$$P(H_k) = P(H_k|A)P(A), \quad k = 1, \cdots, m. \tag{5}$$

If $P(A|H_j)$ and $P(\bar{A}|H_j)$ are positive for more than one j, the above results are identical for all such j.

(b) Consider the case where, for any $j = 1, \cdots, m$, either $P(A|H_j)$ or $P(\bar{A}|H_j)$ is zero. If $P(\bar{A}|H_j) = 0$ for all j, then A is the sure event, $P(A) = 1$. If $P(H_j|A) > 0$ for only a number of H_j while $(H_j|\bar{A}) > 0$ for the remaining H_j, then there is no unique solution for $P(A), P(\bar{A})$ and $P(H_j)$.

Proof: Noting A and $\bar{A}$ are ratio-bridged by H_j and that $P(\bar{A}) = 1 - P(A)$, we can solve for $P(A)$, getting (4) and hence (5). Since the derivation for (4) and (5) is the same for any other j, hence the results should be the same regardless the choice among such j. The first part of (b) is trivial. For the second part we may assume, without loss of generality, only the first k of the $P(H_j|A)$ are positive and only the last $(m-k)$ of the $P(H_j|\bar{A})$ are positive, as shown in the table below, where the 0 and 1 in the cells are based on Lemma 1 and hence the values for the two sets of conditional probabilities are subject to $\sum_{j=1}^{k} a_j = 1$ (conditional on A) and $\sum_{j=k+1}^{m} b_j = 1$ (conditional on $\bar{A}$):

$P(A_i\|H_j)$ \ $P(H_j\|A_i)$	H_1	$\cdots$	H_k	H_{k+1}	$\cdots$	H_m	$P(A_i)$
$A_1 = A$	1 / a_1	$\cdots$	1 / a_k	0 / 0	$\cdots$	0 / 0	q
$A_2 = \bar{A}$	0 / 0	$\cdots$	0 / 0	1 / b_{k+1}	$\cdots$	1 / b_m	$1-q$
$P(H_j)$	π_1	$\cdots$	π_k	π_{k+1}	$\cdots$	π_m	1

The following m equations have m unknowns, q and π_j, as $\sum_{j=1}^{m} \pi_j = 1$,

$$\pi_j = qa_j,\ j = 1, \cdots, k; \quad \pi_j = (1-q)b_j,\ j = k+1, \cdots, m.$$

Note that the sum of the m equations gives 1 on both sides, hence the number of effective equations is less than the number of unknowns and therefore the solution is not unique.

Now let us consider the general case when each hypothesis can lead to n outcome events, $A_1, A_2, \cdots, A_n$, which satisfy the same assumptions (i) to (iii) for H_j. The values of the two sets of $n \times m$ conditional probabilities are to be arranged in the following table where in each cell L_{ij} represents the likelihood $P(A_i|H_j)$, P_{ij} the posterior probability $P(H_j|A_i)$, π_j the prior probability $P(H_j)$ and $q_i = P(A_i)$:

$P(A_i\|H_j)$ \ $P(H_j\|A_i)$	H_1	H_2	$\cdots$	H_j	$\cdots$	H_m	$P(A_i)$
A_1	L_{11} / P_{11}	L_{12} / P_{12}	$\cdots$	L_{1j} / P_{1j}	$\cdots$	L_{1m} / P_{1m}	q_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_i	L_{i1} / P_{i1}	L_{i2} / P_{i2}	$\cdots$	L_{ij} / P_{ij}	$\cdots$	L_{im} / P_{im}	q_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	L_{n1} / P_{n1}	L_{n2} / P_{n2}	$\cdots$	L_{nj} / P_{nj}	$\cdots$	L_{nm} / P_{nm}	q_n
$P(H_j)$	π_1	π_2	$\cdots$	π_j	$\cdots$	π_m	1

Since L_{ij} and P_{ij} in each cell are *simultaneously* zero or non-zero according to Lemma 1, we can form the ratio $r_{ij} = P_{ij}/L_{ij}$ in each positive cell, getting the system of equations,

$$\pi_j/q_i = r_{ij} \equiv P_{ij}/L_{ij}, \quad i = 1, 2, \cdots, n;\ j = 1, \cdots, m. \tag{6}$$

So there are $m+n-2$ effective unknowns, namely q_i and π_j subject to constraints $\sum_{j=1}^{m} \pi_j = 1$ and $\sum_{i=1}^{n} q_i = 1$, in as many number of equations as the number of positive r_{ij}.

When all cells are positive, i.e. all r_{ij} are defined, it is called the **positivity condition** whose counter part in pdf setting for continuous random variables is widely assumed. The next proposition is for this simplest case.

Proposition 2 (IBF under positivity condition in the whole table)
Under the positivity condition, the following IBF are valid:

$$\pi_j = 1/\sum_{i=1}^{n} r_{ij}^{-1},\ j = 1, \cdots, m; \quad q_i = 1/\sum_{j=1}^{m} r_{ij},\ i = 1, \cdots, n. \tag{7}$$

Proof: For each i, sum all the m equations in (6) to get $1/q_i = \sum_{j=1}^{m} r_{ij}$, hence the second equation of (7). Flipping the ratio in (6) to get $q_i/\pi_j = r_{ij}^{-1}$ and then sum them over $i = 1, \cdots, n$ to get the first equation of (7).

Although positivity condition is widely assumed for continuous pdf in applications, it is frequently not realistic for events. The ideas in this article for dealing with situations without positivity condition, including ratio-bridging and ratio-connecting, have been presented in the 1996 Sydney International Statistical Congress by Ng (1996). We first note some obvious properties concerning the combined two-way table of probabilities, which together with Lemma 2, shall form the key to IBF and IBFA in sparse structure of positive cells.

Lemma 3 (Properties for the two-way table of events)
With respect to the $n \times m$ table for events A_i and H_j, the following are true.

(a) Without loss of generality, rows and columns can be permuted any number of times.

(b) Without loss of generality, any row or any column whose cells are all zeros can be discarded.

(c) Given the values of $(\pi_1/\pi_m, \pi_2/\pi_m, \cdots, \pi_{m-1}/\pi_m)$, the set of $(m-1)$ ratios relative to π_m, we have a complete solution for all π_j:

$$\pi_m = \left\{1 + \sum_{i=1}^{m-1} \frac{\pi_j}{\pi_m}\right\}^{-1}, \quad \pi_j = \frac{\pi_j}{\pi_m}\pi_m, \; j = 1, 2, \cdots, m-1. \tag{8}$$

The solution is similar if any one of π_i is used as the common denominator, providing another set of $n-1$ ratios. Furthermore, if the complete consecutive ratios $(\pi_1/\pi_2, \pi_2/\pi_3, \cdots, \pi_{n-1}/\pi_n)$ are given, we can get the ratios against a common denominator by chained multiplications,

$$\frac{\pi_1}{\pi_m} = \prod_{j=1}^{m-1} \frac{\pi_j}{\pi_{j+1}}, \quad \frac{\pi_2}{\pi_m} = \prod_{j=2}^{m-1} \frac{\pi_j}{\pi_{j+1}}, \quad \cdots \quad \frac{\pi_{m-1}}{\pi_m} = \prod_{j=m-1}^{m-1} \frac{\pi_i}{\pi_{i+1}}. \tag{9}$$

Finally, all of the above is analogous for $q_i, i = 1, \cdots, n$.

Note that part (c) says that the key to IBF, or de-conditioning, is whether or not the whole set of H_j, or of A_i, are ratio-connected as defined immediately after Lemma 2, and if yes, it actually provides specific ways of calculating the unconditional probabilities. The following is a special case where the whole set of H_j are ratio-bridged by one event A_i, or vice versa.

Proposition 3 (IBFA under positivity in one row or column)

(a) If the ith row is positive, the inversion of Bayes formula is as follows:

$$\pi_m = \left\{1 + r_{im}^{-1} \sum_{j=1}^{m-1} r_{ij}\right\}^{-1}, \; \pi_j = \pi_m \frac{r_{ij}}{r_{im}}, \; j = 1, \cdots, m-1. \tag{10}$$

$$q_i = \sum_{j=1}^{m} \pi_j L_{ij}, \quad i = 1, \cdots, n. \tag{11}$$

In case of more than one row with positive cells, the above solution is the same regardless the choice among such rows.

(b) If the jth column is positive, the solution of inverting Bayes formula is as follows:

$$q_n = \left\{1 + r_{nj} \sum_{i=1}^{n-1} r_{ij}^{-1}\right\}^{-1}, \quad q_i = q_n \frac{r_{nj}}{r_{ij}}, \ i = 1, \cdots, n-1. \tag{12}$$

$$\pi_j = \sum_{i=1}^{n} q_i P_{ij}, \quad j = 1, \cdots, m. \tag{13}$$

In case of more than one column with positive cells, the above solution does not depend on the choice among such columns.

Proof: For (a), note that every H_j is ratio-bridged to H_m by the same A_i, and hence according to the second equality of (3), $\pi_j/\pi_m = r_{ij}/r_{im}$ $(j = 1, \cdots, m-1)$. By means of (8), we have (10) and hence (11), where the reasoning is the same for any positive row. Part (b) is done in a similar way for a positive column.

Example 1 (IBFA for zigzag paths of positive cells)

The H_j can be ratio-connected through a number of A_i if there is a zigzag path of positive r_{ij} from the first column to the last after permutations of rows and columns. In this case, the consecutive ratios for π_j are readily available for use in (9) of Lemma 3. The same can be said about a path from the first row to the last, or about one corner to the opposite diagonal corner. The following are two examples for illustration, where all well-defined $r_{ij} = P_{ij}/L_{ij}$ are shown, while an empty cell means that the ratio is not defined for a pair of zeros:

	π_1	π_2	π_3	π_4	π_5	π_6
q_1						r_{16}
q_2					r_{25}	
q_3	r_{31}	r_{32}	r_{33}			
q_4			r_{43}		r_{45}	
q_5			r_{53}	r_{54}	r_{55}	r_{56}
q_6		r_{62}				
q_7	r_{71}					

	π_1	π_2	π_3	π_4	π_5	π_6
q_1	r_{11}					
q_2	r_{21}	r_{22}				
q_3		r_{32}	r_{33}			
q_4			r_{43}	r_{44}		
q_5				r_{54}	r_{55}	
q_6				r_{64}	r_{65}	r_{66}
q_7						r_{76}

On the left table, A_3 bridges the first 3 of H_j with consecutive ratios, $\pi_1/\pi_2 = r_{31}/r_{32}$ and $\pi_2/\pi_3 = r_{32}/r_{33}$, and A_5 bridges the last 4 of H_j with consecutive ratios, $\pi_3/\pi_4 = r_{53}/r_{54}$, $\pi_4/\pi_5 = r_{54}/r_{55}$ and $\pi_5/\pi_6 = r_{55}/r_{56}$. So we can plug in (9) and then (8) to get all π_i. For easier discussion in the sequel, it would be convenient to express the ratio-connection process in symbols as follows: [1 : 1, 2, 3] stands for "row 3 bridging the first 3 columns" and $[5 : 3, 4, 5, 6]$ stands for "row 5 bridging the last 4", which together lead to the complete ratio-connection $\{1, 2, 3, 4, 5, 6\}$, since the two ratio-bridged sets have a common member, 3.

On the right table, the connecting process for H_j is: $[2 : 1, 2]$, $[3 : 2, 3]$, $[4 : 3, 4]$, $[5 : 4, 5]$, $[6 : 4, 5, 6]$, cumulatively leading to $\{1, 2, 3, 4, 5, 6\}$; note that $[5 : 4, 5]$ is ignorable due to $[6 : 4, 5, 6]$. For A_i, it is: $[1 : 1, 2]$, $[2 : 2, 3]$, $[3 : 3, 4]$, $[4 : 4, 5, 6]$, $[5 : 5, 6]$ (ignorable), $[6 : 6, 7]$, cumulatively leading to $\{1, 2, 3, 4, 5, 6, 7\}$.

We summarize the procedure as demonstrated in the above example as

Proposition 4 (IBFA for connecting zigzag paths of positive cells)
If, by permutations of rows and columns in the $n \times m$ table, there is a zigzag path of positive cells connecting all columns, then an IBFA for use in Lemma 3(c) is to find the consecutive ratios, $\pi_j/\pi_{j+1} = r_{ij}/r_{i,j+1}$, $j = 1, \cdots, m$, by going through the horizontal segments, numbered by i along the path. If the path connects all rows, it is to get $q_i/q_{i+1} = r_{i+1,j}/r_{ij}$, $i = 1, \cdots, n$, in a similar manner along the vertical segments. If the path connects opposite diagonal corners, an IBFA is to get both π_j/π_{j+1} and q_i/q_{i+1} along the path respectively from the horizontal and vertical segments.

Example 2 (IBFA for scattered positive cells)

If it is not handy to get a table with a zigzag path of positive cells by permutations of rows and columns, we can follow the above process of ratio-connection by means of examining row by row, or column by column. The following work-sheet demonstrates handling a table with a non-zigzag pattern which is amenable for de-conditioning as shown:

	π_1	π_2	π_3	π_4	π_5	π_6	j bridged	Connected	Ratios
q_1				r_{14}		r_{16}	$[4, 6]$	$\{4, 6\}$	$\pi_6/\pi_4 = r_{16}/r_{14}$
q_2		r_{22}	r_{23}		r_{25}		$[2, 3, 5]$	$\{2, 3, 5; 4, 6\}$	$\pi_2/\pi_5 = r_{22}/r_{25}$, $\pi_3/\pi_5 = r_{23}/r_{25}$
q_3						r_{36}			
q_4	r_{41}				r_{45}		$[1, 5]$	$\{2, 3, 5, 1; 4, 6\}$	$\pi_1/\pi_5 = r_{41}/r_{45}$
q_5			r_{53}					$\{2, 3, 5, 1; 4, 6\}$	
q_6		r_{62}			r_{65}		$[2, 5]$		available on 2nd row
q_7	r_{71}			r_{74}			$[1, 4]$	$\{2, 3, 5, 1, 4, 6\}$	$\pi_4/\pi_1 = r_{74}/r_{71}$

Column j:	5	3	1	4	6
i bridged:	$[2, 4, 6]$	$[2, 5]$	$[4, 7]$	$[1, 7]$	$[1, 3]$
Connected:	$\{2, 4, 6\}$	$\{2, 4, 6, 5\}$	$\{2, 4, 6, 5, 7\}$	$\{2, 4, 6, 5, 7, 1\}$	$\{2, 4, 6, 5, 7, 1, 3\}$
Ratios:	$q_4/q_2 = r_{25}/r_{45}$ $q_6/q_2 = r_{25}/r_{65}$	$q_5/q_2 = r_{23}/r_{53}$	$q_7/q_4 = r_{41}/r_{71}$	$q_1/q_7 = r_{74}/r_{14}$	$q_3/q_1 = r_{16}/r_{36}$

The first column on the right side of the table shows subscripts of bridged H_j by A_i, where A_3 and A_5 do not bridge at all. We use short-hand $[4, 6]$ for $[1 : 4, 6]$ and $[2, 3, 5]$ for $[2 : 2, 3, 5]$, etc., since the row no. is clear in a work-sheet like this one, and similarly for the "i bridged" row below the table. Note that a semi-colon separates ratio-connected cluster of events. In row by row process, the two connected clusters, $\{2,3,5,1\}$ and $\{4,6\}$ are connected by the last row where each member of the pair $[1,4]$ intersects one cluster only. For columns, we start with column 5 (bridging more) and ignore column 2 (its bridged set being a subset). The connection is completed through columns in the order $j = 5,3,1,4,6$.

Now the ratios in (8) of Lemma(c) are calculated against the most bridged j, here $j = 5$. As shown in the last column of work-sheet, bridging gives π_6/π_4 via $i = 1$, π_2/π_5 and π_3/π_5 via $i = 2$, π_1/π_5 via $i = 4$, π_4/π_1 via $i = 7$. Connecting $i = 7,4$, we get $(\pi_4/\pi_1)(\pi_1/\pi_5) = \pi_4/\pi_5$; connecting $i = 1,7,4$, we get $(\pi_6/\pi_4)(\pi_4/\pi_1)(\pi_1/\pi_5) = \pi_6/\pi_5$. All 5 ratios to π_5 are ready.

In the work-sheet, the second row below the table shows q_2 being the best common denominator for q_i. As shown on the last row of work-sheet, bridging gives $q_4/q_2 = r_{25}/r_{45}$ and $q_6/q_2 = r_{25}/r_{65}$ via $j = 5$; $q_5/q_2 = r_{23}/r_{53}$ via $j = 3$; $q_7/q_4 = r_{41}/r_{71}$ via $j = 1$; $q_1/q_7 = r_{74}/r_{14}$ via $j = 4$; $q_3/q_1 = r_{16}/r_{36}$ via $j = 6$. Connecting by $j = 1,5$, we get $(q_7/q_4)(q_4/q_2) = q_7/q_2$; by $j = 4,1,5$, we get $(q_1/q_7)(q_7/q_4)(q_4/q_2) = q_1/q_2$; connecting $j = 6$ to that of $i = 4,1,5$, we get $(q_3/q_1)(q_1/q_2) = q_3/q_2$. So all 6 ratios to q_2 are available.

Now suppose that all cells in the last row are zero, i.e. r_{71} and r_{74} are not defined. So we are dealing with a 6×6 table. The procedure by rows stops at the 6th row and there is no ratio-connecting between $\{2,3,5,1\}$ and $\{4,6\}$. Although the ratios within clusters are determined as before, there are infinitely many possible proportions, $a : (1-a), 0 < a < 1$, to be allocated to the two clusters, each being as good as another in reproducing the supposedly compatible P_{ij} and L_{ij} which define r_{ij}. Similarly for the procedure by columns, the two clusters, $\{2,4,6,5\}$ and $\{1,3\}$, are not connected and there are infinitely many solutions.

The IBFA being illustrated by Example 2 is summarized in the following proposition.

Proposition 5 (IBFA for sparse patterns of positive cells)
In case it is not handy to obtain a zigzag path as assumed in the above proposition, the IBFA consists of the following steps for columns:

(1) For efficiency purpose, permute the rows and columns so that the beginning rows and columns have more positive cells. Then set up a work-sheet as shown in Example 2.

(2) In each row, write all the subscripts of ratio-bridged H_j in a square bracket.

(3) Any two ratio-bridged sets of subscripts in different rows are ratio-connected if there is one subscript in common. The bridged set of subscripts on each row is either ratio-connected to some cumulatively connected cluster, or form a new connected cluster by itself. The connected clusters, separated by semi-colons, are put in a curly bracket for each row.

(4) The step (3) is continued until all m subscripts of H_j are ratio-connected; in this case the rows which are not needed in making the complete connection will be ignored in the steps that follow. If all rows are exhausted but there is still no complete connection of all m subscripts, there are more than one IBF solution.

(5) Choose a subscript, say k, that is directly ratio-bridged to other subscripts as many as possible among all rows.

(6) For each bridged set that contains k, get the ratios $\pi_j/\pi_k = r_{ij}/r_{ik}$ where i is the row number of the bridged set and j is any other subscript in the same bridged set with k.

(7) For subscripts of H_j which are not bridged to k, but connected to k through other subscripts, the ratios π_j/π_k for these subscripts are obtained by chained multiplications (or divisions) of ratios involving those other subscripts, with iterated substitutions of previously available ratios if so needed.

The corresponding IBFA based on rows is similar, with $q_i/q_k = r_{kj}/r_{ij}$ for step (6), where q_k is the common denominator and column j bridges row i and row k.

The Bayes formula was developed in a manuscript by Reverend Thomas Bayes and, after his death in 1761, was submitted by his friend to the Royal Society for posthumous publication in 1763. It is still a puzzle as why Bayes, who "was for twenty years a Fellow of the Royal Society" (Fisher 1973, p.8), did not submit his fine essay. Fisher (1973, p.9) wrote: "it seems clear that Bayes had recognized that the postulate proposed in his argument (though not used in his central theorem) would be thought disputable by a critical reader, and there can be little doubt that this was the reason why his treatise was not offered for publication in his own lifetime." Stigler (1983) provided another conjecture. It is a conjecture of the author of this article that, after finishing the manuscript, Bayes recognized the inversion of his formula. His prior probabilities, therefore, could be perceived as the results of reverse-engineering. So he had to think about the implications of the argument and needed more time to re-write his essay (hand-written with feather and ink at that time).

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